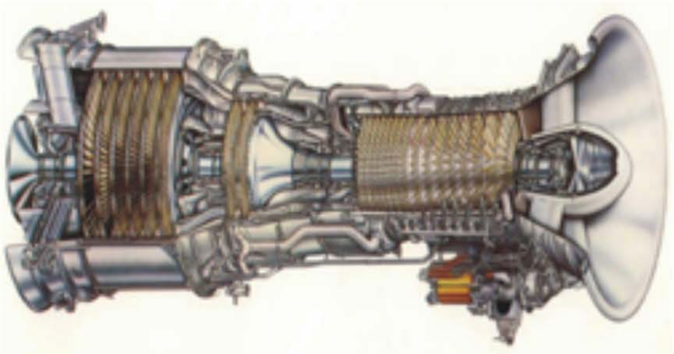
REVISED NINTH EDITION

A Textbook of

FLUID MECHANICS AND HYDRAULIC MACHINES

S.I. Units



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Published by :
LAXMI PUBLICATIONS (P) LTD
113, Golden House, Daryaganj,
New Delhi-110002

Phone : 011-43 53 25 00
Fax : 011-43 53 25 28

www.laxmipublications.com
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Compiled by : **Smt. Nirmal Bansal**

Price : **Rs. 495.00** *Only.*

First Edition : Sept. 1983
Ninth Edition : 2005
Reprint : 2006, 2007, 2008, 2009
Revised Ninth Edition : 2010

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EFM-0559-495-FLUID MECHANICS & HM-BAN **C—**

Typesetted at : Shubham Composer, New Delhi.
Printed at : Repro India Ltd., Mumbai.

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(B) IDEAL FLOW (POTENTIAL FLOW)

1

CHAPTER

PROPERTIES OF FLUIDS

▶ 1.1 INTRODUCTION

Fluid mechanics is that branch of science which deals with the behaviour of the fluids (liquids or gases) at rest as well as in motion. Thus this branch of science deals with the static, kinematics and dynamic aspects of fluids. The study of fluids at rest is called fluid statics. The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics and if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics.

▶ 1.2 PROPERTIES OF FLUIDS

1.2.1 Density or Mass Density. Density or mass density of a fluid is defined as the ratio of the mass of a fluid to its volume. Thus mass per unit volume of a fluid is called density. It is denoted by the symbol ρ (rho). The unit of mass density in SI unit is kg per cubic metre, *i.e.*, kg/m^3. The density of liquids may be considered as constant while that of gases changes with the variation of pressure and temperature.

Mathematically, mass density is written as

$$\rho = \frac{\text{Mass of fluid}}{\text{Volume of fluid}} .$$

The value of density of water is 1 gm/cm^3 or 1000 kg/m^3.

1.2.2 Specific Weight or Weight Density. Specific weight or weight density of a fluid is the ratio between the weight of a fluid to its volume. Thus weight per unit volume of a fluid is called weight density and it is denoted by the symbol w.

Thus mathematically, $$w = \frac{\text{Weight of fluid}}{\text{Volume of fluid}} = \frac{(\text{Mass of fluid}) \times \text{Acceleration due to gravity}}{\text{Volume of fluid}}$$

$$= \frac{\text{Mass of fluid} \times g}{\text{Volume of fluid}}$$

$$= \rho \times g \qquad \left\{\because \frac{\text{Mass of fluid}}{\text{Volume of fluid}} = \rho\right\}$$

$$\therefore \qquad w = \rho g \qquad \text{...(1.1)}$$

The value of specific weight or weight density (w) for water is 9.81×1000 Newton/m^3 in SI units.

1.2.3 Specific Volume. Specific volume of a fluid is defined as the volume of a fluid occupied by a unit mass or volume per unit mass of a fluid is called specific volume. Mathematically, it is expressed as

$$\text{Specific volume} = \frac{\text{Volume of fluid}}{\text{Mass of fluid}} = \frac{1}{\dfrac{\text{Mass of fluid}}{\text{Volume of fluid}}} = \frac{1}{\rho}$$

Thus specific volume is the reciprocal of mass density. It is expressed as m^3/kg. It is commonly applied to gases.

1.2.4 Specific Gravity. Specific gravity is defined as the ratio of the weight density (or density) of a fluid to the weight density (or density) of a standard fluid. For liquids, the standard fluid is taken water and for gases, the standard fluid is taken air. Specific gravity is also called relative density. It is dimensionless quantity and is denoted by the symbol S.

Mathematically, $$S(\text{for liquids}) = \frac{\text{Weight density (density) of liquid}}{\text{Weight density (density) of water}}$$

$$S(\text{for gases}) = \frac{\text{Weight density (density) of gas}}{\text{Weight density (density) of air}}$$

$$\text{Thus weight density of a liquid} = S \times \text{Weight density of water}$$
$$= S \times 1000 \times 9.81 \text{ N/m}^3$$
$$\text{The density of a liquid} = S \times \text{Density of water}$$
$$= S \times 1000 \text{ kg/m}^3. \qquad \text{...(1.1}A\text{)}$$

If the specific gravity of a fluid is known, then the density of the fluid will be equal to specific gravity of fluid multiplied by the density of water. For example, the specific gravity of mercury is 13.6, hence density of mercury = $13.6 \times 1000 = 13600$ kg/m^3.

Problem 1.1 *Calculate the specific weight, density and specific gravity of one litre of a liquid which weighs 7 N.*

Solution. Given :

$$\text{Volume} = 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \qquad \left(\because \ 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \text{ or } 1 \text{ litre} = 1000 \text{ cm}^3\right)$$

$$\text{Weight} = 7 \text{ N}$$

(*i*) Specific weight (w) $$= \frac{\text{Weight}}{\text{Volume}} = \frac{7 \text{ N}}{\left(\dfrac{1}{1000}\right) \text{m}^3} = \mathbf{7000 \text{ N/m}^3}. \textbf{ Ans.}$$

(*ii*) Density (ρ) $$= \frac{w}{g} = \frac{7000}{9.81} \text{ kg/m}^3 = \mathbf{713.5 \text{ kg/m}^3}. \textbf{ Ans.}$$

(*iii*) Specific gravity $$= \frac{\text{Density of liquid}}{\text{Density of water}} = \frac{713.5}{1000} \qquad \{\because \ \text{Density of water} = 1000 \text{ kg/m}^3\}$$

$$= \mathbf{0.7135}. \textbf{ Ans.}$$

Problem 1.2 *Calculate the density, specific weight and weight of one litre of petrol of specific gravity = 0.7*

Solution. Given : Volume = 1 litre = $1 \times 1000\ \text{cm}^3 = \frac{1000}{10^6}\ \text{m}^3 = 0.001\ \text{m}^3$

Sp. gravity $S = 0.7$

(*i*) *Density* (ρ)

Using equation (1.1*A*),

Density (ρ) $= S \times 1000\ \text{kg/m}^3 = 0.7 \times 1000 =$ **700 kg/m³. Ans.**

(*ii*) *Specific weight* (*w*)

Using equation (1.1), $w = \rho \times g = 700 \times 9.81\ \text{N/m}^3 =$ **6867 N/m³. Ans.**

(*iii*) *Weight (W)*

We know that specific weight $= \frac{\text{Weight}}{\text{Volume}}$

or $w = \frac{W}{0.001}$ or $6867 = \frac{W}{0.001}$

∴ $W = 6867 \times 0.001 =$ **6.867 N. Ans.**

1.3 VISCOSITY

Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. When two layers of a fluid, a distance '*dy*' apart, move one over the other at different velocities, say *u* and *u* + *du* as shown in Fig. 1.1, the viscosity together with relative velocity causes a shear stress acting between the fluid layers.

The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer. This shear stress is proportional to the rate of change of velocity with respect to *y*. It is denoted by symbol τ (Tau).

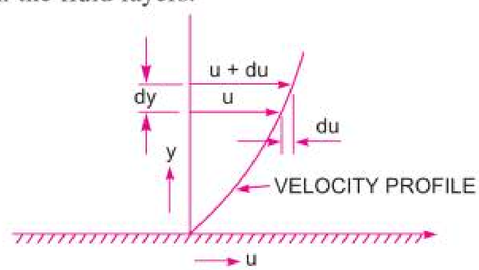

Fig. 1.1 *Velocity variation near a solid boundary.*

Mathematically, $\tau \propto \frac{du}{dy}$

or $$\tau = \mu \frac{du}{dy} \quad \text{...(1.2)}$$

where μ (called *mu*) is the constant of proportionality and is known as the co-efficient of dynamic viscosity or only viscosity. $\frac{du}{dy}$ represents the rate of shear strain or rate of shear deformation or velocity gradient.

From equation (1.2), we have $$\mu = \frac{\tau}{\left(\frac{du}{dy}\right)} \quad \text{...(1.3)}$$

Thus viscosity is also defined as the shear stress required to produce unit rate of shear strain.

1.3.1 Units of Viscosity. The units of viscosity is obtained by putting the dimensions of the quantities in equation (1.3)

$$\mu = \frac{\text{Shear stress}}{\dfrac{\text{Change of velocity}}{\text{Change of distance}}} = \frac{\text{Force/Area}}{\left(\dfrac{\text{Length}}{\text{Time}}\right) \times \dfrac{1}{\text{Length}}}$$

$$= \frac{\text{Force/(Length)}^2}{\dfrac{1}{\text{Time}}} = \frac{\text{Force} \times \text{Time}}{(\text{Length})^2}$$

In MKS system, force is represented by kgf and length by metre (m), in CGS system, force is represented by dyne and length by cm and in SI system force is represented by Newton (N) and length by metre (m).

$$\therefore \quad \text{MKS unit of viscosity} = \frac{\text{kgf-sec}}{\text{m}^2}$$

$$\text{CGS unit of viscosity} = \frac{\text{dyne-sec}}{\text{cm}^2}$$

In the above expression N/m^2 is also known as Pascal which is represented by Pa. Hence N/m^2 = Pa = Pascal

$$\therefore \quad \text{SI unit of viscosity} = \text{Ns/m}^2 = \text{Pa s}.$$

$$\text{SI unit of viscosity} = \frac{\text{Newton-sec}}{\text{m}^2} = \frac{\text{Ns}}{\text{m}^2}$$

The unit of viscosity in CGS is also called Poise which is equal to $\dfrac{\text{dyne-sec}}{\text{cm}^2}$.

The numerical conversion of the unit of viscosity from MKS unit to CGS unit is given below :

$$\frac{\text{one kgf-sec}}{\text{m}^2} = \frac{9.81\ \text{N-sec}}{\text{m}^2} \qquad \{\because\ 1\ \text{kgf} = 9.81\ \text{Newton}\}$$

$$\text{But one Newton} = \text{one kg (mass)} \times \text{one} \left(\frac{\text{m}}{\text{sec}^2}\right) \text{(acceleration)}$$

$$= \frac{(1000\ \text{gm}) \times (100\ \text{cm})}{\text{sec}^2} = 1000 \times 100\ \frac{\text{gm-cm}}{\text{sec}^2}$$

$$= 1000 \times 100\ \text{dyne} \qquad \left\{\because\ \text{dyne} = \text{gm} \times \frac{\text{cm}}{\text{sec}^2}\right\}$$

$$\therefore \quad \frac{\text{one kgf-sec}}{\text{m}^2} = 9.81 \times 100000\ \frac{\text{dyne-sec}}{\text{cm}^2} = 9.81 \times 100000\ \frac{\text{dyne-sec}}{100 \times 100 \times \text{cm}^2}$$

$$= 98.1\ \frac{\text{dyne-sec}}{\text{cm}^2} = 98.1\ \text{poise} \qquad \left\{\because\ \frac{\text{dyne-sec}}{\text{cm}^2} = \text{Poise}\right\}$$

Thus for solving numerical problems, if viscosity is given in poise, it must be divided by 98.1 to get its equivalent numerical value in MKS.

$$\text{But} \quad \frac{\text{one kgf-sec}}{\text{m}^2} = \frac{9.81\ \text{Ns}}{\text{m}^2} = 98.1\ \text{poise}$$

$$\therefore \quad \frac{\text{one Ns}}{\text{m}^2} = \frac{98.1}{9.81}\ \text{poise} = 10\ \text{poise} \quad \text{or} \quad \text{One poise} = \frac{1}{10}\ \frac{\text{Ns}}{\text{m}^2}.$$

Alternate Method. One poise $= \dfrac{\text{dyne} \times \text{s}}{\text{cm}^2} = \left(\dfrac{1\,\text{gm} \times 1\,\text{cm}}{\text{s}^2}\right) \times \dfrac{\text{s}}{\text{cm}^2}$

But dyne $= 1\text{ gm} \times \dfrac{1\,\text{cm}}{\text{s}^2}$

$\therefore$ One poise $= \dfrac{1\,\text{gm}}{\text{s cm}} = \dfrac{\dfrac{1}{1000}\,\text{kg}}{\text{s}\,\dfrac{1}{100}\,\text{m}}$

$= \dfrac{1}{1000} \times 100\,\dfrac{\text{kg}}{\text{sm}} = \dfrac{1}{10}\,\dfrac{\text{kg}}{\text{sm}}$ or $1\,\dfrac{\text{kg}}{\text{sm}} = 10$ poise.

Note. (*i*) In SI units second is represented by 's' and not by 'sec'.

(*ii*) If viscosity is given in poise, it must be divided by 10 to get its equivalent numerical value in SI units. Sometimes a unit of viscosity as centipoise is used where

$$1 \text{ centipoise} = \frac{1}{100} \text{ poise} \quad \text{or} \quad 1 \text{ cP} = \frac{1}{100} \text{ P} \qquad [\text{cP = Centipoise, P = Poise}]$$

The viscosity of water at 20°C is 0.01 poise or 1.0 centipoise.

1.3.2 Kinematic Viscosity.

It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by the Greek symbol (ν) called 'nu'. Thus, mathematically,

$$\nu = \frac{\text{Viscosity}}{\text{Density}} = \frac{\mu}{\rho} \qquad ...(1.4)$$

The units of kinematic viscosity is obtained as

$$\nu = \frac{\text{Units of } \mu}{\text{Units of } \rho} = \frac{\text{Force} \times \text{Time}}{(\text{Length})^2 \times \dfrac{\text{Mass}}{(\text{Length})^3}} = \frac{\text{Force} \times \text{Time}}{\dfrac{\text{Mass}}{\text{Length}}}$$

$$= \frac{\text{Mass} \times \dfrac{\text{Length}}{(\text{Time})^2} \times \text{Time}}{\left(\dfrac{\text{Mass}}{\text{Length}}\right)} \qquad \left\{\begin{aligned} \because\ \text{Force} &= \text{Mass} \times \text{Acc.} \\ &= \text{Mass} \times \frac{\text{Length}}{\text{Time}^2} \end{aligned}\right\}$$

$$= \frac{(\text{Length})^2}{\text{Time}}.$$

In MKS and SI, the unit of kinematic viscosity is metre^2/sec or m^2/sec while in CGS units it is written as cm^2/s. In CGS units, kinematic viscosity is also known as stoke.

Thus, one stoke $= \text{cm}^2/\text{s} = \left(\dfrac{1}{100}\right)^2 \text{m}^2/\text{s} = 10^{-4}\,\text{m}^2/\text{s}$

Centistoke means $= \dfrac{1}{100}$ stoke.

1.3.3 Newton's Law of Viscosity.

It states that the shear stress (τ) on a fluid element layer is directly proportional to the rate of shear strain. The constant of proportionality is called the co-efficient of viscosity. Mathematically, it is expressed as given by equation (1.2) or as

$$\tau = \mu \frac{du}{dy}.$$

Fluids which obey the above relation are known as **Newtonian** fluids and the fluids which do not obey the above relation are called **Non-Newtonian** fluids.

1.3.4 Variation of Viscosity with Temperature. Temperature affects the viscosity. The viscosity of liquids decreases with the increase of temperature while the viscosity of gases increases with the increase of temperature. This is due to reason that the viscous forces in a fluid are due to cohesive forces and molecular momentum transfer. In liquids, the cohesive forces predominates the molecular momentum transfer, due to closely packed molecules and with the increase in temperature, the cohesive forces decreases with the result of decreasing viscosity. But in case of gases the cohesive forces are small and molecular momentum transfer predominates. With the increase in temperature, molecular momentum transfer increases and hence viscosity increases. The relation between viscosity and temperature for liquids and gases are:

(*i*) **For liquids,** $\mu = \mu_0 \left(\dfrac{1}{1 + \alpha t + \beta t^2} \right)$ ...(1.4*A*)

where μ = Viscosity of liquid at t°C, in poise

μ_0 = Viscosity of liquid at 0°C, in poise

α, β = Constants for the liquid

For water, $\mu_0 = 1.79 \times 10^{-3}$ poise, $\alpha = 0.03368$ and $\beta = 0.000221$.

Equation (1.4*A*) shows that with the increase of temperature, the viscosity decreases.

(*ii*) **For a gas,** $\mu = \mu_0 + \alpha t - \beta t^2$ ...(1.4*B*)

where for air $\mu_0 = 0.000017$, $\alpha = 0.000000056$, $\beta = 0.1189 \times 10^{-9}$.

Equation (1.4*B*) shows that with the increase of temperature, the viscosity increases.

1.3.5 Types of Fluids. The fluids may be classified into the following five types :

1. Ideal fluid,
2. Real fluid,
3. Newtonian fluid,
4. Non-Newtonian fluid, and
5. Ideal plastic fluid.

1. Ideal Fluid. A fluid, which is incompressible and is having no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity.

2. Real Fluid. A fluid, which possesses viscosity, is known as real fluid. All the fluids, in actual practice, are real fluids.

3. Newtonian Fluid. A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid.

4. Non-Newtonian Fluid. A real fluid, in which the shear stress is not proportional to the rate of shear strain (or velocity gradient), known as a Non-Newtonian fluid.

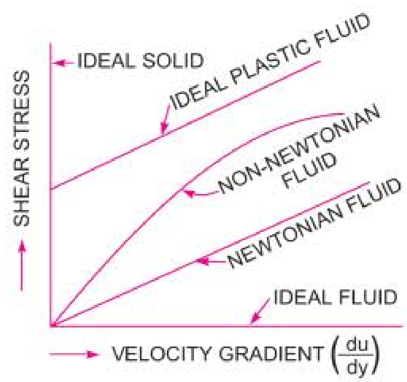

Fig. 1.2 *Types of fluids.*

5. Ideal Plastic Fluid. A fluid, in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or velocity gradient), is known as ideal plastic fluid.

Problem 1.3 *If the velocity distribution over a plate is given by* $u = \frac{2}{3} y - y^2$ *in which u is the velocity in metre per second at a distance y metre above the plate, determine the shear stress at* $y = 0$ *and* $y = 0.15$ *m. Take dynamic viscosity of fluid as 8.63 poises.*

Solution. Given : $u = \frac{2}{3}y - y^2 \quad \therefore \quad \frac{du}{dy} = \frac{2}{3} - 2y$

$$\left(\frac{du}{dy}\right)_{\text{at } y=0} \text{ or } \left(\frac{du}{dy}\right)_{y=0} = \frac{2}{3} - 2(0) = \frac{2}{3} = 0.667$$

Also $$\left(\frac{du}{dy}\right)_{\text{at } y=0.15} \text{ or } \left(\frac{du}{dy}\right)_{y=0.15} = \frac{2}{3} - 2 \times .15 = .667 - .30 = 0.367$$

Value of $\mu = 8.63$ poise $= \frac{8.63}{10}$ SI units $= 0.863$ N s/m^2

Now shear stress is given by equation (1.2) as $\tau = \mu \frac{du}{dy}$.

(*i*) Shear stress at $y = 0$ is given by

$$\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0} = 0.863 \times 0.667 = \mathbf{0.5756\ N/m^2}.\ \textbf{Ans.}$$

(*ii*) Shear stress at $y = 0.15$ m is given by

$$(\tau)_{y=0.15} = \mu \left(\frac{du}{dy}\right)_{y=0.15} = 0.863 \times 0.367 = \mathbf{0.3167\ N/m^2}.\ \textbf{Ans.}$$

Problem 1.4 *A plate 0.025 mm distant from a fixed plate, moves at 60 cm/s and requires a force of 2 N per unit area i.e., 2 N/m^2 to maintain this speed. Determine the fluid viscosity between the plates.*

Solution. Given :

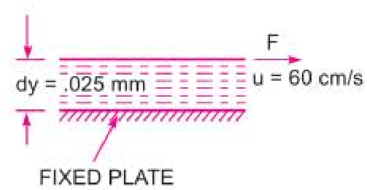

Fig. 1.3

Distance between plates, $dy = .025$ mm $= .025 \times 10^{-3}$ m

Velocity of upper plate, $u = 60$ cm/s $= 0.6$ m/s

Force on upper plate, $F = 2.0 \frac{\text{N}}{\text{m}^2}$.

This is the value of shear stress *i.e.*, τ

Let the fluid viscosity between the plates is μ.

Using the equation (1.2), we have $\tau = \mu \frac{du}{dy}$.

where du = Change of velocity $= u - 0 = u = 0.60$ m/s

dy = Change of distance $= .025 \times 10^{-3}$ m

τ = Force per unit area $= 2.0 \frac{\text{N}}{\text{m}^2}$

$$\therefore \quad 2.0 = \mu \frac{0.60}{.025 \times 10^{-3}} \quad \therefore \quad \mu = \frac{2.0 \times .025 \times 10^{-3}}{0.60} = 8.33 \times 10^{-5} \frac{\text{Ns}}{\text{m}^2}$$

$$= 8.33 \times 10^{-5} \times 10 \text{ poise} = \mathbf{8.33 \times 10^{-4}\ poise}.\ \textbf{Ans.}$$

Problem 1.5 *A flat plate of area 1.5 × 10^6 mm^2 is pulled with a speed of 0.4 m/s relative to another plate located at a distance of 0.15 mm from it. Find the force and power required to maintain this speed, if the fluid separating them is having viscosity as 1 poise.*

Solution. Given :

Area of the plate, $A = 1.5 \times 10^6 \text{ mm}^2 = 1.5 \text{ m}^2$

Speed of plate relative to another plate, $du = 0.4$ m/s

Distance between the plates, $dy = 0.15 \text{ mm} = 0.15 \times 10^{-3}$ m

Viscosity $\mu = 1 \text{ poise} = \frac{1}{10} \frac{\text{Ns}}{\text{m}^2}$.

Using equation (1.2) we have $\tau = \mu \frac{du}{dy} = \frac{1}{10} \times \frac{0.4}{.15 \times 10^{-3}} = 266.66 \frac{\text{N}}{\text{m}^2}$

(*i*) ∴ Shear force, $F = \tau \times \text{area} = 266.66 \times 1.5 =$ **400 N. Ans.**

(*ii*) Power* required to move the plate at the speed 0.4 m/sec

$= F \times u = 400 \times 0.4 =$ **160 W. Ans.**

Problem 1.6 *Determine the intensity of shear of an oil having viscosity = 1 poise. The oil is used for lubricating the clearance between a shaft of diameter 10 cm and its journal bearing. The clearance is 1.5 mm and the shaft rotates at 150 r.p.m.*

Solution. Given : $\mu = 1 \text{ poise} = \frac{1}{10} \frac{\text{N s}}{\text{m}^2}$

Dia. of shaft, $D = 10 \text{ cm} = 0.1$ m

Distance between shaft and journal bearing,

$dy = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m

Speed of shaft, $N = 150$ r.p.m.

Tangential speed of shaft is given by

$$u = \frac{\pi DN}{60} = \frac{\pi \times 0.1 \times 150}{60} = 0.785 \text{ m/s}$$

Using equation (1.2), $\tau = \mu \frac{du}{dy}$,

where du = change of velocity between shaft and bearing $= u - 0 = u$

$$= \frac{1}{10} \times \frac{0.785}{1.5 \times 10^{-3}} = \mathbf{52.33 \text{ N/m}^2}. \text{ Ans.}$$

Problem 1.7 *Calculate the dynamic viscosity of an oil, which is used for lubrication between a square plate of size 0.8 m × 0.8 m and an inclined plane with angle of inclination 30° as shown in Fig. 1.4. The weight of the square plate is 300 N and it slides down the inclined plane with a uniform velocity of 0.3 m/s. The thickness of oil film is 1.5 mm.*

Solution. Given :

Area of plate, $A = 0.8 \times 0.8 = 0.64 \text{ m}^2$

Angle of plane, $\theta = 30°$

Weight of plate, $W = 300$ N

Velocity of plate, $u = 0.3$ m/s

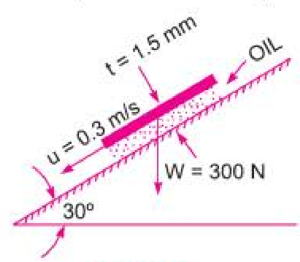

Fig. 1.4

* Power = $F \times u$ N m/s = $F \times u$ W (∵ Nm/s = Watt)

Thickness of oil film, $t = dy = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m

Let the viscosity of fluid between plate and inclined plane is μ.

Component of weight W, along the plane $= W \cos 60° = 300 \cos 60° = 150$ N

Thus the shear force, F, on the bottom surface of the plate = 150 N

and shear stress, $$\tau = \frac{F}{\text{Area}} = \frac{150}{0.64} \text{ N/m}^2$$

Now using equation (1.2), we have

$$\tau = \mu \frac{du}{dy}$$

where du = change of velocity $= u - 0 = u = 0.3$ m/s

$$dy = t = 1.5 \times 10^{-3} \text{ m}$$

$$\therefore \quad \frac{150}{0.64} = \mu \frac{0.3}{1.5 \times 10^{-3}}$$

$$\therefore \quad \mu = \frac{150 \times 1.5 \times 10^{-3}}{0.64 \times 0.3} = 1.17 \text{ N s/m}^2 = 1.17 \times 10 = \textbf{11.7 poise. Ans.}$$

Problem 1.8 *Two horizontal plates are placed 1.25 cm apart, the space between them being filled with oil of viscosity 14 poises. Calculate the shear stress in oil if upper plate is moved with a velocity of 2.5 m/s.*

Solution. Given :

Distance between plates, $dy = 1.25 \text{ cm} = 0.0125$ m

Viscosity, $\mu = 14 \text{ poise} = \frac{14}{10} \text{ N s/m}^2$

Velocity of upper plate, $u = 2.5$ m/sec.

Shear stress is given by equation (1.2) as, $\tau = \mu \frac{du}{dy}$

where du = Change of velocity between plates $= u - 0 = u = 2.5$ m/sec.

$$dy = 0.0125 \text{ m.}$$

$$\therefore \quad \tau = \frac{14}{10} \times \frac{2.5}{.0125} = \textbf{280 N/m}^2\textbf{. Ans.}$$

Problem 1.9 *The space between two square flat parallel plates is filled with oil. Each side of the plate is 60 cm. The thickness of the oil film is 12.5 mm. The upper plate, which moves at 2.5 metre per sec requires a force of 98.1 N to maintain the speed. Determine :*

(i) the dynamic viscosity of the oil in poise, and

(ii) the kinematic viscosity of the oil in stokes if the specific gravity of the oil is 0.95.

Solution. Given :

Each side of a square plate $= 60 \text{ cm} = 0.60$ m

$\therefore$ Area, $A = 0.6 \times 0.6 = 0.36 \text{ m}^2$

Thickness of oil film, $dy = 12.5 \text{ mm} = 12.5 \times 10^{-3}$ m

Velocity of upper plate, $u = 2.5$ m/sec

$\therefore$ Change of velocity between plates, $du = 2.5$ m/sec

Force required on upper plate, $F = 98.1$ N

$\therefore$ Shear stress, $$\tau = \frac{\text{Force}}{\text{Area}} = \frac{F}{A} = \frac{98.1 \text{ N}}{0.36 \text{ m}^2}$$

(*i*) Let μ = Dynamic viscosity of oil

Using equation (1.2), $$\tau = \mu \frac{du}{dy} \text{ or } \frac{98.1}{0.36} = \mu \times \frac{2.5}{12.5 \times 10^{-3}}$$

$\therefore$ $$\mu = \frac{98.1}{0.36} \times \frac{12.5 \times 10^{-3}}{2.5} = 1.3635 \frac{\text{Ns}}{\text{m}^2} \qquad \left(\because \frac{1 \text{ Ns}}{\text{m}^2} = 10 \text{ poise}\right)$$

$$= 1.3635 \times 10 = \mathbf{13.635 \text{ poise.}} \text{ Ans.}$$

(*ii*) Sp. gr. of oil, S = 0.95

Let ν = kinematic viscosity of oil

Using equation (1.1*A*),

Mass density of oil, $\rho = S \times 1000 = 0.95 \times 1000 = 950$ kg/m^3

Using the relation, $\nu = \dfrac{\mu}{\rho}$, we get $$\nu = \frac{1.3635 \left(\frac{\text{Ns}}{\text{m}^2}\right)}{950} = .001435 \text{ m}^2\text{/sec} = .001435 \times 10^4 \text{ cm}^2\text{/s}$$

$$= \mathbf{14.35 \text{ stokes.}} \text{ Ans.} \qquad (\because \text{cm}^2\text{/s} = \text{stoke})$$

Problem 1.10 *Find the kinematic viscosity of an oil having density 981 kg/m^3. The shear stress at a point in oil is 0.2452 N/m^2 and velocity gradient at that point is 0.2 per second.*

Solution. Given :

Mass density, $\rho = 981$ kg/m^3

Shear stress, $\tau = 0.2452$ N/m^2

Velocity gradient, $\dfrac{du}{dy} = 0.2$ s

Using the equation (1.2), $$\tau = \mu \frac{du}{dy} \text{ or } 0.2452 = \mu \times 0.2$$

$\therefore$ $$\mu = \frac{0.2452}{0.200} = 1.226 \text{ Ns/m}^2$$

Kinematic viscosity ν is given by

$\therefore$ $$\nu = \frac{\mu}{\rho} = \frac{1.226}{981} = .125 \times 10^{-2} \text{ m}^2\text{/sec}$$

$$= 0.125 \times 10^{-2} \times 10^4 \text{ cm}^2\text{/s} = 0.125 \times 10^2 \text{ cm}^2\text{/s}$$

$$= 12.5 \text{ cm}^2\text{/s} = \mathbf{12.5 \text{ stoke.}} \text{ Ans.} \qquad (\because \text{cm}^2\text{/s} = \text{stoke})$$

Problem 1.11 *Determine the specific gravity of a fluid having viscosity 0.05 poise and kinematic viscosity 0.035 stokes.*

Solution. Given :

Viscosity, $\mu = 0.05 \text{ poise} = \dfrac{0.05}{10}$ N s/m^2

Kinematic viscosity, $\nu = 0.035$ stokes
$= 0.035 \text{ cm}^2/\text{s}$ $\{\because \text{Stoke} = \text{cm}^2/\text{s}\}$
$= 0.035 \times 10^{-4} \text{ m}^2/\text{s}$

Using the relation $\nu = \dfrac{\mu}{\rho}$, we get $0.035 \times 10^{-4} = \dfrac{0.05}{10} \times \dfrac{1}{\rho}$

$$\therefore \quad \rho = \frac{0.05}{10} \times \frac{1}{0.035 \times 10^{-4}} = 1428.5 \text{ kg/m}^3$$

$$\therefore \quad \text{Sp. gr. of liquid} = \frac{\text{Density of liquid}}{\text{Density of water}} = \frac{1428.5}{1000} = 1.4285 \simeq \mathbf{1.43.}\ \textbf{Ans.}$$

Problem 1.12 *Determine the viscosity of a liquid having kinematic viscosity 6 stokes and specific gravity 1.9.*

Solution. Given :

Kinematic viscosity $\nu = 6$ stokes $= 6 \text{ cm}^2/\text{s} = 6 \times 10^{-4} \text{ m}^2/\text{s}$

Sp. gr. of liquid $= 1.9$

Let the viscosity of liquid $= \mu$

$$\text{Now sp. gr. of a liquid} = \frac{\text{Density of the liquid}}{\text{Density of water}}$$

or
$$1.9 = \frac{\text{Density of liquid}}{1000}$$

$$\therefore \text{ Density of liquid} = 1000 \times 1.9 = 1900 \ \frac{\text{kg}}{\text{m}^3}$$

$\therefore$ Using the relation $\nu = \dfrac{\mu}{\rho}$, we get

$$6 \times 10^{-4} = \frac{\mu}{1900}$$

or
$$\mu = 6 \times 10^{-4} \times 1900 = 1.14 \text{ Ns/m}^2$$
$$= 1.14 \times 10 = \mathbf{11.40\ poise.}\ \textbf{Ans.}$$

Problem 1.13 *The velocity distribution for flow over a flat plate is given by $u = \frac{3}{4}y - y^2$ in which u is the velocity in metre per second at a distance y metre above the plate. Determine the shear stress at y = 0.15 m. Take dynamic viscosity of fluid as 8.6 poise.*

Solution. Given : $u = \dfrac{3}{4}y - y^2$

$$\therefore \quad \frac{du}{dy} = \frac{3}{4} - 2y$$

At $y = 0.15$,
$$\frac{du}{dy} = \frac{3}{4} - 2 \times 0.15 = 0.75 - 0.30 = 0.45$$

Viscosity, $\mu = 8.5 \text{ poise} = \dfrac{8.5}{10} \dfrac{\text{N s}}{\text{m}^2}$ $\left(\because 10 \text{ poise} = 1 \dfrac{\text{Ns}}{\text{m}^2}\right)$

Using equation (1.2), $\tau = \mu \dfrac{du}{dy} = \dfrac{8.5}{10} \times 0.45 \dfrac{\text{N}}{\text{m}^2} = \mathbf{0.3825} \dfrac{\mathbf{N}}{\mathbf{m^2}}$. **Ans.**

Problem 1.14 *The dynamic viscosity of an oil, used for lubrication between a shaft and sleeve is 6 poise. The shaft is of diameter 0.4 m and rotates at 190 r.p.m. Calculate the power lost in the bearing for a sleeve length of 90 mm. The thickness of the oil film is 1.5 mm.*

Solution. Given :

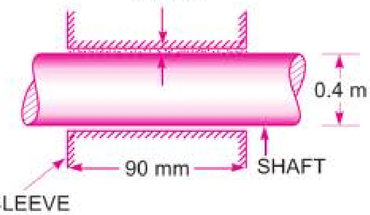

Fig. 1.5

Viscosity $\mu = 6$ poise

$$= \frac{6}{10} \frac{\text{N s}}{\text{m}^2} = 0.6 \frac{\text{N s}}{\text{m}^2}$$

Dia. of shaft, $D = 0.4$ m

Speed of shaft, $N = 190$ r.p.m

Sleeve length, $L = 90 \text{ mm} = 90 \times 10^{-3}$ m

Thickness of oil film, $t = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m

Tangential velocity of shaft, $u = \dfrac{\pi D N}{60} = \dfrac{\pi \times 0.4 \times 190}{60} = 3.98$ m/s

Using the relation $\tau = \mu \dfrac{du}{dy}$

where du = Change of velocity $= u - 0 = u = 3.98$ m/s

dy = Change of distance $= t = 1.5 \times 10^{-3}$ m

$$\tau = 10 \times \frac{3.98}{1.5 \times 10^{-3}} = 1592 \text{ N/m}^2$$

This is shear stress on shaft

$\therefore$ Shear force on the shaft, F = Shear stress $\times$ Area

$$= 1592 \times \pi D \times L = 1592 \times \pi \times .4 \times 90 \times 10^{-3} = 180.05 \text{ N}$$

Torque on the shaft, $T = \text{Force} \times \dfrac{D}{2} = 180.05 \times \dfrac{0.4}{2} = 36.01$ Nm

$\therefore$ *Power lost $= \dfrac{2\pi NT}{60} = \dfrac{2\pi \times 190 \times 36.01}{60} = \mathbf{716.48\ W}$. **Ans.**

Problem 1.15 *If the velocity profile of a fluid over a plate is parabolic with the vertex 20 cm from the plate, where the velocity is 120 cm/sec. Calculate the velocity gradients and shear stresses at a distance of 0, 10 and 20 cm from the plate, if the viscosity of the fluid is 8.5 poise.*

Solution. Given :

Distance of vertex from plate = 20 cm

Velocity at vertex, $u = 120$ cm/sec

Viscosity, $\mu = 8.5 \text{ poise} = \dfrac{8.5}{10} \dfrac{\text{N s}}{\text{m}^2} = 0.85.$

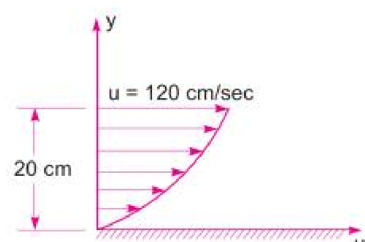

Fig. 1.6

* Power in S.I. unit $= T * \omega = T \times \dfrac{2\pi N}{60}$ Watt $= \dfrac{2\pi NT}{60}$ Watt

The velocity profile is given parabolic and equation of velocity profile is

$$u = ay^2 + by + c \quad ...(i)$$

where a, b and c are constants. Their values are determined from boundary conditions as :

(a) at $y = 0$, $u = 0$

(b) at $y = 20$ cm, $u = 120$ cm/sec

(c) at $y = 20$ cm, $\frac{du}{dy} = 0$.

Substituting boundary condition (a) in equation (i), we get

$$c = 0.$$

Boundary condition (b) on substitution in (i) gives

$$120 = a(20)^2 + b(20) = 400a + 20b \quad ...(ii)$$

Boundary condition (c) on substitution in equation (i) gives

$$\frac{du}{dy} = 2ay + b \quad ...(iii)$$

or $$0 = 2 \times a \times 20 + b = 40a + b$$

Solving equations (ii) and (iii) for a and b

From equation (iii), $\quad b = -40a$

Substituting this value in equation (ii), we get

$$120 = 400a + 20 \times (-40a) = 400a - 800a = -400a$$

$$\therefore \quad a = \frac{120}{-400} = -\frac{3}{10} = -0.3$$

$$\therefore \quad b = -40 \times (-0.3) = 12.0$$

Substituting the values of a, b and c in equation (i),

$$u = -0.3y^2 + 12y.$$

Velocity Gradient

$$\frac{du}{dy} = -0.3 \times 2y + 12 = -0.6y + 12$$

at $y = 0$, Velocity gradient, $\left(\frac{du}{dy}\right)_{y=0} = -0.6 \times 0 + 12 =$ **12/s. Ans.**

at $y = 10$ cm, $\left(\frac{du}{dy}\right)_{y=10} = -0.6 \times 10 + 12 = -6 + 12 =$ **6/s. Ans.**

at $y = 20$ cm, $\left(\frac{du}{dy}\right)_{y=20} = -0.6 \times 20 + 12 = -12 + 12 =$ **0. Ans.**

Shear Stresses

Shear stress is given by, $\quad \tau = \mu \frac{du}{dy}$

(*i*) Shear stress at $y = 0$, $\tau = \mu \left(\dfrac{du}{dy}\right)_{y=0} = 0.85 \times 12.0 = 10.2 \text{ N/m}^2$.

(*ii*) Shear stress at $y = 10$, $\tau = \mu \left(\dfrac{du}{dy}\right)_{y=10} = 0.85 \times 6.0 = 5.1 \text{ N/m}^2$.

(*iii*) Shear stress at $y = 20$, $\tau = \mu \left(\dfrac{du}{dy}\right)_{y=20} = 0.85 \times 0 = \mathbf{0}$. **Ans.**

Problem 1.16 *A Newtonian fluid is filled in the clearance between a shaft and a concentric sleeve. The sleeve attains a speed of 50 cm/s, when a force of 40 N is applied to the sleeve parallel to the shaft. Determine the speed if a force of 200 N is applied.*

Solution. Given : Speed of sleeve, $u_1 = 50$ cm/s

when force, $F_1 = 40$ N.

Let speed of sleeve is u_2 when force, $F_2 = 200$ N.

Using relation $$\tau = \mu \frac{du}{dy}$$

where τ = Shear stress $= \dfrac{\text{Force}}{\text{Area}} = \dfrac{F}{A}$

du = Change of velocity $= u - 0 = u$

dy = Clearance $= y$

$$\therefore \quad \frac{F}{A} = \mu \frac{u}{y}$$

$$\therefore \quad F = \frac{A\mu u}{y} \propto u \qquad \{\because \text{ A, } \mu \text{ and } y \text{ are constant}\}$$

$$\therefore \quad \frac{F_1}{u_1} = \frac{F_2}{u_2}$$

Substituting values, we get $\dfrac{40}{50} = \dfrac{200}{u_2}$

$$\therefore \quad u_2 = \frac{50 \times 200}{40} = 50 \times 5 = \mathbf{250 \text{ cm/s}}.$$ **Ans.**

Problem 1.17 *A 15 cm diameter vertical cylinder rotates concentrically inside another cylinder of diameter 15.10 cm. Both cylinders are 25 cm high. The space between the cylinders is filled with a liquid whose viscosity is unknown. If a torque of 12.0 Nm is required to rotate the inner cylinder at 100 r.p.m., determine the viscosity of the fluid.*

Solution. Given :

Diameter of cylinder $= 15$ cm $= 0.15$ m

Dia. of outer cylinder $= 15.10$ cm $= 0.151$ m

Length of cylinders, $L = 25$ cm $= 0.25$ m

Torque, $T = 12.0$ Nm

Speed, $N = 100$ r.p.m.

Let the viscosity $= \mu$

Tangential velocity of cylinder, $u = \dfrac{\pi DN}{60} = \dfrac{\pi \times 0.15 \times 100}{60} = 0.7854$ m/s

Surface area of cylinder, $A = \pi D \times L = \pi \times 0.15 \times 0.25 = .1178\ \text{m}^2$

Now using relation $\tau = \mu \dfrac{du}{dy}$

where $du = u - 0 = u = .7854$ m/s

$$dy = \frac{0.151 - 0.150}{2}\ \text{m} = .0005\ \text{m}$$

$$\tau = \frac{\mu \times .7854}{.0005}$$

$\therefore$ Shear force, $F = \text{Shear stress} \times \text{Area} = \dfrac{\mu \times .7854}{.0005} \times .1178$

$\therefore$ Torque, $T = F \times \dfrac{D}{2}$

$$12.0 = \frac{\mu \times .7854}{.0005} \times .1178 \times \frac{.15}{2}$$

$$\therefore \quad \mu = \frac{12.0 \times .0005 \times 2}{.7854 \times .1178 \times .15} = 0.864\ \text{N s/m}^2$$

$= 0.864 \times 10 =$ **8.64 poise. Ans.**

Problem 1.18 *Two large plane surfaces are 2.4 cm apart. The space between the surfaces is filled with glycerine. What force is required to drag a very thin plate of surface area 0.5 square metre between the two large plane surfaces at a speed of 0.6 m/s, if :*

(i) the thin plate is in the middle of the two plane surfaces, and

(ii) the thin plate is at a distance of 0.8 cm from one of the plane surfaces ? Take the dynamic viscosity of glycerine = 8.10×10^{-1} N s/m^2.

Solution. Given :

Distance between two large surfaces = 2.4 cm

Area of thin plate, $A = 0.5\ \text{m}^2$

Velocity of thin plate, $u = 0.6$ m/s

Viscosity of glycerine, $\mu = 8.10 \times 10^{-1}$ N s/m^2

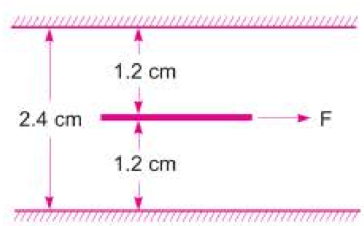

Fig. 1.7 (*a*)

Case I. When the thin plate is in the middle of the two plane surfaces [Refer to Fig. 1.7 (*a*)]

Let F_1 = Shear force on the upper side of the thin plate

F_2 = Shear force on the lower side of the thin plate

F = Total force required to drag the plate

Then $F = F_1 + F_2$

The shear stress (τ_1) on the upper side of the thin plate is given by equation,

$$\tau_1 = \mu\left(\frac{du}{dy}\right)_1$$

where du = Relative velocity between thin plate and upper large plane surface
= 0.6 m/sec
dy = Distance between thin plate and upper large plane surface
= 1.2 cm = 0.012 m (plate is a thin one and hence thickness of plate is neglected)

$$\therefore \qquad \tau_1 = 8.10 \times 10^{-1} \times \left(\frac{0.6}{.012}\right) = 40.5 \text{ N/m}^2$$

Now shear force, F_1 = Shear stress × Area
$= \tau_1 \times A = 40.5 \times 0.5 = 20.25$ N

Similarly shear stress (τ_2) on the lower side of the thin plate is given by

$$\tau_2 = \mu\left(\frac{du}{dy}\right)_2 = 8.10 \times 10^{-1} \times \left(\frac{0.6}{0.012}\right) = 40.5 \text{ N/m}^2$$

∴ Shear force, $F_2 = \tau_2 \times \text{A} = 40.5 \times 0.5 = 20.25$ N

∴ Total force, $F = F_1 + F_2 = 20.25 + 20.25 =$ **40.5 N. Ans.**

Case II. When the thin plate is at a distance of 0.8 cm from one of the plane surfaces [Refer to Fig. 1.7 (*b*)].

Let the thin plate is at a distance 0.8 cm from the lower plane surface.

Then distance of the plate from the upper plane surface
= 2.4 – 0.8 = 1.6 cm = .016 m
(Neglecting thickness of the plate)

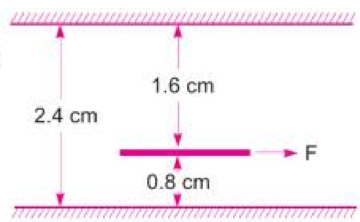

Fig. 1.7 (*b*)

The shear force on the upper side of the thin plate,

$$F_1 = \text{Shear stress} \times \text{Area} = \tau_1 \times A$$

$$= \mu\left(\frac{du}{dy}\right)_1 \times A = 8.10 \times 10^{-1} \times \left(\frac{0.6}{0.016}\right) \times 0.5 = 15.18 \text{ N}$$

The shear force on the lower side of the thin plate,

$$F_2 = \tau_2 \times A = \mu\left(\frac{du}{dy}\right)_2 \times A$$

$$= 8.10 \times 10^{-1} \times \left(\frac{0.6}{0.8/100}\right) \times 0.5 = 30.36 \text{ N}$$

∴ Total force required = $F_1 + F_2 = 15.18 + 30.36 =$ **45.54 N. Ans.**

Problem 1.19 *A vertical gap 2.2 cm wide of infinite extent contains a fluid of viscosity 2.0 N s/m² and specific gravity 0.9. A metallic plate 1.2 m × 1.2 m × 0.2 cm is to be lifted up with a constant velocity of 0.15 m/sec, through the gap. If the plate is in the middle of the gap, find the force required. The weight of the plate is 40 N.*

Solution. Given :

Width of gap = 2.2 cm, viscosity, $\mu = 2.0$ N s/m²

Sq. gr. of fluid = 0.9

$\therefore$ Weight density of fluid

$$= 0.9 \times 1000 = 900 \text{ kgf/m}^3 = 900 \times 9.81 \text{ N/m}^3$$

$$(\because 1 \text{ kgf} = 9.81 \text{ N})$$

Volume of plate $= 1.2 \text{ m} \times 1.2 \text{ m} \times 0.2 \text{ cm}$

$$= 1.2 \times 1.2 \times .002 \text{ m}^3 = .00288 \text{ m}^3$$

Thickness of plate $= 0.2$ cm

Velocity of plate $= 0.15$ m/sec

Weight of plate $= 40$ N.

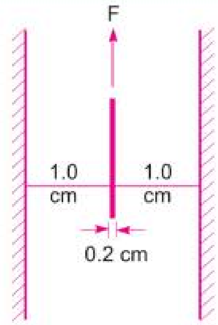

Fig. 1.8

When plate is in the middle of the gap, the distance of the plate from vertical surface of the gap

$$= \left(\frac{\text{Width of gap} - \text{Thickness of plate}}{2}\right)$$

$$= \frac{(2.2 - 0.2)}{2} = 1 \text{ cm} = .01 \text{ m}.$$

Now the shear force on the left side of the metallic plate,

$$F_1 = \text{Shear stress} \times \text{Area}$$

$$= \mu \left(\frac{du}{dy}\right)_1 \times \text{Area} = 2.0 \times \left(\frac{0.15}{.01}\right) \times 1.2 \times 1.2 \text{ N}$$

$$(\because \text{Area} = 1.2 \times 1.2 \text{ m}^2)$$

$$= 43.2 \text{ N}.$$

Similarly, the shear force on the right side of the metallic plate,

$$F_2 = \text{Shear stress} \times \text{Area} = 2.0 \times \left(\frac{0.15}{.01}\right) \times 1.2 \times 1.2 = 43.2 \text{ N}$$

$\therefore$ Total shear force $= F_1 + F_2 = 43.2 + 43.2 = 86.4$ N.

In this case the weight of plate (which is acting vertically downward) and upward thrust is also to be taken into account.

The upward thrust = Weight of fluid displaced

$$= (\text{Weight density of fluid}) \times \text{Volume of fluid displaced}$$

$$= 9.81 \times 900 \times .00288 \text{ N}$$

$$(\because \text{Volume of fluid displaced} = \text{Volume of plate} = .00288)$$

$$= 25.43 \text{ N}.$$

The net force acting in the downward direction due to weight of the plate and upward thrust

$$= \text{Weight of plate} - \text{Upward thrust} = 40 - 25.43 = 14.57 \text{ N}$$

$\therefore$ Total force required to lift the plate up

= Total shear force + 14.57 = 86.4 + 14.57 = **100.97 N. Ans.**

▶ 1.4 THERMODYNAMIC PROPERTIES

Fluids consist of liquids or gases. But gases are compressible fluids and hence thermodynamic properties play an important role. With the change of pressure and temperature, the gases undergo large variation in density. The relationship between pressure (absolute), specific volume and temperature (absolute) of a gas is given by the equation of state as

$$p\,\forall = RT \text{ or } \frac{p}{\rho} = RT \qquad \text{...(1.5)}$$

where p = Absolute pressure of a gas in N/m^2

$\forall$ = Specific volume = $\frac{1}{\rho}$

R = Gas constant

T = Absolute temperature in °K

ρ = Density of a gas.

1.4.1 Dimension of R. The gas constant, R, depends upon the particular gas. The dimension of R is obtained from equation (1.5) as

$$R = \frac{p}{\rho T}$$

(*i*) In MKS units

$$R = \frac{\text{kgf/m}^2}{\left(\frac{\text{kg}}{\text{m}^3}\right)°\text{K}} = \frac{\text{kgf-m}}{\text{kg °K}}$$

(*ii*) In SI units, p is expressed in Newton/m^2 or N/m^2.

$$\therefore \qquad R = \frac{\text{N/m}^2}{\frac{\text{kg}}{\text{m}^3} \times \text{K}} = \frac{\text{Nm}}{\text{kg-K}} = \frac{\text{Joule}}{\text{kg-K}} \quad [\text{Joule} = \text{Nm}]$$

$$= \frac{\text{J}}{\text{kg-K}}$$

For air,

$$R \text{ in MKS} = 29.3\ \frac{\text{kgf-m}}{\text{kg °K}}$$

$$R \text{ in SI} = 29.3 \times 9.81\ \frac{\text{Nm}}{\text{kg°K}} = 287\ \frac{\text{J}}{\text{kg-K}}$$

1.4.2 Isothermal Process. If the change in density occurs at constant temperature, then the process is called isothermal and relationship between pressure (p) and density (ρ) is given by

$$\frac{p}{\rho} = \text{Constant} \qquad \text{...(1.6)}$$

1.4.3 Adiabatic Process. If the change in density occurs with no heat exchange to and from the gas, the process is called adiabatic. And if no heat is generated within the gas due to friction, the relationship between pressure and density is given by

$$\frac{p}{\rho^k} = \text{Constant} \qquad \text{...(1.7)}$$

where k = Ratio of specific heat of a gas at constant pressure and constant volume.

= 1.4 for air.

1.4.4 Universal Gas Constant

Let m = Mass of a gas in kg

$\forall$ = Volume of gas of mass m

p = Absolute pressure

T = Absolute temperature

Then, we have
$$p\forall = mRT \quad \text{...(1.8)}$$
where R = Gas constant.

Equation (1.8) can be made universal, *i.e.*, applicable to all gases if it is expressed in **mole-basis**.

Let n = Number of moles in volume of a gas

$\forall$ = Volume of the gas

$$M = \frac{\text{Mass of the gas molecules}}{\text{Mass of a hydrogen atom}}$$

m = Mass of a gas in kg

Then, we have $n \times M = m$.

Substituting the value of m in equation (1.8), we get
$$p\forall = n \times M \times RT \quad \text{...(1.9)}$$

The product $M \times R$ is called universal gas constant and is equal to 848 $\dfrac{\text{kgf-m}}{\text{kg-mole °K}}$ in MKS units and 8314 J/kg-mole K in SI units.

One kilogram mole is defined as the product of one kilogram mass of the gas and its molecular weight.

Problem 1.20 *A gas weighs 16 N/m^3 at 25°C and at an absolute pressure of 0.25 N/mm^2. Determine the gas constant and density of the gas.*

Solution. Given :

Weight density, $w = 16$ N/m^2

Temperature, $t = 25$°C

$\therefore$ $T = 273 + t = 273 + 25 = 288$°K

$p = 0.25$ N/mm^2 (abs.) $= 0.25 \times 10^6$ N/m^2 $= 25 \times 10^4$ N/m^2

(*i*) Using relation $w = \rho g$, density is obtained as
$$\rho = \frac{w}{g} = \frac{16}{9.81} = \mathbf{1.63\ kg/m^3}. \textbf{ Ans.}$$

(*ii*) Using equation (1.5), $\dfrac{p}{\rho} = RT$

$$\therefore \quad R = \frac{p}{\rho T} = \frac{25 \times 10^4}{1.63 \times 288} = \mathbf{532.55\ \frac{Nm}{kg\ K}}. \textbf{ Ans.}$$

Problem 1.21 *A cylinder of 0.6 m^3 in volume contains air at 50°C and 0.3 N/mm^2 absolute pressure. The air is compressed to 0.3 m^3. Find (i) pressure inside the cylinder assuming isothermal process and (ii) pressure and temperature assuming adiabatic process. Take k = 1.4.*

Solution. Given :

Initial volume, $\forall_1 = 0.6$ m^3

Temperature $t_1 = 50°C$

$\therefore$ $T_1 = 273 + 50 = 323°K$

Pressure $p_1 = 0.3 \text{ N/mm}^2 = 0.3 \times 10^6 \text{ N/m}^2 = 30 \times 10^4 \text{ N/m}^2$

Final volume $\forall_2 = 0.3 \text{ m}^3$

$k = 1.4$

(*i*) **Isothermal process :**

Using equation (1.6), $\frac{p}{\rho}$ = Constant or $p\forall$ = Constant.

$$\therefore \quad p_1\forall_1 = p_2\forall_2$$

$$\therefore \quad p_2 = \frac{p_1\forall_1}{\forall_2} = \frac{30 \times 10^4 \times 0.6}{0.3} = 0.6 \times 10^6 \text{ N/m}^2 = \mathbf{0.6\ N/mm^2}. \text{ Ans.}$$

(*ii*) **Adiabatic process :**

Using equation (1.7), $\frac{p}{\rho^k}$ = Constant or $p\ \forall^k$ = Constant

$$\therefore \quad p_1\forall_1^k = p_2\forall_2^k.$$

$$\therefore \quad p_2 = p_1\frac{\forall_1^k}{\forall_2^k} = 30 \times 10^4 \times \left(\frac{0.6}{0.3}\right)^{1.4} = 30 \times 10^4 \times 2^{1.4}$$

$$= 0.791 \times 10^6 \text{ N/m}^2 = \mathbf{0.791\ N/mm^2}. \text{ Ans.}$$

For temperature, using equation (1.5), we get

$$p\forall = RT \text{ and also } p\ \forall^k = \text{Constant}$$

$$\therefore \quad p = \frac{RT}{\forall} \text{ and } \frac{RT}{\forall} \times \forall^k = \text{Constant}$$

or $$RT\forall^{k-1} = \text{Constant}$$

or $$T\ \forall^{k-1} = \text{Constant} \qquad \{\because R \text{ is also constant}\}$$

$$\therefore \quad T_1\ \forall_1^{\ k-1} = T_2\ \forall_2^{k-1}$$

$$\therefore \quad T_2 = T_1\left(\frac{\forall_1}{\forall_2}\right)^{k-1} = 323\left(\frac{0.6}{0.3}\right)^{1.4-1.0} = 323 \times 2^{0.4} = 426.2°K$$

$$\therefore \quad t_2 = 426.2 - 273 = \mathbf{153.2°C}. \text{ Ans.}$$

Problem 1.22 *Calculate the pressure exerted by 5 kg of nitrogen gas at a temperature of 10°C if the volume is 0.4 m³. Molecular weight of nitrogen is 28. Assume, ideal gas laws are applicable.*

Solution. Given :

Mass of nitrogen = 5 kg

Temperature, $t = 10°C$

$\therefore$ $T = 273 + 10 = 283°K$

Volume of nitrogen, $\forall = 0.4 \text{ m}^3$

Molecular weight = 28

Using equation (1.9), we have $p\forall = n \times M \times RT$

where $M \times R$ = Universal gas constant = $8314 \dfrac{\text{Nm}}{\text{kg-mole }^\circ\text{K}}$

and one kg-mole = (kg-mass) × Molecular weight = (kg-mass) × 28

$$\therefore \qquad R \text{ for nitrogen} = \frac{8314}{28} = 296.9 \frac{\text{Nm}}{\text{kg}^\circ\text{K}}$$

The gas laws for nitrogen is $p\forall = mRT$, where R = Characteristic gas constant

or $$p \times 0.4 = 5 \times 296.9 \times 283$$

$$\therefore \qquad p = \frac{5 \times 296.9 \times 283}{0.4} = 1050283.7 \text{ N/m}^2 = \mathbf{1.05 \text{ N/mm}^2}. \textbf{ Ans.}$$

▶ 1.5 COMPRESSIBILITY AND BULK MODULUS

Compressibility is the reciprocal of the bulk modulus of elasticity, K which is defined as the ratio of compressive stress to volumetric strain.

Consider a cylinder fitted with a piston as shown in Fig. 1.9.

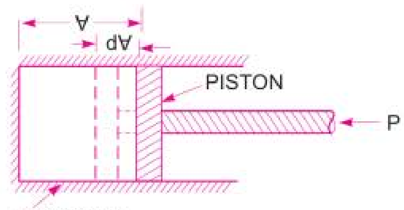

Fig. 1.9

Let $\forall$ = Volume of a gas enclosed in the cylinder

p = Pressure of gas when volume is $\forall$

Let the pressure is increased to $p + dp$, the volume of gas decreases from $\forall$ to $\forall - d\forall$.

Then increase in pressure $= dp$ kgf/m^2

Decrease in volume $= d\forall$

$\therefore$ Volumetric strain $= -\dfrac{d\forall}{\forall}$

– ve sign means the volume decreases with increase of pressure.

$$\therefore \quad \text{Bulk modulus} \qquad K = \frac{\text{Increase of pressure}}{\text{Volumetric strain}}$$

$$= \frac{dp}{\dfrac{-d\forall}{\forall}} = \frac{-dp}{d\forall}\forall \qquad \text{...(1.10)}$$

$$\text{Compressibility} = \frac{1}{K} \qquad \text{...(1.11)}$$

Relationship between Bulk Modulus (K) and Pressure (p) for a Gas

The relationship between bulk modulus of elasticity (K) and pressure for a gas for two different processes of compression are as :

(*i*) **For Isothermal Process.** Equation (1.6) gives the relationship between pressure (p) and density (ρ) of a gas as

$$\frac{p}{\rho} = \text{Constant}$$

or $$p\forall = \text{Constant} \qquad \left\{\because \forall = \frac{1}{\rho}\right\}$$

Differentiating this equation, we get (p and $\forall$ both are variables)

$$pd\forall + \forall dp = 0 \quad \text{or} \quad pd\forall = -\forall dp \quad \text{or} \quad p = \frac{-\forall dp}{d\forall}$$

Substituting this value in equation (1.10), we get

$$K = p \qquad ...(1.12)$$

(*ii*) **For Adiabatic Process.** Using equation (1.7) for adiabatic process

$$\frac{p}{\rho^k} = \text{Constant or } p\ \forall^k = \text{Constant}$$

Differentiating, we get $pd(\forall^k) + \forall^k(dp) = 0$

or $$p \times k \times \forall^{k-1}\, d\forall + \forall^k\, dp = 0$$

or $$pkd\forall + \forall dp = 0 \qquad \text{[Cancelling } \forall^{k-1} \text{ to both sides]}$$

or $$pkd\forall = -\forall dp \quad \text{or} \quad pk = -\frac{\forall dp}{d\forall}$$

Hence from equation (1.10), we have

$$K = pk \qquad ...(1.13)$$

where K = Bulk modulus and k = Ratio of specific heats.

Problem 1.23 *Determine the bulk modulus of elasticity of a liquid, if the pressure of the liquid is increased from 70 N/cm² to 130 N/cm². The volume of the liquid decreases by 0.15 per cent.*

Solution. Given :

Initial pressure $= 70\ \text{N/cm}^2$

Final pressure $= 130\ \text{N/cm}^2$

$\therefore$ dp = Increase in pressure $= 130 - 70 = 60\ \text{N/cm}^2$

Decrease in volume $= 0.15\%$

$$\therefore \qquad -\frac{d\forall}{\forall} = +\frac{0.15}{100}$$

Bulk modulus, K is given by equation (1.10) as

$$K = \frac{dp}{-\frac{d\forall}{\forall}} = \frac{60\ \text{N/cm}^2}{\frac{.15}{100}} = \frac{60 \times 100}{.15} = \mathbf{4 \times 10^4\ N/cm^2}. \textbf{ Ans.}$$

Problem 1.24 *What is the bulk modulus of elasticity of a liquid which is compressed in a cylinder from a volume of 0.0125 m³ at 80 N/cm² pressure to a volume of 0.0124 m³ at 150 N/cm² pressure ?*

Solution. Given :

Initial volume, $\forall = 0.0125\ \text{m}^3$

Final volume $= 0.0124\ \text{m}^3$

$\therefore$ Decrease in volume, $d\forall = .0125 - .0124 = .0001\ \text{m}^3$

$$\therefore \qquad -\frac{d\forall}{\forall} = \frac{.0001}{.0125}$$

Initial pressure $= 80 \text{ N/cm}^2$

Final pressure $= 150 \text{ N/cm}^2$

$\therefore$ Increase in pressure, $dp = (150 - 80) = 70 \text{ N/cm}^2$

Bulk modulus is given by equation (1.10) as

$$K = \frac{dp}{-\frac{d\forall}{\forall}} = \frac{70}{\frac{.0001}{.0125}} = 70 \times 125 \text{ N/cm}^2$$

$$= \mathbf{8.75 \times 10^3 \text{ N/cm}^2}. \text{ Ans.}$$

▶ 1.6 SURFACE TENSION AND CAPILLARITY

Surface tension is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension. The magnitude of this force per unit length of the free surface will have the same value as the surface energy per unit area. It is denoted by Greek letter σ (called sigma). In MKS units, it is expressed as kgf/m while in SI units as N/m.

The phenomenon of surface tension is explained by Fig. 1.10. Consider three molecules *A*, *B*, *C* of a liquid in a mass of liquid. The molecule *A* is attracted in all directions equally by the surrounding molecules of the liquid. Thus the resultant force acting on the molecule *A* is zero. But the molecule *B*, which is situated near the free surface, is acted upon by upward and downward forces which are unbalanced. Thus a net resultant force on molecule *B* is acting in the downward direction. The molecule *C*, situated on the free surface of liquid, does experience a resultant downward force. All the molecules on the free surface experience a downward force. Thus the free surface of the liquid acts like a very thin film under tension of the surface of the liquid act as though it is an elastic membrane under tension.

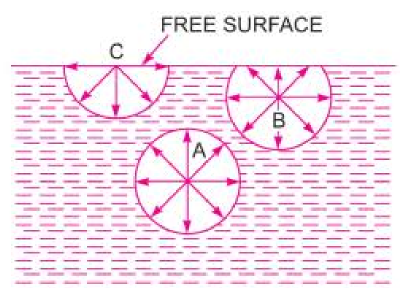

Fig. 1.10 *Surface tension.*

1.6.1 Surface Tension on Liquid Droplet. Consider a small spherical droplet of a liquid of radius '*r*'. On the entire surface of the droplet, the tensile force due to surface tension will be acting.

Let σ = Surface tension of the liquid

p = Pressure intensity inside the droplet (in excess of the outside pressure intensity)

d = Dia. of droplet.

Let the droplet is cut into two halves. The forces acting on one half (say left half) will be

(*i*) tensile force due to surface tension acting around the circumference of the cut portion as shown in Fig. 1.11 (*b*) and this is equal to

$$= \sigma \times \text{Circumference}$$

$$= \sigma \times \pi d$$

(*ii*) pressure force on the area $\frac{\pi}{4}d^2 = p \times \frac{\pi}{4} d^2$ as shown in Fig. 1.11 (*c*). These two forces will be equal and opposite under equilibrium conditions, *i.e.*,

$$p \times \frac{\pi}{4} d^2 = \sigma \times \pi d$$

or
$$p = \frac{\sigma \times \pi d}{\frac{\pi}{4} \times d^2} = \frac{4\sigma}{d} \qquad \text{...(1.14)}$$

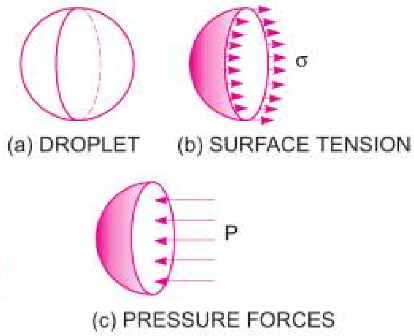

Fig. 1.11 *Forces on droplet.*

Equation (1.14) shows that with the decrease of diameter of the droplet, pressure intensity inside the droplet increases.

1.6.2 Surface Tension on a Hollow Bubble. A hollow bubble like a soap bubble in air has two surfaces in contact with air, one inside and other outside. Thus two surfaces are subjected to surface tension. In such case, we have

$$p \times \frac{\pi}{4} d^2 = 2 \times (\sigma \times \pi d)$$

$\therefore$
$$p = \frac{2\sigma\pi d}{\frac{\pi}{4} d^2} = \frac{8\sigma}{d} \qquad \text{...(1.15)}$$

1.6.3 Surface Tension on a Liquid Jet. Consider a liquid jet of diameter '*d*' and length 'L' as shown in Fig. 1.12.

Let p = Pressure intensity inside the liquid jet above the outside pressure

σ = Surface tension of the liquid.

Consider the equilibrium of the semi jet, we have

Force due to pressure $= p \times$ area of semi jet

$= p \times L \times d$

Force due to surface tension $= \sigma \times 2L.$

Equating the forces, we have

$$p \times L \times d = \sigma \times 2L$$

$\therefore$
$$p = \frac{\sigma \times 2L}{L \times d} \qquad \text{...(1.16)}$$

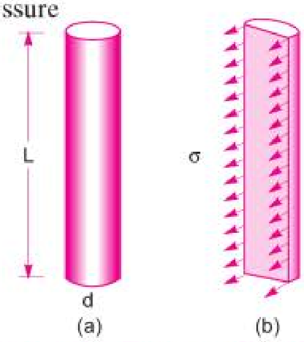

Fig. 1.12 *Forces on liquid jet.*

Problem 1.25 *The surface tension of water in contact with air at 20°C is 0.0725 N/m. The pressure inside a droplet of water is to be 0.02 N/cm*^{2} *greater than the outside pressure. Calculate the diameter of the droplet of water.*

Solution. Given :

Surface tension, $\sigma = 0.0725$ N/m

Pressure intensity, p in excess of outside pressure is

$$p = 0.02 \text{ N/cm}^2 = 0.02 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

Let d = dia. of the droplet

Using equation (1.14), we get $p = \frac{4\sigma}{d}$ or $0.02 \times 10^4 = \frac{4 \times 0.0725}{d}$

$$\therefore \quad d = \frac{4 \times 0.0725}{0.02 \times (10)^4} = .00145 \text{ m} = .00145 \times 1000 = \mathbf{1.45\ mm.}\ \textbf{Ans.}$$

Problem 1.26 *Find the surface tension in a soap bubble of 40 mm diameter when the inside pressure is 2.5 N/m² above atmospheric pressure.*

Solution. Given :

Dia. of bubble, $d = 40 \text{ mm} = 40 \times 10^{-3}$ m

Pressure in excess of outside, $p = 2.5$ N/m²

For a soap bubble, using equation (1.15), we get

$$p = \frac{8\sigma}{d} \quad \text{or} \quad 2.5 = \frac{8 \times \sigma}{40 \times 10^{-3}}$$

$$\sigma = \frac{2.5 \times 40 \times 10^{-3}}{8} \text{ N/m} = \mathbf{0.0125\ N/m.}\ \textbf{Ans.}$$

Problem 1.27 *The pressure outside the droplet of water of diameter 0.04 mm is 10.32 N/cm² (atmospheric pressure). Calculate the pressure within the droplet if surface tension is given as 0.0725 N/m of water.*

Solution. Given :

Dia. of droplet, $d = 0.04 \text{ mm} = .04 \times 10^{-3}$ m

Pressure outside the droplet $= 10.32 \text{ N/cm}^2 = 10.32 \times 10^4 \text{ N/m}^2$

Surface tension, $\sigma = 0.0725$ N/m

The pressure inside the droplet, in excess of outside pressure is given by equation (1.14)

or
$$p = \frac{4\sigma}{d} = \frac{4 \times 0.0725}{.04 \times 10^{-3}} = 7250 \text{ N/m}^2 = \frac{7250 \text{ N}}{10^4 \text{ cm}^2} = 0.725 \text{ N/cm}^2$$

$\therefore$ Pressure inside the droplet $= p +$ Pressure outside the droplet

$= 0.725 + 10.32 = \mathbf{11.045\ N/cm^2}$. **Ans.**

1.6.4 Capillarity. Capillarity is defined as a phenomenon of rise or fall of a liquid surface in a small tube relative to the adjacent general level of liquid when the tube is held vertically in the liquid. The rise of liquid surface is known as capillary rise while the fall of the liquid surface is known as capillary depression. It is expressed in terms of cm or mm of liquid. Its value depends upon the specific weight of the liquid, diameter of the tube and surface tension of the liquid.

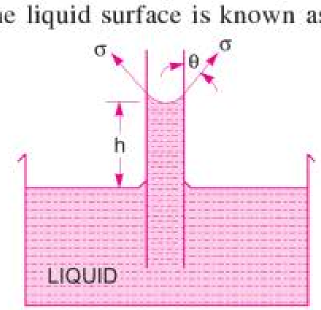

Fig. 1.13 *Capillary rise.*

Expression for Capillary Rise. Consider a glass tube of small diameter 'd' opened at both ends and is inserted in a liquid, say water. The liquid will rise in the tube above the level of the liquid.

Let h = height of the liquid in the tube. Under a state of equilibrium, the weight of liquid of height h is balanced by the force at the surface of the liquid in the tube. But the force at the surface of the liquid in the tube is due to surface tension.

Let σ = Surface tension of liquid

θ = Angle of contact between liquid and glass tube.

The weight of liquid of height h in the tube = (Area of tube $\times h) \times \rho \times g$

$$= \frac{\pi}{4} d^2 \times h \times \rho \times g \qquad ...(1.17)$$

where ρ = Density of liquid

Vertical component of the surface tensile force

$$= (\sigma \times \text{Circumference}) \times \cos\theta$$
$$= \sigma \times \pi d \times \cos\theta \qquad ...(1.18)$$

For equilibrium, equating (1.17) and (1.18), we get

$$\frac{\pi}{4} d^2 \times h \times \rho \times g = \sigma \times \pi d \times \cos\theta$$

or
$$h = \frac{\sigma \times \pi d \times \cos\theta}{\frac{\pi}{4} d^2 \times \rho \times g} = \frac{4\sigma\cos\theta}{\rho \times g \times d} \qquad ...(1.19)$$

The value of θ between water and clean glass tube is approximately equal to zero and hence cos θ is equal to unity. Then rise of water is given by

$$h = \frac{4\sigma}{\rho \times g \times d} \qquad ... (1.20)$$

Expression for Capillary Fall. If the glass tube is dipped in mercury, the level of mercury in the tube will be lower than the general level of the outside liquid as shown in Fig. 1.14.

Let h = Height of depression in tube.

Then in equilibrium, two forces are acting on the mercury inside the tube. First one is due to surface tension acting in the downward direction and is equal to $\sigma \times \pi d \times \cos\theta$.

Second force is due to hydrostatic force acting upward and is equal to intensity of pressure at a depth 'h' × Area

$$= p \times \frac{\pi}{4} d^2 = \rho g \times h \times \frac{\pi}{4} d^2 \; \{\because p = \rho g h\}$$

Equating the two, we get

$$\sigma \times \pi d \times \cos\theta = \rho g h \times \frac{\pi}{4} d^2$$

$$\therefore \qquad h = \frac{4\sigma\cos\theta}{\rho g d} \qquad ...(1.21)$$

Value of θ for mercury and glass tube is 128°.

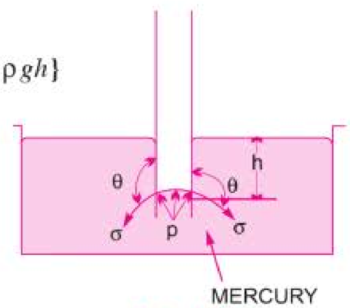

Fig. 1.14

Problem 1.28 *Calculate the capillary rise in a glass tube of 2.5 mm diameter when immersed vertically in (a) water and (b) mercury. Take surface tensions σ = 0.0725 N/m for water and σ = 0.52 N/m for mercury in contact with air. The specific gravity for mercury is given as 13.6 and angle of contact = 130°.*

Solution. Given :

Dia. of tube, $d = 2.5 \text{ mm} = 2.5 \times 10^{-3}$ m

Surface tension, σ for water = 0.0725 N/m

σ for mercury = 0.52 N/m

Sp. gr. of mercury = 13.6

$\therefore$ Density $= 13.6 \times 1000$ kg/m^3.

(*a*) **Capillary rise for water ($\theta = 0^o$)**

Using equation (1.20), we get $h = \dfrac{4\sigma}{\rho \times g \times d} = \dfrac{4 \times 0.0725}{1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= .0118$ m $=$ **1.18 cm. Ans.**

(*b*) **For mercury**

Angle of contact between mercury and glass tube, $\theta = 130°$

Using equation (1.21), we get $h = \dfrac{4\sigma \cos\theta}{\rho \times g \times d} = \dfrac{4 \times 0.52 \times \cos 130°}{13.6 \times 1000 \times 9.81 \times 2.5 \times 10^{-3}}$

$= -.004$ m $= -$ **0.4 cm. Ans.**

The negative sign indicates the capillary depression.

Problem 1.29 *Calculate the capillary effect in millimetres in a glass tube of 4 mm diameter, when immersed in (i) water, and (ii) mercury. The temperature of the liquid is 20°C and the values of the surface tension of water and mercury at 20°C in contact with air are 0.073575 N/m and 0.51 N/m respectively. The angle of contact for water is zero and that for mercury is 130°. Take density of water at 20°C as equal to 998 kg/m^3.*

Solution. Given :

Dia. of tube, $d = 4$ mm $= 4 \times 10^{-3}$ m

The capillary effect (*i.e.*, capillary rise or depression) is given by equation (1.20) as

$$h = \frac{4\sigma \cos\theta}{\rho \times g \times d}$$

where σ = surface tension in N/m

θ = angle of contact, and ρ = density

(*i*) **Capillary effect for water**

$\sigma = 0.073575$ N/m, $\theta = 0°$

$\rho = 998$ kg/m^3 at 20°C

$\therefore$ $h = \dfrac{4 \times 0.073575 \times \cos 0°}{998 \times 9.81 \times 4 \times 10^{-3}} = 7.51 \times 10^{-3}$ m $=$ **7.51 mm. Ans.**

(*ii*) **Capillary effect for mercury**

$\sigma = 0.51$ N/m, $\theta = 130°$ and

$\rho =$ sp. gr. $\times 1000 = 13.6 \times 1000 = 13600$ kg/m^2

$\therefore$ $h = \dfrac{4 \times 0.51 \times \cos 130°}{13600 \times 9.81 \times 4 \times 10^{-3}} = -2.46 \times 10^{-3}$ m $=$ **– 2.46 mm. Ans.**

The negative sign indicates the capillary depression.

Problem 1.30 *The capillary rise in the glass tube is not to exceed 0.2 mm of water. Determine its minimum size, given that surface tension for water in contact with air = 0.0725 N/m.*

Solution. Given :

Capillary rise, $h = 0.2$ mm $= 0.2 \times 10^3$ m

Surface tension, $\sigma = 0.0725$ N/m

Let dia. of tube $= d$

The angle θ for water $= 0^{\circ}$

Density (ρ) for water $= 1000 \text{ kg/m}^3$

Using equation (1.20), we get

$$h = \frac{4\sigma}{\rho \times g \times d} \text{ or } 0.2 \times 10^{-3} = \frac{4 \times 0.0725}{1000 \times 9.81 \times d}$$

$$\therefore \quad d = \frac{4 \times 0.0725}{1000 \times 9.81 \times .2 \times 10^{-3}} = 0.148 \text{ m} = \textbf{14.8 cm. Ans.}$$

Thus minimum diameter of the tube should be 14.8 cm.

Problem 1.31 *Find out the minimum size of glass tube that can be used to measure water level if the capillary rise in the tube is to be restricted to 2 mm. Consider surface tension of water in contact with air as 0.073575 N/m.*

Solution. Given :

Capillary rise, $h = 2.0 \text{ mm} = 2.0 \times 10^{-3} \text{ m}$

Surface tension, $\sigma = 0.073575 \text{ N/m}$

Let dia. of tube $= d$

The angle θ for water $= 0^{\circ}$

The density for water, $\rho = 1000 \text{ kg/m}^3$

Using equation (1.20), we get

$$h = \frac{4\sigma}{\rho \times g \times d} \text{ or } 2.0 \times 10^{-3} = \frac{4 \times 0.073575}{1000 \times 9.81 \times d}$$

$$\therefore \quad d = \frac{4 \times 0.073575}{1000 \times 9.81 \times 2 \times 10^{-3}} = 0.015 \text{ m} = \textbf{1.5 cm. Ans.}$$

Thus minimum diameter of the tube should be 1.5 cm.

Problem 1.32 *An oil of viscosity 5 poise is used for lubrication between a shaft and sleeve. The diameter of the shaft is 0.5 m and it rotates at 200 r.p.m. Calculate the power lost in oil for a sleeve length of 100 mm. The thickness of oil film is 1.0 mm.*

Solution. Given :

Viscosity, $\mu = 5 \text{ poise}$

$$= \frac{5}{10} = 0.5 \text{ N s/m}^2$$

Dia. of shaft, $D = 0.5 \text{ m}$

Speed of shaft, $N = 200 \text{ r.p.m.}$

Sleeve length, $L = 100 \text{ mm} = 100 \times 10^{-3} \text{ m} = 0.1 \text{ m}$

Thickness of oil film, $t = 1.0 \text{ mm} = 1 \times 10^{-3} \text{ m}$

Tangential velocity of shaft, $u = \frac{\pi DN}{60} = \frac{\pi \times 0.5 \times 200}{60} = 5.235 \text{ m/s}$

Using the relation, $\tau = \mu \frac{du}{dy}$

where, du = Change of velocity = $u - 0 = u = 5.235$ m/s

dy = Change of distance = $t = 1 \times 10^{-3}$ m

$$\therefore \qquad \tau = \frac{0.5 \times 5.235}{1 \times 10^{-3}} = 2617.5 \text{ N/m}^2$$

This is the shear stress on the shaft

$\therefore$ Shear force on the shaft, F = Shear stress × Area = $2617.5 \times \pi D \times L$ $\quad (\because \text{ Area} = \pi D \times L)$

$$= 2617.5 \times \pi \times 0.5 \times 0.1 = 410.95 \text{ N}$$

Torque on the shaft, $\qquad T = \text{Force} \times \dfrac{D}{2} = 410.95 \times \dfrac{0.5}{2} = 102.74$ Nm

$$\therefore \qquad \text{Power* lost} = T \times \omega \text{ Watts} = T \times \frac{2\pi N}{60} \text{ W}$$

$$= 102.74 \times \frac{2\pi \times 200}{60} = 2150 \text{ W} = \mathbf{2.15 \text{ kW}}. \text{ Ans.}$$

▶ 1.7 VAPOUR PRESSURE AND CAVITATION

A change from the liquid state to the gaseous state is known as vaporization. The vaporization (which depends upon the prevailing pressure and temperature condition) occurs because of continuous escaping of the molecules through the free liquid surface.

Consider a liquid (say water) which is confined in a closed vessel. Let the temperature of liquid is 20°C and pressure is atmospheric. This liquid will vaporise at 100°C. When vaporization takes place, the molecules escapes from the free surface of the liquid. These vapour molecules get accumulated in the space between the free liquid surface and top of the vessel. These accumulated vapours exert a pressure on the liquid surface. This pressure is known as **vapour pressure** of the liquid or this is the pressure at which the liquid is converted into vapours.

Again consider the same liquid at 20°C at atmospheric pressure in the closed vessel. If the pressure above the liquid surface is reduced by some means, the boiling temperature will also reduce. If the pressure is reduced to such an extent that it becomes equal to or less than the vapour pressure, the boiling of the liquid will start, though the temperature of the liquid is 20°C. Thus a liquid may boil even at ordinary temperature, if the pressure above the liquid surface is reduced so as to be equal or less than the vapour pressure of the liquid at that temperature.

Now consider a flowing liquid in a system. If the pressure at any point in this flowing liquid becomes equal to or less than the vapour pressure, the vaporization of the liquid starts. The bubbles of these vapours are carried by the flowing liquid into the region of high pressure where they collapse, giving rise to high impact pressure. The pressure developed by the collapsing bubbles is so high that the material from the adjoining boundaries gets eroded and cavities are formed on them. This phenomenon is known as **cavitation.**

Hence the cavitation is the phenomenon of formation of vapour bubbles of a flowing liquid in a region where the pressure of the liquid falls below the vapour pressure and sudden collapsing of these vapour bubbles in a region of higher pressure. When the vapour bubbles collapse, a very high pressure is created. The metallic surfaces, above which the liquid is flowing, is subjected to these high pressures, which cause pitting action on the surface. Thus cavities are formed on the metallic surface and hence the name is cavitation.

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* Power in case of S.I. Unit = $T \times \omega$ or $\dfrac{2\pi NT}{60}$ Watts or $\dfrac{2\pi NT}{60{,}000}$ kW. The angular velocity $\omega = \dfrac{2\pi N}{60}$.

HIGHLIGHTS

1. The weight density or specific weight of a fluid is equal to weight per unit volume. It is also equal to,
$$w = \rho \times g.$$
2. Specific volume is the reciprocal of mass density.
3. The shear stress is proportional to the velocity gradient $\frac{du}{dy}$. Mathematically, $\tau = \mu \frac{du}{dy}$.
4. Kinematic viscosity ν is given by $\nu = \frac{\mu}{\rho}$.
5. Poise and stokes are the units of viscosity and kinematic viscosity respectively.
6. To convert the unit of viscosity from poise to MKS units, poise should be divided by 98.1 and to convert poise into SI units, the poise should be divided by 10. SI unit of viscosity is Ns/m^2 or Pa s, where N/m^2 = Pa = Pascal.
7. For a perfect gas, the equation of state is $\frac{p}{\rho} = RT$

   where R = gas constant and for air = $29.3 \frac{\text{kgf-m}}{\text{kg}\,°\text{K}}$ = 287 J/kg °K.
8. For isothermal process, $\frac{p}{\rho}$ = Constant whereas for adiabatic process, $\frac{p}{\rho^k}$ = constant.
9. Bulk modulus of elasticity is given as $K = \frac{-dp}{\left(\frac{d\forall}{\forall}\right)}$.
10. Compressibility is the reciprocal of bulk modulus of elasticity or $= \frac{1}{K}$.
11. Surface tension is expressed in N/m or dyne/cm. The relation between surface tension (σ) and difference of pressure (p) between the inside and outside of a liquid drop is given as $p = \frac{4\sigma}{d}$

    For a soap bubble, $p = \frac{8\sigma}{d}$.

    For a liquid jet, $p = \frac{2\sigma}{d}$.
12. Capillary rise or fall of a liquid is given by $h = \frac{4\sigma\cos\theta}{wd}$.

    The value of θ for water is taken equal to zero and for mercury equal to 128°.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define the following fluid properties :
   Density, weight density, specific volume and specific gravity of a fluid.
2. Differentiate between : (*i*) Liquids and gases, (*ii*) Real fluids and ideal fluids, (*iii*) Specific weight and specific volume of a fluid.
3. What is the difference between dynamic viscosity and kinematic viscosity ? State their units of measurements.

4. Explain the terms : (*i*) Dynamic viscosity, and (*ii*) Kinematic viscosity. Give their dimensions.
5. State the Newton's law of viscosity and give examples of its application.
6. Enunciate Newton's law of viscosity. Explain the importance of viscosity in fluid motion. What is the effect of temperature on viscosity of water and that of air?
7. Define Newtonian and Non-Newtonian fluids.
8. What do you understand by terms : (*i*) Isothermal process, (*ii*) Adiabatic process, and (*iii*) Universal-gas constant.
9. Define compressibility. Prove that compressibility for a perfect gas undergoing isothermal compression is $\frac{1}{p}$ while for a perfect gas undergoing isentropic compression is $\frac{1}{wp}$.
10. Define surface tension. Prove that the relationship between surface tension and pressure inside a droplet of liquid in excess of outside pressure is given by $p = \frac{4\sigma}{d}$.
11. Explain the phenomenon of capillarity. Obtain an expression for capillary rise of a liquid.
12. (*a*) Distinguish between ideal fluids and real fluids. Explain the importance of compressibility in fluid flow.
    (*b*) Define the terms : density, specific volume, specific gravity, vacuum pressure, compressible and incompressible fluids. (*R.G.P. Vishwavidyalaya, Bhopal S 2002*)
13. Define and explain Newton's law of viscosity.
14. Convert 1 kg/s-m dynamic viscosity in poise.
15. Why does the viscosity of a gas increases with the increase in temperature while that of a liquid decreases with increase in temperature ?
16. (*a*) How does viscosity of a fluid vary with temperature ?
    (*b*) Cite examples where surface tension effects play a prominent role. (*J.N.T.U., Hyderabad S 2002*)
17. (*i*) Develop the expression for the relation between gauge pressure *P* inside a droplet of liquid and the surface tension.
    (*ii*) Explain the following :
    Newtonian and Non-Newtonian fluids, vapour pressure, and compressibility.
    (*R.G.P.V., Bhopal S 2001*)

(B) NUMERICAL PROBLEMS

1. One litre of crude oil weighs 9.6 N. Calculate its specific weight, density and specific gravity.
   [**Ans.** 9600 N/m^3, 978.6 kg/m^3, 0.978]
2. The velocity distribution for flow over a flat plate is given by $u = \frac{3}{2}\ y - y^{3/2}$, where u is the point velocity in metre per second at a distance y metre above the plate. Determine the shear stress at $y = 9$ cm. Assume dynamic viscosity as 8 poise. (*Nagpur University*) [**Ans.** 0.839 N/m^2]
3. A plate 0.025 mm distant from a fixed plate, moves at 50 cm/s and requires a force of 1.471 N/m^2 to maintain this speed. Determine the fluid viscosity between the plates in the poise. [**Ans.** 7.357×10^{-4}]
4. Determine the intensity of shear of an oil having viscosity = 1.2 poise and is used for lubrication in the clearance between a 10 cm diameter shaft and its journal bearing. The clearance is 1.0 mm and shaft rotates at 200 r.p.m. [**Ans.** 125.56 N/m^2]
5. Two plates are placed at a distance of 0.15 mm apart. The lower plate is fixed while the upper plate having surface area 1.0 m^2 is pulled at 0.3 m/s. Find the force and power required to maintain this speed, if the fluid separating them is having viscosity 1.5 poise. [**Ans.** 300 N, 89.8 W]
6. An oil film of thickness 1.5 mm is used for lubrication between a square plate of size 0.9 m × 0.9 m and an inclined plane having an angle of inclination 20°. The weight of the square is 392.4 N and it slides down the plane with a uniform velocity of 0.2 m/s. Find the dynamic viscosity of the oil. [**Ans.** 12.42 poise]

7. In a stream of glycerine in motion, at a certain point the velocity gradient is 0.25 metre per sec per metre. The mass density of fluid is 1268.4 kg per cubic metre and kinematic viscosity is 6.30×10^{-4} square metre per second. Calculate the shear stress at the point. [**Ans.** 0.2 N/m^2]

8. Find the kinematic viscosity of an oil having density 980 kg/m^2 when at a certain point in the oil, the shear stress is 0.25 N/m^2 and velocity gradient is 0.3/s. $\left[\textbf{Ans. } 0.000849 \dfrac{\text{m}^2}{\text{sec}} \text{ or } 8.49 \text{ stokes}\right]$

9. Determine the specific gravity of a fluid having viscosity 0.07 poise and kinematic viscosity 0.042 stokes. [**Ans.** 1.667]

10. Determine the viscosity of a liquid having kinematic viscosity 6 stokes and specific gravity 2.0. [**Ans.** 11.99 poise]

11. If the velocity distribution of a fluid over a plate is given by $u = (3/4)\, y - y^2$, where u is the velocity in metre per second at a distance of y metres above the plate, determine the shear stress at $y = 0.15$ metre. Take dynamic viscosity of the fluid as 8.5×10^{-5} kg-sec/m^2. [**Ans.** 3.825×10^{-5} kgf/m^2]

12. An oil of viscosity 5 poise is used for lubrication between a shaft and sleeve. The diameter of shaft is 0.5 m and it rotates at 200 r.p.m. Calculate the power lost in the oil for a sleeve length of 100 mm. The thickness of the oil film is 1.0 mm. [**Ans.** 2.15 kW]

13. The velocity distribution over a plate is given by $u = \dfrac{2}{3}\, y - y^2$ in which u is the velocity in m/sec at a distance of y m above the plate. Determine the shear stress at $y = 0$, 0.1 and 0.2 m. Take $\mu = 6$ poise. [**Ans.** 0.4, 0.028 and 0.159 N/m^2]

14. In question 13, find the distance in metres above the plate, at which the shear stress is zero. [**Ans.** 0.333 m]

15. The velocity profile of a viscous fluid over a plate is parabolic with vertex 20 cm from the plate, where the velocity is 120 cm/s. Calculate the velocity gradient and shear stress at distances of 0, 5 and 15 cm from the plate, given the viscosity of the fluid = 6 poise. [**Ans.** 12/s, 7.18 N/m^2; 9/s, 5.385 N/m^2 ; 3/s, 1.795 N/m^2]

16. The weight of a gas is given as 17.658 N/m^3 at 30°C and at an absolute pressure of 29.43 N/cm^2. Determine the gas constant and also the density of the gas. $\left[\textbf{Ans. } \dfrac{1.8 \text{ kg}}{\text{m}^3}, \dfrac{539.55 \text{ N m}}{\text{kg}^\circ \text{K}}\right]$

17. A cylinder of 0.9 m^3 in volume contains air at 0°C and 39.24 N/cm^2 absolute pressure. The air is compressed to 0.45 m^3. Find (*i*) the pressure inside the cylinder assuming isothermal process, (*ii*) pressure and temperature assuming adiabatic process. Take $k = 1.4$ for air. [**Ans.** (*i*) 78.48 N/cm^2, (*ii*) 103.5 N/m^2, 140°C]

18. Calculate the pressure exerted by 4 kg mass of nitrogen gas at a temperature of 15°C if the volume is 0.35 m^3. Molecular weight of nitrogen is 28. [**Ans.** 97.8 N/cm^2]

19. The pressure of a liquid is increased from 60 N/cm^2 to 100 N/cm^2 and volume decreases by 0.2 per cent. Determine the bulk modulus of elasticity. [**Ans.** 2×10^4 N/cm^2]

20. Determine the bulk modulus of elasticity of a fluid which is compressed in a cylinder from a volume of 0.009 m^3 at 70 N/cm^2 pressure to a volume of 0.0085 m^3 at 270 N/cm^2 pressure. [**Ans.** 3.6×10^3 N/cm^2]

21. The surface tension of water in contact with air at 20°C is given as 0.0716 N/m. The pressure inside a droplet of water is to be 0.0147 N/cm^2 greater than the outside pressure, calculate the diameter of the droplet of water. [**Ans.** 1.94 mm]

22. Find the surface tension in a soap bubble of 30 mm diameter when the inside pressure is 1.962 N/m^2 above atmosphere. [**Ans.** 0.00735 N/m]

23. The surface tension of water in contact with air is given as 0.0725 N/m. The pressure outside the droplet of water of diameter 0.02 mm is atmospheric $\left(10.32 \dfrac{\text{N}}{\text{cm}^2}\right)$. Calculate the pressure within the droplet of water. [**Ans.** 11.77 N/cm^2]

24. Calculate the capillary rise in a glass tube of 3.0 mm diameter when immersed vertically in (*a*) water, and (*b*) mercury. Take surface tensions for mercury and water as 0.0725 N/m and 0.52 N/m respectively in contact with air. Specific gravity for mercury is given as 13.6. [**Ans.** 0.966 cm, 0.3275 cm]

25. The capillary rise in the glass tube used for measuring water level is not to exceed 0.5 mm. Determine its minimum size, given that surface tension for water in contact with air = 0.07112 N/m. [**Ans.** 5.8 cm]

26. (**SI Units**). One litre of crude oil weighs 9.6 N. Calculate its specific weight, density and specific gravity. [**Ans.** 9600 N/m^3; 979.6 kg/m^3 ; 0.9786]

27. (**SI Units**). A piston 796 mm diameter and 200 mm long works in a cylinder of 800 mm diameter. If the annular space is filled with a lubricating oil of viscosity 5 cp (centi-poise), calculate the speed of descent of the piston in vertical position. The weight of the piston and axial load are 9.81 N. [**Ans.** 7.84 m/s]

28. (**SI Units**). Find the capillary rise of water in a tube 0.03 cm diameter. The surface tension of water is 0.0735 N/m. [**Ans.** 9.99 cm]

29. Calculate the specific weight, density and specific gravity of two litres of a liquid which weight 15 N. [**Ans.** 7500 N/m^3, 764.5 kg/m^3, 0.764]

30. A 150 mm diameter vertical cylinder rotates concentrically inside another cylinder of diameter 151 mm. Both the cylinders are of 250 mm height. The space between the cylinders is filled with a liquid of viscosity 10 poise. Determine the torque required to rotate the inner cylinder at 100 r.p.m. [**Ans.** 13.87 Nm]

31. A shaft of diameter 120 mm is rotating inside a journal bearing of diameter 122 mm at a speed of 360 r.p.m. The space between the shaft and the bearing is filled with a lubricating oil of viscosity 6 poise. Find the power absorbed in oil if the length of bearing is 100 mm. [**Ans.** 115.73 W]

32. A shaft of diameter 100 mm is rotating inside a journal bearing of diameter 102 mm at a space of 360 r.p.m. The space between the shaft and bearing is filled with a lubricating oil of viscosity 5 poise. The length of the bearing is 200 mm. Find the power absorbed in the lubricating oil. [**Ans.** 111.58 W]

33. Assuming that the bulk modulus of elasticity of water is 2.07×10^6 kN/m^2 at standard atmospheric conditions, determine the increase of pressure necessary to produce 1% reduction in volume at the same temperature.

[**Hint.** $K = 2.07 \times 10^6 \text{ kN/m}^2 ; \dfrac{-d\forall}{\forall} = \dfrac{1}{100} = 0.01.$

Increase in pressure $(dp) = K \times \left(\dfrac{-d\forall}{\forall}\right) = 2.07 \times 10^6 \times 0.01 = 2.07 \times 10^4 \text{ kN/m}^2.$]

34. A square plate of size 1 m × 1 m and weighing 350 N slides down an inclined plane with a uniform velocity of 1.5 m/s. The inclined plane is laid on a slope of 5 vertical to 12 horizontal and has an oil film of 1 mm thickness. Calculate the dynamic viscosity of oil. [*J.N.T.U., Hyderabad, S 2002*]

[**Hint.** $A = 1 \times 1 = 1 \text{ m}^2$, W = 350 N, $u = 1.5$ m/s, $\tan\theta = \dfrac{5}{12} = \dfrac{BC}{AB}$

Component of weight along the plane = W × sin θ

where $\sin\theta = \dfrac{BC}{AC} = \dfrac{5}{13}$ $\quad \left(\because \; AC = \sqrt{AB^2 + BC^2} = \sqrt{12^2 + 5^2} = 13\right)$

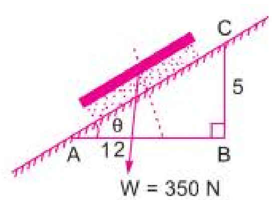


Fig. 1.15

$\therefore \quad F = W \sin\theta = 350 \times \dfrac{5}{13} = 134.615$

Now $\quad \tau = \mu \dfrac{du}{dy}$, where $du = u - 0 = u = 1.5$ m/s and $dy = 1 \text{ mm} = 1 \times 10^{-3}$ m

or $\quad \dfrac{F}{A} = \mu \dfrac{du}{dy}, \;\; \therefore \mu = \dfrac{F}{A} \times \dfrac{dy}{du} = \dfrac{134.615}{1} \times \dfrac{1 \times 10^{-3}}{1.5} = 0.0897 \dfrac{\text{Ns}}{\text{m}^2} = 0.897$ poise]

2

CHAPTER

PRESSURE AND ITS MEASUREMENT

▶ 2.1 FLUID PRESSURE AT A POINT

Consider a small area dA in large mass of fluid. If the fluid is stationary, then the force exerted by the surrounding fluid on the area dA will always be perpendicular to the surface dA. Let dF is the force acting on the area dA in the normal direction. Then the ratio of $\frac{dF}{dA}$ is known as the intensity of pressure or simply pressure and this ratio is represented by p. Hence mathematically the pressure at a point in a fluid at rest is

$$p = \frac{dF}{dA}.$$

If the force (F) is uniformly distributed over the area (A), then pressure at any point is given by

$$p = \frac{F}{A} = \frac{\text{Force}}{\text{Area}}.$$

∴ Force or pressure force, $F = p \times A$.

The units of pressure are : (*i*) kgf/m^2 and kgf/cm^2 in MKS units, (*ii*) Newton/m^2 or N/m^2 and N/mm^2 in SI units. N/m^2 is known as Pascal and is represented by Pa. Other commonly used units of pressure are :

$$\text{kPa} = \text{kilo pascal} = 1000 \text{ N/m}^2$$
$$\text{bar} = 100 \text{ kPa} = 10^5 \text{ N/m}^2.$$

▶ 2.2 PASCAL'S LAW

It states that the pressure or intensity of pressure at a point in a static fluid is equal in all directions. This is proved as :

The fluid element is of very small dimensions *i.e.*, dx, dy and ds.

Consider an arbitrary fluid element of wedge shape in a fluid mass at rest as shown in Fig. 2.1. Let the width of the element perpendicular to the plane of paper is unity and p_x,

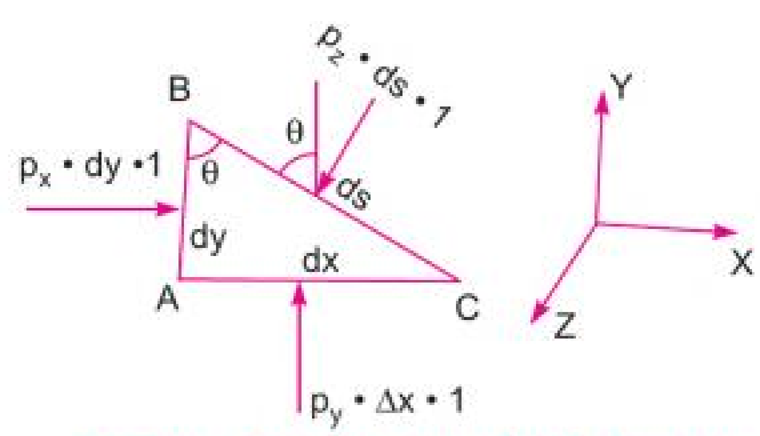


Fig. 2.1 *Forces on a fluid element.*

p_y and p_z are the pressures or intensity of pressure acting on the face AB, AC and BC respectively. Let $\angle ABC = \theta$. Then the forces acting on the element are :

1. Pressure forces normal to the surfaces, and
2. Weight of element in the vertical direction.

The forces on the faces are :

Force on the face AB $= p_x \times$ Area of face AB

$= p_x \times dy \times 1$

Similarly force on the face $AC = p_y \times dx \times 1$

Force on the face BC $= p_z \times ds \times 1$

Weight of element $=$ (Mass of element) $\times g$

$$= (\text{Volume} \times \rho) \times g = \left(\frac{AB \times AC}{2} \times 1\right) \times \rho \times g,$$

where ρ = density of fluid.

Resolving the forces in x-direction, we have

$$p_x \times dy \times 1 - p\,(ds \times 1) \sin (90° - \theta) = 0$$

or $$p_x \times dy \times 1 - p_z\, ds \times 1 \cos \theta = 0.$$

But from Fig. 2.1, $ds \cos \theta = AB = dy$

$\therefore$ $$p_x \times dy \times 1 - p_z \times dy \times 1 = 0$$

or $$p_x = p_z \quad \text{...(2.1)}$$

Similarly, resolving the forces in y-direction, we get

$$p_y \times dx \times 1 - p_z \times ds \times 1 \cos (90° - \theta) - \frac{dx \times dy}{2} \times 1 \times \rho \times g = 0$$

or $$p_y \times dx - p_z\, ds \sin \theta - \frac{dxdy}{2} \times \rho \times g = 0.$$

But $ds \sin \theta = dx$ and also the element is very small and hence weight is negligible.

$\therefore$ $$p_y dx - p_z \times dx = 0$$

or $$p_y = p_z \quad \text{...(2.2)}$$

From equations (2.1) and (2.2), we have

$$p_x = p_y = p_z \quad \text{...(2.3)}$$

The above equation shows that the pressure at any point in x, y and z directions is equal.

Since the choice of fluid element was completely arbitrary, which means the pressure at any point is the same in all directions.

▶ 2.3 PRESSURE VARIATION IN A FLUID AT REST

The pressure at any point in a fluid at rest is obtained by the Hydrostatic Law which states that the rate of increase of pressure in a vertically downward direction must be equal to the specific weight of the fluid at that point. This is proved as :

Consider a small fluid element as shown in Fig. 2.2

Let ΔA = Cross-sectional area of element

ΔZ = Height of fluid element

p = Pressure on face AB

Z = Distance of fluid element from free surface.

The forces acting on the fluid element are :

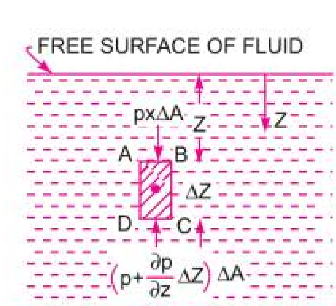


Fig. 2.2 *Forces on a fluid element.*

1. Pressure force on $AB = p \times \Delta A$ and acting perpendicular to face AB in the downward direction.

2. Pressure force on $CD = \left(p + \frac{\partial p}{\partial Z}\Delta Z\right) \times \Delta A$, acting perpendicular to face CD, vertically upward direction.

3. Weight of fluid element = Density × g × Volume = $\rho \times g \times (\Delta A \times \Delta Z)$.

4. Pressure forces on surfaces BC and AD are equal and opposite. For equilibrium of fluid element, we have

$$p\Delta A - \left(p + \frac{\partial p}{\partial Z}\Delta Z\right)\Delta A + \rho \times g \times (\Delta A \times \Delta Z) = 0$$

or
$$p\Delta A - p\Delta A - \frac{\partial p}{\partial Z}\Delta Z\Delta A + \rho \times g \times \Delta A \times Z = 0$$

or
$$-\frac{\partial p}{\partial Z}\Delta Z\Delta A + \rho \times g \times \Delta A\Delta Z = 0$$

or
$$\frac{\partial p}{\partial Z}\Delta Z\Delta A = \rho \times g \times \Delta A\Delta Z \quad \text{or} \quad \frac{\partial p}{\partial Z} = \rho \times g \text{ [cancelling } \Delta A\Delta Z \text{ on both sides]}$$

$$\therefore \quad \frac{\partial p}{\partial Z} = \rho \times g = w \qquad (\because \rho \times g = w) \qquad ...(2.4)$$

where w = Weight density of fluid.

Equation (2.4) states that rate of increase of pressure in a vertical direction is equal to weight density of the fluid at that point. This is **Hydrostatic Law.**

By integrating the above equation (2.4) for liquids, we get

$$\int dp = \int \rho g dZ$$

or
$$p = \rho g Z \qquad ...(2.5)$$

where p is the pressure above atmospheric pressure and Z is the height of the point from free surfaces.

From equation (2.5), we have $\quad Z = \frac{p}{\rho \times g} \qquad ...(2.6)$

Here Z is called **pressure head.**

Problem 2.1 *A hydraulic press has a ram of 30 cm diameter and a plunger of 4.5 cm diameter. Find the weight lifted by the hydraulic press when the force applied at the plunger is 500 N.*

Solution. Given :

Dia. of ram, $D = 30$ cm $= 0.3$ m

Dia. of plunger, $d = 4.5$ cm $= 0.045$ m

Force on plunger, $F = 500$ N

Find weight lifted $= W$

Area of ram, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4}(0.3)^2 = 0.07068 \text{ m}^2$

Area of plunger, $a = \frac{\pi}{4} d^2 = \frac{\pi}{4}(0.045)^2 = .00159 \text{ m}^2$

Pressure intensity due to plunger

$$= \frac{\text{Force on plunger}}{\text{Area of plunger}} = \frac{F}{a} = \frac{500}{.00159} \text{ N/m}^2.$$

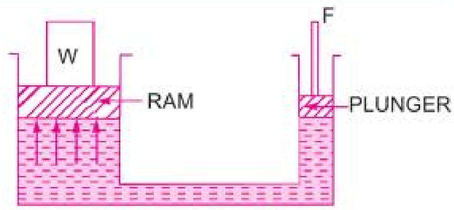


Fig. 2.3

Due to Pascal's law, the intensity of pressure will be equally transmitted in all directions. Hence the pressure intensity at the ram

$$= \frac{500}{.00159} = 314465.4 \text{ N/m}^2$$

But pressure intensity at ram $= \frac{\text{Weight}}{\text{Area of ram}} = \frac{W}{A} = \frac{W}{.07068} \text{ N/m}^2$

$$\frac{W}{.07068} = 314465.4$$

$\therefore$ Weight $= 314465.4 \times .07068 = 22222 \text{ N} = $ **22.222 kN. Ans.**

Problem 2.2 *A hydraulic press has a ram of 20 cm diameter and a plunger of 3 cm diameter. It is used for lifting a weight of 30 kN. Find the force required at the plunger.*

Solution. Given :

Dia. of ram, $D = 20 \text{ cm} = 0.2 \text{ m}$

$\therefore$ Area of ram, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$

Dia. of plunger $d = 3 \text{ cm} = 0.03 \text{ m}$

$\therefore$ Area of plunger, $a = \frac{\pi}{4} (.03)^2 = 7.068 \times 10^{-4} \text{ m}^2$

Weight lifted, $W = 30 \text{ kN} = 30 \times 1000 \text{ N} = 30000 \text{ N}.$

See Fig. 2.3.

Pressure intensity developed due to plunger $= \frac{\text{Force}}{\text{Area}} = \frac{F}{a}.$

By Pascal's Law, this pressure is transmitted equally in all directions

Hence pressure transmitted at the ram $= \frac{F}{a}$

$\therefore$ Force acting on ram = Pressure intensity × Area of ram

$$= \frac{F}{a} \times A = \frac{F \times .0314}{7.068 \times 10^{-4}} \text{ N}$$

But force acting on ram = Weight lifted = 30000 N

$$\therefore \quad 30000 = \frac{F \times .0314}{7.068 \times 10^{-4}}$$

$$\therefore \quad F = \frac{30000 \times 7.068 \times 10^{-4}}{.0314} = \mathbf{675.2 \text{ N.}} \textbf{ Ans.}$$

Problem 2.3 *Calculate the pressure due to a column of 0.3 of (a) water, (b) an oil of sp. gr. 0.8, and (c) mercury of sp. gr. 13.6. Take density of water, $\rho = 1000 \text{ kg/m}^3$.*

Solution. Given :

Height of liquid column, $Z = 0.3 \text{ m}.$

The pressure at any point in a liquid is given by equation (2.5) as

$$p = \rho g Z$$

(*a*) For water, $\rho = 1000 \text{ kg/m}^3$

$\therefore$ $p = \rho g Z = 1000 \times 9.81 \times 0.3 = 2943 \text{ N/m}^2$

$$= \frac{2943}{10^4} \text{ N/cm}^2 = \mathbf{0.2943 \text{ N/cm}^2}. \textbf{ Ans.}$$

(*b*) For oil of sp. gr. 0.8,

From equation (1.1*A*), we know that the density of a fluid is equal to specific gravity of fluid multiplied by density of water.

$\therefore$ Density of oil, $\rho_0 = \text{Sp. gr. of oil} \times \text{Density of water}$ (ρ_0 = Density of oil)

$$= 0.8 \times \rho = 0.8 \times 1000 = 800 \text{ kg/m}^3$$

Now pressure, $p = \rho_0 \times g \times Z$

$$= 800 \times 9.81 \times 0.3 = 2354.4 \frac{\text{N}}{\text{m}^2} = \frac{2354.4}{10^4} \frac{\text{N}}{\text{cm}^2}.$$

$$= \mathbf{0.2354 \frac{N}{cm^2}}. \textbf{ Ans.}$$

(*c*) For mercury, sp. gr. = 13.6

From equation (1.1*A*) we know that the density of a fluid is equal to specific gravity of fluid multiplied by density of water

$\therefore$ Density of mercury, $\rho_s = \text{Specific gravity of mercury} \times \text{Density of water}$

$$= 13.6 \times 1000 = 13600 \text{ kg/m}^3$$

$\therefore$ $p = \rho_s \times g \times Z$

$$= 13600 \times 9.81 \times 0.3 = 40025 \frac{\text{N}}{\text{m}^2}$$

$$= \frac{40025}{10^4} = \mathbf{4.002 \frac{N}{cm^2}}. \textbf{ Ans.}$$

Problem 2.4 *The pressure intensity at a point in a fluid is given 3.924 N/cm². Find the corresponding height of fluid when the fluid is : (a) water, and (b) oil of sp. gr. 0.9.*

Solution. Given :

Pressure intensity, $p = 3.924 \frac{\text{N}}{\text{cm}^2} = 3.924 \times 10^4 \frac{\text{N}}{\text{m}^2}.$

The corresponding height, Z, of the fluid is given by equation (2.6) as

$$Z = \frac{p}{\rho \times g}$$

(*a*) For water, $\rho = 1000 \text{ kg/m}^3$

$\therefore$ $Z = \frac{p}{\rho \times g} = \frac{3.924 \times 10^4}{1000 \times 9.81} = \mathbf{4 \text{ m of water}}. \textbf{ Ans.}$

(*b*) For oil, sp. gr. = 0.9

$\therefore$ Density of oil $\rho_0 = 0.9 \times 1000 = 900 \text{ kg/m}^3$

$\therefore$ $Z = \frac{p}{\rho_0 \times g} = \frac{3.924 \times 10^4}{900 \times 9.81} = \mathbf{4.44 \text{ m of oil}}. \textbf{ Ans.}$

Problem 2.5 *An oil of sp. gr. 0.9 is contained in a vessel. At a point the height of oil is 40 m. Find the corresponding height of water at the point.*

Solution. Given :

Sp. gr. of oil, $S_0 = 0.9$

Height of oil, $Z_0 = 40$ m

Density of oil, ρ_0 = Sp. gr. of oil × Density of water = 0.9 × 1000 = 900 kg/m^3

Intensity of pressure, $p = \rho_0 \times g \times Z_0 = 900 \times 9.81 \times 40 \ \frac{\text{N}}{\text{m}^2}$

$$\therefore \text{ Corresponding height of water} = \frac{p}{\text{Density of water} \times g}$$

$$= \frac{900 \times 9.81 \times 40}{1000 \times 9.81} = 0.9 \times 40 = \textbf{36 m of water. Ans.}$$

Problem 2.6 *An open tank contains water upto a depth of 2 m and above it an oil of sp. gr. 0.9 for a depth of 1 m. Find the pressure intensity (i) at the interface of the two liquids, and (ii) at the bottom of the tank.*

Solution. Given :

Height of water, $Z_1 = 2$ m

Height of oil, $Z_2 = 1$ m

Sp. gr. of oil, $S_0 = 0.9$

Density of water, $\rho_1 = 1000$ kg/m^3

Density of oil, ρ_2 = Sp. gr. of oil × Density of water = 0.9 × 1000 = 900 kg/m^3

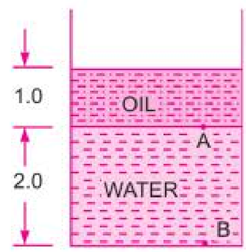


Fig. 2.4

Pressure intensity at any point is given by

$$p = \rho \times g \times Z.$$

(*i*) At interface, *i.e.*, at *A*

$$p = \rho_2 \times g \times 1.0$$
$$= 900 \times 9.81 \times 1.0$$
$$= 8829 \ \frac{\text{N}}{\text{m}^2} = \frac{8829}{10^4} = \textbf{0.8829 N/cm}^2\textbf{. Ans.}$$

(*ii*) At the bottom, *i.e.*, at *B*

$$p = \rho_2 \times gZ_2 + \rho_1 \times g \times Z_1 = 900 \times 9.81 \times 1.0 + 1000 \times 9.81 \times 2.0$$
$$= 8829 + 19620 = 28449 \text{ N/m}^2 = \frac{28449}{10^4} \text{ N/cm}^2 = \textbf{2.8449 N/cm}^2\textbf{. Ans.}$$

Problem 2.7 *The diameters of a small piston and a large piston of a hydraulic jack are 3 cm and 10 cm respectively. A force of 80 N is applied on the small piston. Find the load lifted by the large piston when :*

(a) the pistons are at the same level.

(b) small piston is 40 cm above the large piston.

The density of the liquid in the jack is given as 1000 kg/m^3.

Solution. Given :

Dia. of small piston, $d = 3$ cm

$\therefore$ Area of small piston, $a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times (3)^2 = 7.068 \text{ cm}^2$

Dia. of large piston, $D = 10$ cm

$\therefore$ Area of larger piston, $A = \frac{p}{4} \times (10)^2 = 78.54 \text{ cm}^2$

Force on small piston, $F = 80$ N

Let the load lifted $= W$.

(*a*) **When the pistons are at the same level**

Pressure intensity on small piston

$$\frac{F}{a} = \frac{80}{7.068} \text{ N/cm}^2$$

This is transmitted equally on the large piston.

$\therefore$ Pressure intensity on the large piston

$$= \frac{80}{7.068}$$

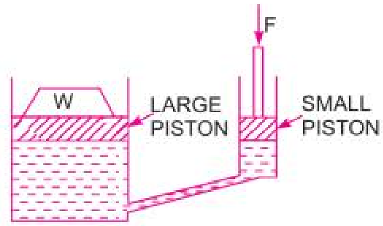


Fig. 2.5

$\therefore$ Force on the large piston $= \text{Pressure} \times \text{Area}$

$$= \frac{80}{7.068} \times 78.54 \text{ N} = \mathbf{888.96 \text{ N.}}$$ **Ans.**

(*b*) **When the small piston is 40 cm above the large piston**

Pressure intensity on the small piston

$$= \frac{F}{a} = \frac{80}{7.068} \frac{\text{N}}{\text{cm}^2}$$

$\therefore$ Pressure intensity at section *A-A*

$$= \frac{F}{a} + \text{Pressure intensity due to height of 40 cm of liquid.}$$

But pressure intensity due to 40 cm of liquid

$$= \rho \times g \times h = 1000 \times 9.81 \times 0.4 \text{ N/m}^2$$

$$= \frac{1000 \times 9.81 \times .40}{10^4} \text{ N/cm}^2 = 0.3924 \text{ N/cm}^2$$

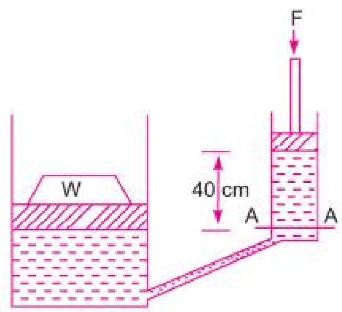


Fig. 2.6

$\therefore$ Pressure intensity at section *A-A*

$$= \frac{80}{7.068} + 0.3924$$

$$= 11.32 + 0.3924 = 11.71 \text{ N/cm}^2$$

$\therefore$ Pressure intensity transmitted to the large piston $= 11.71 \text{ N/cm}^2$

$\therefore$ Force on the large piston = Pressure $\times$ Area of the large piston

$$= 11.71 \times A = 11.71 \times 78.54 = 919.7 \text{ N.}$$

▶ 2.4 ABSOLUTE, GAUGE, ATMOSPHERIC AND VACUUM PRESSURES

The pressure on a fluid is measured in two different systems. In one system, it is measured above the absolute zero or complete vacuum and it is called the absolute pressure and in other system, pressure is measured above the atmospheric pressure and it is called gauge pressure. Thus :

1. Absolute pressure is defined as the pressure which is measured with reference to absolute vacuum pressure.

2. Gauge pressure is defined as the pressure which is measured with the help of a pressure measuring instrument, in which the atmospheric pressure is taken as datum. The atmospheric pressure on the scale is marked as zero.

3. Vacuum pressure is defined as the pressure below the atmospheric pressure.

The relationship between the absolute pressure, gauge pressure and vacuum pressure are shown in Fig. 2.7.

Mathematically :

(*i*) Absolute pressure

= Atmospheric pressure + Gauge pressure

or $$p_{ab} = p_{atm} + p_{gauge}$$

(*ii*) Vacuum pressure

= Atmospheric pressure – Absolute pressure.

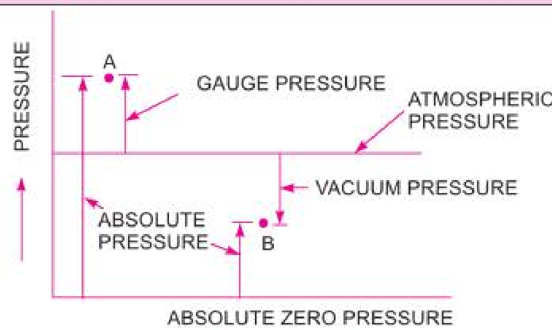


Fig. 2.7 *Relationship between pressures.*

Note. (*i*) The atmospheric pressure at sea level at 15°C is 101.3 kN/m^2 or 10.13 N/cm^2 in SI unit. In case of MKS units, it is equal to 1.033 kgf/cm^2.

(*ii*) The atmospheric pressure head is 760 mm of mercury or 10.33 m of water.

Problem 2.8 *What are the gauge pressure and absolute pressure at a point 3 m below the free surface of a liquid having a density of 1.53 × 10^3 kg/m^3 if the atmospheric pressure is equivalent to 750 mm of mercury ? The specific gravity of mercury is 13.6 and density of water = 1000 kg/m^3.*

Solution. Given :

Depth of liquid, $Z_1 = 3$ m

Density of liquid, $\rho_1 = 1.53 \times 10^3$ kg/m^3

Atmospheric pressure head, $Z_0 = 750$ mm of Hg

$$= \frac{750}{1000} = 0.75 \text{ m of Hg}$$

∴ Atmospheric pressure, $p_{atm} = \rho_0 \times g \times Z_0$

where ρ_0 = Density of Hg = Sp. gr. of mercury × Density of water = 13.6 × 1000 kg/m^3

and Z_0 = Pressure head in terms of mercury.

$$\therefore \quad p_{atm} = (13.6 \times 1000) \times 9.81 \times 0.75 \text{ N/m}^2 \qquad (\because \; Z_0 = 0.75)$$

$$= 100062 \text{ N/m}^2$$

Pressure at a point, which is at a depth of 3 m from the free surface of the liquid is given by,

$$p = \rho_1 \times g \times Z_1$$

$$= (1.53 \times 1000) \times 9.81 \times 3 = 45028 \text{ N/m}^2$$

∴ Gauge pressure, p = **45028 N/m^2. Ans.**

Now absolute pressure = Gauge pressure + Atmospheric pressure

= 45028 + 100062 = **145090 N/m^2. Ans.**

▶ 2.5 MEASUREMENT OF PRESSURE

The pressure of a fluid is measured by the following devices :

1. Manometers
2. Mechanical Gauges.

2.5.1 Manometers. Manometers are defined as the devices used for measuring the pressure at a point in a fluid by balancing the column of fluid by the same or another column of the fluid. They are classified as :

(*a*) Simple Manometers, (*b*) Differential Manometers.

2.5.2 Mechanical Gauges. Mechanical gauges are defined as the devices used for measuring the pressure by balancing the fluid column by the spring or dead weight. The commonly used mechanical pressure gauges are :

(*a*) Diaphragm pressure gauge,
(*b*) Bourdon tube pressure gauge,
(*c*) Dead-weight pressure gauge, and
(*d*) Bellows pressure gauge.

▶ 2.6 SIMPLE MANOMETERS

A simple manometer consists of a glass tube having one of its ends connected to a point where pressure is to be measured and other end remains open to atmosphere. Common types of simple manometers are :

1. Piezometer,`
2. U-tube Manometer, and
3. Single Column Manometer.

2.6.1 Piezometer. It is the simplest form of manometer used for measuring gauge pressures. One end of this manometer is connected to the point where pressure is to be measured and other end is open to the atmosphere as shown in Fig. 2.8. The rise of liquid gives the pressure head at that point. If at a point A, the height of liquid say water is h in piezometer tube, then pressure at A

$$= \rho \times g \times h \ \frac{\text{N}}{\text{m}^2}.$$

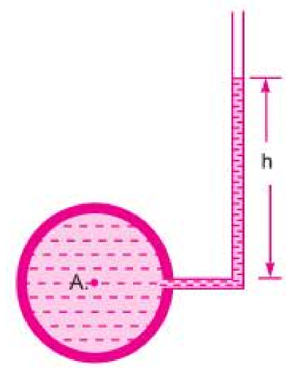


Fig. 2.8 *Piezometer.*

2.6.2 U-tube Manometer. It consists of glass tube bent in U-shape, one end of which is connected to a point at which pressure is to be measured and other end remains open to the atmosphere as shown in Fig. 2.9. The tube generally contains mercury or any other liquid whose specific gravity is greater than the specific gravity of the liquid whose pressure is to be measured.

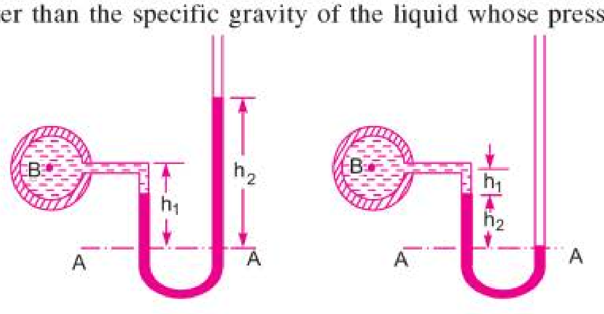


Fig. 2.9 *U-tube Manometer.*

(*a*) **For Gauge Pressure.** Let B is the point at which pressure is to be measured, whose value is p. The datum line is A-A.

Let h_1 = Height of light liquid above the datum line
h_2 = Height of heavy liquid above the datum line
S_1 = Sp. gr. of light liquid
ρ_1 = Density of light liquid = $1000 \times S_1$
S_2 = Sp. gr. of heavy liquid
ρ_2 = Density of heavy liquid = $1000 \times S_2$

As the pressure is the same for the horizontal surface. Hence pressure above the horizontal datum line *A-A* in the left column and in the right column of U-tube manometer should be same.

Pressure above *A-A* in the left column $= p + \rho_1 \times g \times h_1$

Pressure above *A-A* in the right column $= \rho_2 \times g \times h_2$

Hence equating the two pressures $p + \rho_1 g h_1 = \rho_2 g h_2$

$$\therefore \quad p = (\rho_2 g h_2 - \rho_1 \times g \times h_1). \quad ...(2.7)$$

(*b*) **For Vacuum Pressure.** For measuring vacuum pressure, the level of the heavy liquid in the manometer will be as shown in Fig. 2.9 (*b*). Then

Pressure above *A-A* in the left column $= \rho_2 g h_2 + \rho_1 g h_1 + p$

Pressure head in the right column above *A-A* $= 0$

$$\therefore \quad \rho_2 g h_2 + \rho_1 g h_1 + p = 0$$

$$\therefore \quad p = -(\rho_2 g h_2 + \rho_1 g h_1). \quad ...(2.8)$$

Problem 2.9 *The right limb of a simple U-tube manometer containing mercury is open to the atmosphere while the left limb is connected to a pipe in which a fluid of sp. gr. 0.9 is flowing. The centre of the pipe is 12 cm below the level of mercury in the right limb. Find the pressure of fluid in the pipe if the difference of mercury level in the two limbs is 20 cm.*

Solution. Given :

Sp. gr. of fluid, $S_1 = 0.9$

$\therefore$ Density of fluid, $\rho_1 = S_1 \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$

Sp. gr. of mercury, $S_2 = 13.6$

$\therefore$ Density of mercury, $\rho_2 = 13.6 \times 1000 \text{ kg/m}^3$

Difference of mercury level, $h_2 = 20 \text{ cm} = 0.2 \text{ m}$

Height of fluid from *A-A*, $h_1 = 20 - 12 = 8 \text{ cm} = 0.08 \text{ m}$

Let p = Pressure of fluid in pipe

Equating the pressure above *A-A*, we get

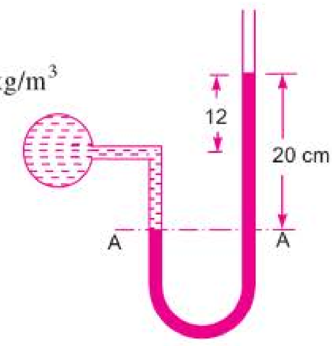


Fig. 2.10

$$p + \rho_1 g h_1 = \rho_2 g h_2$$

or $$p + 900 \times 9.81 \times 0.08 = 13.6 \times 1000 \times 9.81 \times .2$$

$$p = 13.6 \times 1000 \times 9.81 \times .2 - 900 \times 9.81 \times 0.08$$

$$= 26683 - 706 = 25977 \text{ N/m}^2 = \mathbf{2.597 \text{ N/cm}^2}. \text{ Ans.}$$

Problem 2.10 *A simple U-tube manometer containing mercury is connected to a pipe in which a fluid of sp. gr. 0.8 and having vacuum pressure is flowing. The other end of the manometer is open to atmosphere. Find the vacuum pressure in pipe, if the difference of mercury level in the two limbs is 40 cm and the height of fluid in the left from the centre of pipe is 15 cm below.*

Solution. Given :

Sp. gr. of fluid, $S_1 = 0.8$

Sp. gr. of mercury, $S_2 = 13.6$

Density of fluid, $\rho_1 = 800$

Density of mercury, $\rho_2 = 13.6 \times 1000$

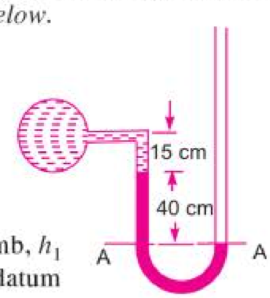


Fig. 2.11

Difference of mercury level, $h_2 = 40 \text{ cm} = 0.4 \text{ m}$. Height of liquid in left limb, $h_1 = 15 \text{ cm} = 0.15 \text{ m}$. Let the pressure in pipe $= p$. Equating pressure above datum line *A-A*, we get

$$\rho_2 g h_2 + \rho_1 g h_1 + p = 0$$

$$\therefore \quad p = -[\rho_2 g h_2 + \rho_1 g h_1]$$
$$= -[13.6 \times 1000 \times 9.81 \times 0.4 + 800 \times 9.81 \times 0.15]$$
$$= -[53366.4 + 1177.2] = -54543.6 \text{ N/m}^2 = \mathbf{-5.454 \text{ N/cm}^2}. \textbf{ Ans.}$$

Problem 2.11 *A U-Tube manometer is used to measure the pressure of water in a pipe line, which is in excess of atmospheric pressure. The right limb of the manometer contains mercury and is open to atmosphere. The contact between water and mercury is in the left limb. Determine the pressure of water in the main line, if the difference in level of mercury in the limbs of U-tube is 10 cm and the free surface of mercury is in level with the centre of the pipe. If the pressure of water in pipe line is reduced to 9810 N/m², calculate the new difference in the level of mercury. Sketch the arrangements in both cases.*

Solution. Given :

Difference of mercury = 10 cm = 0.1 m

The arrangement is shown in Fig. 2.11 (*a*)

Ist Part

Let p_A = (pressure of water in pipe line (*i.e.*, at point *A*)

The points *B* and *C* lie on the same horizontal line. Hence pressure at *B* should be equal to pressure at *C*. But pressure at *B*

= Pressure at *A* + Pressure due to 10 cm (or 0.1 m) of water

$= p_A + \rho \times g \times h$

where $\rho = 1000 \text{ kg/m}^3$ and $h = 0.1$ m

$= p_A + 1000 \times 9.81 \times 0.1$

$= p_A + 981 \text{ N/m}^2$ ...(*i*)

Pressure at *C* = Pressure at *D* + Pressure due to 10 cm of mercury

$= 0 + \rho_0 \times g \times h_0$

where ρ_0 for mercury $= 13.6 \times 1000 \text{ kg/m}^3$

and $h_0 = 10 \text{ cm} = 0.1 \text{ m}$

$\therefore$ Pressure at $C = 0 + (13.6 \times 1000) \times 9.81 \times 0.1$

$= 13341.6$ N ...(*ii*)

But pressure at *B* is equal to pressure at *C*. Hence equating the equations (*i*) and (*ii*), we get

$$p_A + 981 = 13341.6$$

$$\therefore \quad p_A = 13341.6 - 981$$

$$= \mathbf{12360.6 \frac{N}{m^2}}. \textbf{ Ans.}$$

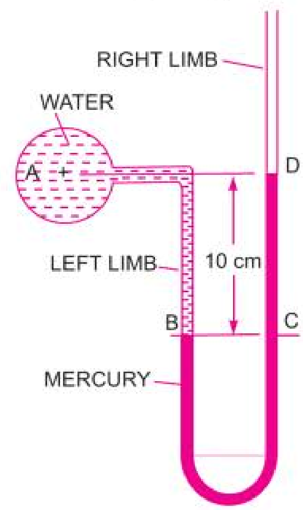


Fig. 2.11 (*a*)

IInd Part

Given, $p_A = 9810 \text{ N/m}^2$

Find new difference of mercury level. The arrangement is shown in Fig. 2.11 (*b*). In this case the pressure at *A* is 9810 N/m² which is less than the 12360.6 N/m². Hence mercury in left limb will rise. The rise of mercury in left limb will be equal to the fall of mercury in right limb as the total volume of mercury remains same.

Let *x* = Rise of mercury in left limb in cm

Then fall of mercury in right limb = *x* cm

The points *B*, *C* and *D* show the initial conditions whereas points *B**, *C** and *D** show the final conditions.

The pressure at B^* = Pressure at C^*

or Pressure at A + Pressure due to $(10 - x)$ cm of water

= Pressure at D^* + Pressure due to $(10 - 2x)$ cm of mercury

or $$p_A + \rho_1 \times g \times h_1 = p_D^* + \rho_2 \times g \times h_2$$

or $$1910 + 1000 \times 9.81 \times \left(\frac{10-x}{100}\right)$$

$$= 0 + (13.6 \times 1000) \times 9.81 \times \left(\frac{10-2x}{100}\right)$$

Dividing by 9.81, we get

or $$1000 + 100 - 10x = 1360 - 272x$$

or $$272x - 10x = 1360 - 1100$$

or $$262x = 260$$

$$\therefore \quad x = \frac{260}{262} = 0.992 \text{ cm}$$

$\therefore$ New difference of mercury $= 10 - 2x$ cm $= 10 - 2 \times 0.992$

$= \mathbf{8.016}$ **cm. Ans.**

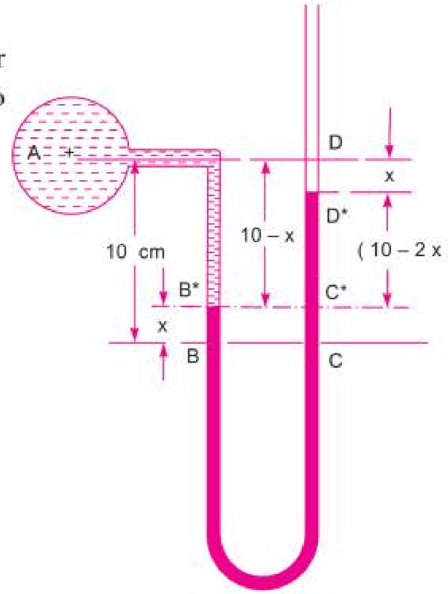


Fig. 2.11 (*b*)

Problem 2.12 *Fig. 2.12 shows a conical vessel having its outlet at A to which a U-tube manometer is connected. The reading of the manometer given in the figure shows when the vessel is empty. Find the reading of the manometer when the vessel is completely filled with water.*

Solution. Vessel is empty. Given :

Difference of mercury level $h_2 = 20$ cm

Let h_1 = Height of water above X-X

Sp. gr. of mercury, $S_2 = 13.6$

Sp. gr. of water, $S_1 = 1.0$

Density of mercury, $\rho_2 = 13.6 \times 1000$

Density of water, $\rho_1 = 1000$

Equating the pressure above datum line X-X, we have

$$\rho_2 \times g \times h_2 = \rho_1 \times g \times h_1$$

or $$13.6 \times 1000 \times 9.81 \times 0.2 = 1000 \times 9.81 \times h_1$$

$$h_1 = 2.72 \text{ m of water.}$$

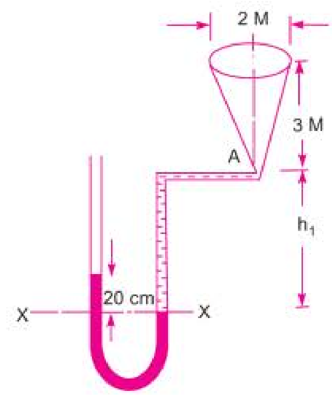


Fig. 2.12

Vessel is full of water. When vessel is full of water, the pressure in the right limb will increase and mercury level in the right limb will go down. Let the distance through which mercury goes down in the right limb be, y cm as shown in Fig. 2.13. The mercury will rise in the left by a distance of y cm. Now the datum line is Z-Z. Equating the pressure above the datum line Z-Z.

Pressure in left limb = Pressure in right limb

$$13.6 \times 1000 \times 9.81 \times (0.2 + 2y/100)$$

$$= 1000 \times 9.81 \times (3 + h_1 + y/100)$$

or $\quad 13.6 \times (0.2 + 2y/100) = (3 + 2.72 + y/100) \quad (\because \; h_1 = 2.72 \text{ cm})$

or $\quad 2.72 + 27.2y/100 = 3 + 2.72 + y/100$

or $\quad (27.2y - y)/100 = 3.0$

or $\quad 26.2y = 3 \times 100 = 300$

$\therefore \quad y = \dfrac{300}{26.2} = 11.45$ cm

The difference of mercury level in two limbs

$= (20 + 2y)$ cm of mercury

$= 20 + 2 \times 11.45 = 20 + 22.90$

$= 42.90$ cm of mercury

$\therefore$ Reading of manometer = **42.90 cm. Ans.**

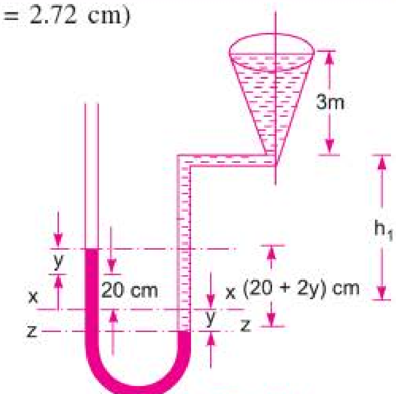


Fig. 2.13

Problem 2.13 *A pressure gauge consists of two cylindrical bulbs B and C each of 10 sq. cm cross-sectional area, which are connected by a U-tube with vertical limbs each of 0.25 sq. cm cross-sectional area. A red liquid of specific gravity 0.9 is filled into C and clear water is filled into B, the surface of separation being in the limb attached to C. Find the displacement of the surface of separation when the pressure on the surface in C is greater than that in B by an amount equal to 1 cm head of water.*

Solution. Given :

Area of each bulb B and C, $\quad A = 10 \text{ cm}^2$

Area of each vertical limb, $\quad a = 0.25 \text{ cm}^2$

Sp. gr. of red liquid $\quad = 0.9 \quad \therefore$ Its density $= 900 \text{ kg/m}^3$

Let $\quad X\text{-}X$ = Initial separation level

h_C = Height of red liquid above X-X

h_B = Height of water above X-X

Pressure above X-X in the left limb $= 1000 \times 9.81 \times h_B$

Pressure above X-X in the right limb $= 900 \times 9.81 \times h_C$

Equating the two pressure, we get

$1000 \times 9.81 \times h_B = 900 \times 9.81 \times h_C$

$\therefore \quad h_B = 0.9\, h_C \quad \quad ...(i)$

When the pressure head over the surface in C is increased by 1 cm of water, let the separation level falls by an amount equal to Z. Then Y-Y becomes the final separation level.

Now fall in surface level of C multiplied by cross-sectional area of bulb C must be equal to the fall in separation level multiplied by cross-sectional area of limb.

$\therefore$ Fall in surface level of C

$$= \frac{\text{Fall in separation level} \times a}{A}$$

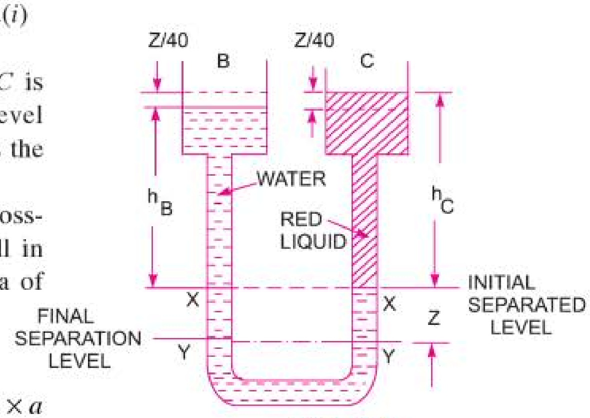


Fig. 2.14

$$= \frac{Z \times a}{A} = \frac{Z \times 0.25}{10} = \frac{Z}{40}.$$

Also fall in surface level of C

$$= \text{Rise in surface level of } B$$

$$= \frac{Z}{40}$$

The pressure of 1 cm (or 0.01 m) of water = $\rho gh = 1000 \times 9.81 \times 0.01 = 98.1 \text{ N/m}^2$

Consider final separation level Y-Y

Pressure above Y-Y in the left limb = $1000 \times 9.81 \left(Z + h_B + \frac{Z}{40}\right)$

Pressure above Y-Y in the right limb = $900 \times 9.81 \left(Z + h_C - \frac{Z}{40}\right) + 98.1$

Equating the two pressure, we get

$$1000 \times 9.81 \left(Z + h_B + \frac{Z}{40}\right) = \left(Z + h_C - \frac{Z}{40}\right) 900 \times 9.81 + 98.1$$

Dividing by 9.81, we get

$$1000 \left(Z + h_B + \frac{Z}{40}\right) = 900 \left(Z + h_C - \frac{Z}{40}\right) + 10$$

Dividing by 1000, we get $Z + h_B + \frac{Z}{40} = 0.9 \left(Z + h_C - \frac{Z}{40}\right) + 0.01$

But from equation (i), $\qquad h_B = 0.9\, h_C$

$$\therefore \qquad Z + 0.9\, h_C + \frac{Z}{40} = \frac{39Z}{40} \times 0.9 + 0.9\, h_C + 0.01$$

or
$$\frac{41Z}{40} = \frac{39}{40} \times .9Z + .01$$

or
$$Z\left(\frac{41}{40} - \frac{39 \times .9}{40}\right) = .01 \quad \text{or} \quad Z\left(\frac{41 - 35.1}{40}\right) = .01$$

$$\therefore \qquad Z = \frac{40 \times 0.01}{5.9} = \mathbf{0.0678\ m = 6.78\ cm.}\ \text{Ans.}$$

2.6.3 Single Column Manometer.

Single column manometer is a modified form of a U-tube manometer in which a reservoir, having a large cross-sectional area (about 100 times) as compared to the area of the tube is connected to one of the limbs (say left limb) of the manometer as shown in Fig. 2.15. Due to large cross-sectional area of the reservoir, for any variation in pressure, the change in the liquid level in the reservoir will be very small which may be neglected and hence the pressure is given by the height of liquid in the other limb. The other limb may be vertical or inclined. Thus there are two types of single column manometer as :

1. Vertical Single Column Manometer.
2. Inclined Single Column Manometer.

1. Vertical Single Column Manometer

Fig. 2.15 shows the vertical single column manometer. Let X-X be the datum line in the reservoir and in the right limb of the manometer, when it is not connected to the pipe. When the manometer is connected to the pipe, due to high pressure at A, the heavy liquid in the reservoir will be pushed downward and will rise in the right limb.

Let Δh = Fall of heavy liquid in reservoir

h_2 = Rise of heavy liquid in right limb

h_1 = Height of centre of pipe above X-X

p_A = Pressure at A, which is to be measured

A = Cross-sectional area of the reservoir

a = Cross-sectional area of the right limb

S_1 = Sp. gr. of liquid in pipe

S_2 = Sp. gr. of heavy liquid in reservoir and right limb

ρ_1 = Density of liquid in pipe

ρ_2 = Density of liquid in reservoir

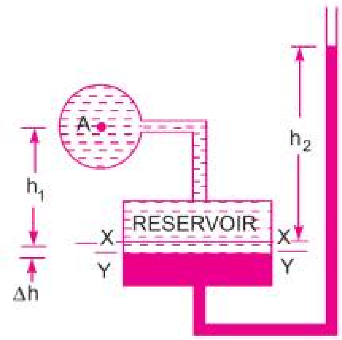


Fig. 2.15 *Vertical single column manometer.*

Fall of heavy liquid in reservoir will cause a rise of heavy liquid level in the right limb.

$$\therefore \quad A \times \Delta h = a \times h_2$$

$$\therefore \quad \Delta h = \frac{a \times h_2}{A} \quad \text{...(i)}$$

Now consider the datum line Y-Y as shown in Fig. 2.15. Then pressure in the right limb above Y-Y.

$$= \rho_2 \times g \times (\Delta h + h_2)$$

Pressure in the left limb above Y-$Y = \rho_1 \times g \times (\Delta h + h_1) + p_A$

Equating these pressures, we have

$$\rho_2 \times g \times (\Delta h + h_2) = \rho_1 \times g \times (\Delta h + h_1) + p_A$$

or

$$p_A = \rho_2 g\,(\Delta h + h_2) - \rho_1 g(\Delta h + h_1)$$

$$= \Delta h[\rho_2 g - \rho_1 g] + h_2\rho_2 g - h_1\rho_1 g$$

But from equation (*i*), $\Delta h = \dfrac{a \times h_2}{A}$

$$\therefore \quad p_A = \frac{a \times h_2}{A}\,[\rho_2 g - \rho_1 g] + h_2\rho_2 g - h_1\rho_1 g \quad \text{...(2.9)}$$

As the area A is very large as compared to a, hence ratio $\dfrac{a}{A}$ becomes very small and can be neglected.

Then $p_A = h_2\rho_2 g - h_1\rho_1 g$ ...(2.10)

From equation (2.10), it is clear that as h_1 is known and hence by knowing h_2 or rise of heavy liquid in the right limb, the pressure at A can be calculated.

2. Inclined Single Column Manometer

Fig. 2.16 shows the inclined single column manometer. This manometer is more sensitive. Due to inclination the distance moved by the heavy liquid in the right limb will be more.

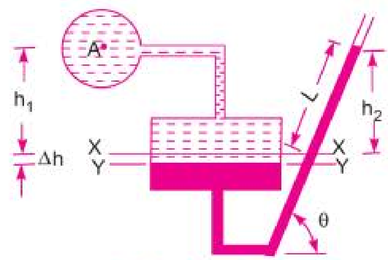


Fig. 2.16 *Inclined single column manometer.*

Let L = Length of heavy liquid moved in right limb from X-X

θ = Inclination of right limb with horizontal

h_2 = Vertical rise of heavy liquid in right limb from X-$X = L \times \sin\theta$

From equation (2.10), the pressure at A is

$$p_A = h_2\rho_2 g - h_1\rho_1 g.$$

Substituting the value of h_2, we get

$$p_A = \sin\theta \times \rho_2 g - h_1\rho_1 g. \qquad ...(2.11)$$

Problem 2.14 *A single column manometer is connected to a pipe containing a liquid of sp. gr. 0.9 as shown in Fig. 2.17. Find the pressure in the pipe if the area of the reservoir is 100 times the area of the tube for the manometer reading shown in Fig. 2.17. The specific gravity of mercury is 13.6.*

Solution. Given :

Sp. gr. of liquid in pipe, $S_1 = 0.9$

$\therefore$ Density $\rho_1 = 900$ kg/m^3

Sp. gr. of heavy liquid, $S_2 = 13.6$

Density, $\rho_2 = 13.6 \times 1000$

$$\frac{\text{Area of reservoir}}{\text{Area of right limb}} = \frac{A}{a} = 100$$

Height of liquid, $h_1 = 20$ cm $= 0.2$ m

Rise of mercury in right limb,

$h_2 = 40$ cm $= 0.4$ m

Let p_A = Pressure in pipe

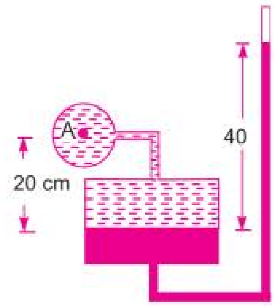


Fig. 2.17

Using equation (2.9), we get

$$p_A = \frac{a}{A}\, h_2[\rho_2 g - \rho_1 g] + h_2\rho_2 g - h_1\rho_1 g$$

$$= \frac{1}{100} \times 0.4[13.6 \times 1000 \times 9.81 - 900 \times 9.81] + 0.4 \times 13.6 \times 1000 \times 9.81 - 0.2 \times 900 \times 9.81$$

$$= \frac{0.4}{100}\,[133416 - 8829] + 53366.4 - 1765.8$$

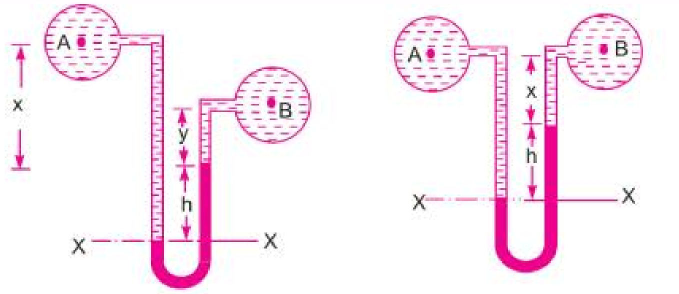
$= 533.664 + 53366.4 - 1765.8$ N/m^2 $= 52134$ N/m^2 $=$ **5.21 N/cm^2. Ans.**

2.7 DIFFERENTIAL MANOMETERS

Differential manometers are the devices used for measuring the difference of pressures between two points in a pipe or in two different pipes. A differential manometer consists of a U-tube, containing a heavy liquid, whose two ends are connected to the points, whose difference of pressure is to be measured. Most commonly types of differential manometers are :

1. U-tube differential manometer and
2. Inverted U-tube differential manometer.

2.7.1 U-tube Differential Manometer. Fig. 2.18 shows the differential manometers of U-tube type.



(a)Two pipes at different levels (b) A and B are at the same level

Fig. 2.18 *U-tube differential manometers.*

In Fig. 2.18 (*a*), the two points *A* and *B* are at different level and also contains liquids of different sp. gr. These points are connected to the U-tube differential manometer. Let the pressure at *A* and *B* are p_A and p_B.

Let h = Difference of mercury level in the U-tube.

y = Distance of the centre of *B*, from the mercury level in the right limb.

x = Distance of the centre of *A*, from the mercury level in the right limb.

ρ_1 = Density of liquid at *A*.

ρ_2 = Density of liquid at *B*.

ρ_g = Density of heavy liquid or mercury.

Taking datum line at *X-X*.

Pressure above *X-X* in the left limb $= \rho_1 g(h + x) + p_A$

where p_A = pressure at *A*.

Pressure above *X-X* in the right limb $= \rho_g \times g \times h + \rho_2 \times g \times y + p_B$

where p_B = Pressure at *B*.

Equating the two pressure, we have

$$\rho_1 g(h + x) + p_A = \rho_g \times g \times h + \rho_2 g y + p_B$$

$$\therefore \quad p_A - p_B = \rho_g \times g \times h + \rho_2 g y - \rho_1 g(h + x)$$

$$= h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x \qquad \text{...(2.12)}$$

$\therefore$ Difference of pressure at *A* and $B = h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x$

In Fig. 2.18 (*b*), the two points *A* and *B* are at the same level and contains the same liquid of density ρ_1. Then

Pressure above *X-X* in right limb $= \rho_g \times g \times h + \rho_1 \times g \times x + p_B$

Pressure above *X-X* in left limb $= \rho_1 \times g \times (h + x) + p_A$

Equating the two pressure

$$\rho_g \times g \times h + \rho_1 g x + p_B = \rho_1 \times g \times (h + x) + p_A$$

$$\therefore \quad p_A - p_B = \rho_g \times g \times h + \rho_1 g x - \rho_1 g(h + x)$$

$$= g \times h(\rho_g - \rho_1). \qquad \text{...(2.13)}$$

Problem 2.15 *A pipe contains an oil of sp. gr. 0.9. A differential manometer connected at the two points A and B shows a difference in mercury level as 15 cm. Find the difference of pressure at the two points.*

Solution. Given :

Sp. gr. of oil, $S_1 = 0.9$ $\therefore$ Density, $\rho_1 = 0.9 \times 1000 = 900$ kg/m^3

Difference in mercury level, $h = 15$ cm $= 0.15$ m

Sp. gr. of mercury, $S_g = 13.6$ $\therefore$ Density, $\rho_g = 13.6 \times 1000$ kg/m^3

The difference of pressure is given by equation (2.13)

or $$p_A - p_B = g \times h(\rho_g - \rho_1)$$
$$= 9.81 \times 0.15\ (13600 - 900) = \mathbf{18688\ N/m^2}.\ \textbf{Ans.}$$

Problem 2.16 *A differential manometer is connected at the two points A and B of two pipes as shown in Fig. 2.19. The pipe A contains a liquid of sp. gr. = 1.5 while pipe B contains a liquid of sp. gr. = 0.9. The pressures at A and B are 1 kgf/cm² and 1.80 kgf/cm² respectively. Find the difference in mercury level in the differential manometer.*

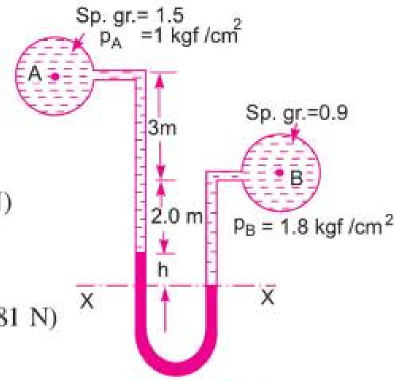


Fig. 2.19

Solution. Given :

Sp. gr. of liquid at A, $S_1 = 1.5$ $\therefore$ $\rho_1 = 1500$

Sp. gr. of liquid at B, $S_2 = 0.9$ $\therefore$ $\rho_2 = 900$

Pressure at A, $p_A = 1$ kgf/cm$^2 = 1 \times 10^4$ kgf/m^2
$= 10^4 \times 9.81$ N/m^2 ($\because$ 1 kgf = 9.81 N)

Pressure at B, $p_B = 1.8$ kgf/cm^2
$= 1.8 \times 10^4$ kgf/m^2
$= 1.8 \times 10^4 \times 9.81$ N/m^2 ($\because$ 1 kgf = 9.81 N)

Density of mercury $= 13.6 \times 1000$ kg/m^3

Taking X-X as datum line.

Pressure above X-X in the left limb

$$= 13.6 \times 1000 \times 9.81 \times h + 1500 \times 9.81 \times (2 + 3) + p_A$$
$$= 13.6 \times 1000 \times 9.81 \times h + 7500 \times 9.81 + 9.81 \times 10^4$$

Pressure above X-X in the right limb $= 900 \times 9.81 \times (h + 2) + p_B$
$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Equating the two pressure, we get

$$13.6 \times 1000 \times 9.81h + 7500 \times 9.81 + 9.81 \times 10^4$$
$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Dividing by 1000×9.81, we get

$$13.6h + 7.5 + 10 = (h + 2.0) \times .9 + 18$$

or $$13.6h + 17.5 = 0.9h + 1.8 + 18 = 0.9h + 19.8$$

or $$(13.6 - 0.9)h = 19.8 - 17.5 \text{ or } 12.7h = 2.3$$

$\therefore$ $$h = \frac{2.3}{12.7} = 0.181 \text{ m} = \mathbf{18.1\ cm}.\ \textbf{Ans.}$$

Problem 2.17 *A differential manometer is connected at the two points A and B as shown in Fig. 2.20. At B air pressure is 9.81 N/cm² (abs), find the absolute pressure at A.*

Solution. Given :

Air pressure at $B = 9.81$ N/cm^2

or $$p_B = 9.81 \times 10^4 \text{ N/m}^2$$

Density of oil $= 0.9 \times 1000 = 900 \text{ kg/m}^3$
Density of mercury $= 13.6 \times 1000 \text{ kg/m}^3$
Let the pressure at A is p_A
Taking datum line at X-X
Pressure above X-X in the right limb

$$= 1000 \times 9.81 \times 0.6 + p_B$$
$$= 5886 + 98100 = 103986$$

Pressure above X-X in the left limb

$$= 13.6 \times 1000 \times 9.81 \times 0.1 + 900 \times 9.81 \times 0.2 + p_A$$
$$= 13341.6 + 1765.8 + p_A$$

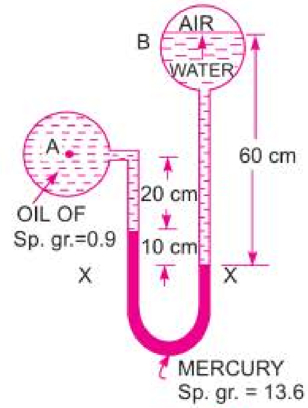


Fig. 2.20

Equating the two pressure heads

$$103986 = 13341.6 + 1765.8 + p_A$$
$$\therefore \quad p_A = 103986 - 15107.4 = 88876.8$$
$$\therefore \quad p_A = 88876.8 \text{ N/m}^2 = \frac{88876.8\text{N}}{10000 \text{ cm}^2} = 8.887 \frac{\text{N}}{\text{cm}^2}.$$

$\therefore$ Absolute pressure at A = **8.887 N/cm²**. **Ans.**

2.7.2 Inverted U-tube Differential Manometer. It consists of an inverted U-tube, containing a light liquid. The two ends of the tube are connected to the points whose difference of pressure is to be measured. It is used for measuring difference of low pressures. Fig. 2.21 shows an inverted U-tube differential manometer connected to the two points A and B. Let the pressure at A is more than the pressure at B.

Let h_1 = Height of liquid in left limb below the datum line X-X
h_2 = Height of liquid in right limb
h = Difference of light liquid
ρ_1 = Density of liquid at A
ρ_2 = Density of liquid at B
ρ_s = Density of light liquid
p_A = Pressure at A
p_B = Pressure at B.

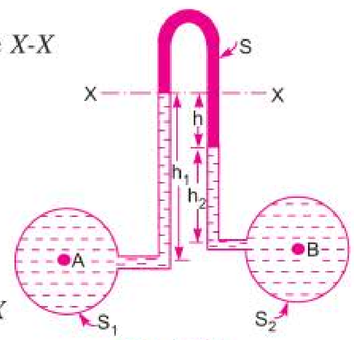


Fig. 2.21

Taking X-X as datum line. Then pressure in the left limb below X-X

$$= p_A - \rho_1 \times g \times h_1.$$

Pressure in the right limb below X-X

$$= p_B - \rho_2 \times g \times h_2 - \rho_s \times g \times h$$

Equating the two pressure

$$p_A - \rho_1 \times g \times h_1 = p_B - \rho_2 \times g \times h_2 - \rho_s \times g \times h$$

or
$$p_A - p_B = \rho_1 \times g \times h_1 - \rho_2 \times g \times h_2 - \rho_s \times g \times h. \quad \text{...(2.14)}$$

Problem 2.18 *Water is flowing through two different pipes to which an inverted differential manometer having an oil of sp. gr. 0.8 is connected. The pressure head in the pipe A is 2 m of water, find the pressure in the pipe B for the manometer readings as shown in Fig. 2.22.*

Solution. Given :

Pressure head at $A = \frac{p_A}{\rho g} = 2$ m of water

$$\therefore \quad p_A = \rho \times g \times 2 = 1000 \times 9.81 \times 2 = 19620 \text{ N/m}^2$$

Fig. 2.22 shows the arrangement. Taking X-X as datum line.

Pressure below X-X in the left limb $= p_A - \rho_1 \times g \times h_1$

$$= 19620 - 1000 \times 9.81 \times 0.3 = 16677 \text{ N/m}^2.$$

Pressure below X-X in the right limb

$$= p_B - 1000 \times 9.81 \times 0.1 - 800 \times 9.81 \times 0.12$$

$$= p_B - 981 - 941.76 = p_B - 1922.76$$

Equating the two pressure, we get

$$16677 = p_B - 1922.76$$

or $$p_B = 16677 + 1922.76 = 18599.76 \text{ N/m}^2$$

or $$p_B = \mathbf{1.8599 \text{ N/cm}^2}. \text{ Ans.}$$

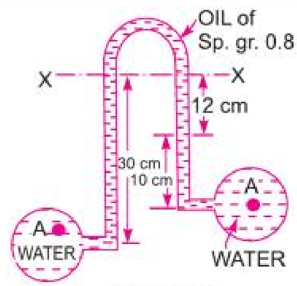


Fig. 2.22

Problem 2.19 *In Fig. 2.23, an inverted differential manometer is connected to two pipes A and B which convey water. The fluid in manometer is oil of sp. gr. 0.8. For the manometer readings shown in the figure, find the pressure difference between A and B.*

Solution. Given :

Sp. gr. of oil $= 0.8 \quad \therefore \quad \rho_s = 800 \text{ kg/m}^3$

Difference of oil in the two limbs

$$= (30 + 20) - 30 = 20 \text{ cm}$$

Taking datum line at X-X

Pressure in the left limb below X-X

$$= p_A - 1000 \times 9.81 \times 0$$

$$= p_A - 2943$$

Pressure in the right limb below X-X

$$= p_B - 1000 \times 9.81 \times 0.3 - 800 \times 9.81 \times 0.2$$

$$= p_B - 2943 - 1569.6 = p_B - 4512.6$$

Equating the two pressure $p_A - 2943 = p_B - 4512.6$

$$\therefore \quad p_B - p_A = 4512.6 - 2943 = \mathbf{1569.6 \text{ N/m}^2}. \text{ Ans.}$$

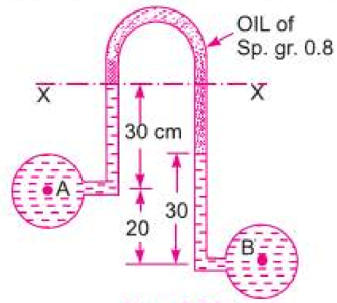


Fig. 2.23

Problem 2.20 *Find out the differential reading 'h' of an inverted U-tube manometer containing oil of specific gravity 0.7 as the manometric fluid when connected across pipes A and B as shown in Fig. 2.24 below, conveying liquids of specific gravities 1.2 and 1.0 and immiscible with manometric fluid. Pipes A and B are located at the same level and assume the pressures at A and B to be equal.*

Solution. Given :

Fig. 2.24 shows the arrangement. Taking X-X as datum line.

Let p_A = Pressure at A

p_B = Pressure at B

Density of liquid in pipe A = Sp. gr. $\times$ 1000

$= 1.2 \times 1000$

$= 1200 \text{ kg/m}^2$

Density of liquid in pipe B $= 1 \times 1000 = 1000 \text{ kg/m}^3$

Density of oil $= 0.7 \times 1000 = 700 \text{ kg/m}^3$

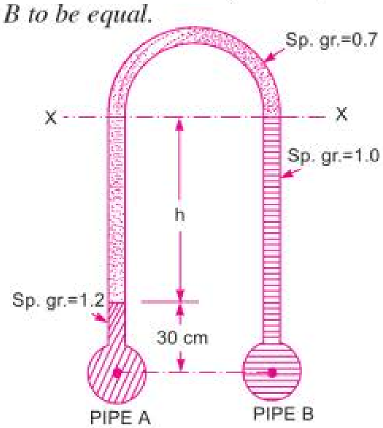


Fig. 2.24

Now pressure below X-X in the left limb

$$= p_A - 1200 \times 9.81 \times 0.3 - 700 \times 9.81 \times h$$

Pressure below X-X in the right limb

$$= p_B - 1000 \times 9.81 \times (h + 0.3)$$

Equating the two pressure, we get

$$p_A - 1200 \times 9.81 \times 0.3 - 700 \times 9.81 \times h = p_B - 1000 \times 9.81\ (h + 0.3)$$

But $\qquad p_A = p_B$ (given)

$$\therefore \quad - 1200 \times 9.81 \times 0.3 - 700 \times 9.81 \times h = - 1000 \times 9.81\ (h + 0.3)$$

Dividing by 1000×9.81

$$- 1.2 \times 0.3 - 0.7h = - (h + 0.3)$$

or $$0.3 \times 1.2 + 0.7h = h + 0.3 \text{ or } 0.36 - 0.3 = h - 0.7h = 0.3h$$

$$\therefore \quad h = \frac{0.36 - 0.30}{0.30} = \frac{0.06}{0.30} \text{ m}$$

$$= \frac{1}{5} \text{ m} = \frac{1}{5} \times 100 = \textbf{20 cm. Ans.}$$

Problem 2.21 *An inverted U-tube manometer is connected to two horizontal pipes A and B through which water is flowing. The vertical distance between the axes of these pipes is 30 cm. When an oil of specific gravity 0.8 is used as a gauge fluid, the vertical heights of water columns in the two limbs of the inverted manometer (when measured from the respective centre lines of the pipes) are found to be same and equal to 35 cm. Determine the difference of pressure between the pipes.*

Solution. Given :

Specific gravity of measuring liquid = 0.8

The arrangement is shown in Fig. 2.24 (*a*).

Let $\quad p_A$ = pressure at A

$\qquad p_B$ = pressure at B.

The points C and D lie on the same horizontal line.

Hence pressure at C should be equal to pressure at D.

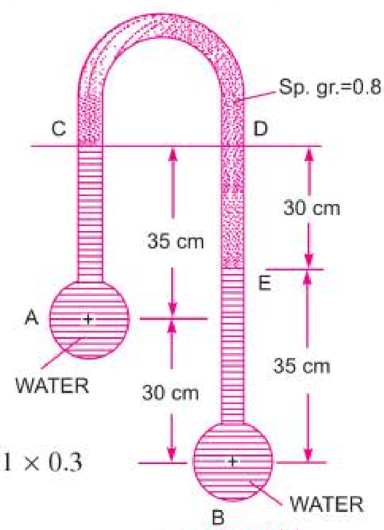


Fig. 2.24 (*a*)

But pressure at C $\quad = p_A - \rho g\, h$

$$= p_A - 1000 \times 9.81 \times (0.35)$$

And pressure at D $\quad = p_B - \rho_1 g h_1 - \rho_2 g h_2$

$$= p_B - 1000 \times 9.81 \times (0.35) - 800 \times 9.81 \times 0.3$$

But pressure at C = pressure at D

$$\therefore \quad p_A - 1000 \times 9.81 \times .35$$

$$= p_B - 1000 \times 9.81 \times 0.35 - 800 \times 9.81 \times 0.3$$

or $$800 \times 9.81 \times 0.3 = p_B - p_A$$

or $$p_B - p_A = 800 \times 9.81 \times 0.3 = \mathbf{2354.4\ \frac{N}{m^2}}. \textbf{ Ans.}$$

▶ 2.8 PRESSURE AT A POINT IN COMPRESSIBLE FLUID

For compressible fluids, density (ρ) changes with the change of pressure and temperature. Such problems are encountered in aeronautics, oceanography and meteorology where we are concerned with atmospheric* air where density, pressure and temperature changes with elevation. Thus for fluids with variable density, equation (2.4) cannot be integrated, unless the relationship between p and ρ is known. For gases the equation of state is

$$\frac{p}{\rho} = RT \qquad \text{...(2.15)}$$

or

$$\rho = \frac{p}{RT}$$

Now equation (2.4) is

$$\frac{dp}{dZ} = w = \rho g = \frac{p}{RT} \times g$$

$$\therefore \quad \frac{dp}{p} = \frac{g}{RT} dZ \qquad \text{...(2.16)}$$

In equation (2.4), Z is measured vertically downward. But if Z is measured vertically up, then $\frac{dp}{dZ} = -\rho g$ and hence equation (2.16) becomes

$$\frac{dp}{p} = \frac{-g}{RT}\, dZ \qquad \text{...(2.17)}$$

2.8.1 Isothermal Process. **Case I.** If temperature T is constant which is true for **isothermal process**, equation (2.17) can be integrated as

$$\int_{p_0}^{p} \frac{dp}{p} = -\int_{Z_0}^{Z} \frac{g}{RT}\, dz = -\frac{g}{RT}\int_{Z_0}^{Z} dz$$

or

$$\log \frac{p}{p_0} = \frac{-g}{RT}\,[Z - Z_0]$$

where p_0 is the pressure where height is Z_0. If the datum line is taken at Z_0, then $Z_0 = 0$ and p_0 becomes the pressure at datum line.

$$\therefore \quad \log \frac{p}{p_0} = \frac{-g}{RT}\, Z$$

$$\frac{p}{p_0} = e^{-gZ/RT}$$

or pressure at a height Z is given by $p = p_0 e^{-gZ/RT}$ ...(2.18)

2.8.2 Adiabatic Process. If temperature T is not constant but the process follows adiabatic law then the relation between pressure and density is given by

$$\frac{p}{\rho^k} = \text{Constant} = C \qquad \text{...}(i)$$

* The standard atmospheric pressure, temperature and density referred to STP at the sea-level are : Pressure = 101.325 kN/m^2 ; Temperature = 15°C and Density = 1.225 kg/m^3.

where k is ratio of specific constant.

$$\therefore \qquad \rho^k = \frac{p}{C}$$

or $$\rho = \left(\frac{p}{C}\right)^{1/k} \qquad \ldots(ii)$$

Then equation (2.4) for Z measured vertically up becomes,

$$\frac{dp}{dZ} = -\rho g = -\left(\frac{p}{C}\right)^{1/k} g$$

or $$\frac{dp}{\left(\frac{p}{C}\right)^{1/k}} = -gdZ \text{ or } C^{1/k}\frac{dp}{\frac{1}{p^k}} = -gdZ$$

Integrating, we get $\displaystyle\int_{p_0}^{p} C^{1/k}\, p^{-1/k}\, dp = \int_{Z_0}^{Z} -gdZ$

or $$C^{1/k}\left[\frac{p^{-1/k+1}}{-\frac{1}{k}+1}\right]_{p_0}^{p} = -g\,[Z]_{Z_0}^{Z}$$

or $$\left[\frac{C^{1/k}\, p^{-1/k+1}}{-\frac{1}{k}+1}\right]_{p_0}^{p} = -g\,[Z]_{Z_0}^{Z} \qquad [C \text{ is a constant, can be taken inside}]$$

But from equation (i), $$C^{1/k} = \left(\frac{p}{\rho^k}\right)^{1/k} = \frac{p^{1/k}}{\rho}$$

Substituting this value of $C^{1/k}$ above, we get

$$\left[\frac{p^{1/k}}{\rho}\frac{p^{-1/k+1}}{-\frac{1}{k}+1}\right]_{p_0}^{p} = -g[Z - Z_0]$$

or $$\left[\frac{p^{\frac{1-1+k}{k}}}{\rho^{\frac{k-1}{k}}}\right]_{p_0}^{p} = -g[Z - Z_0] \text{ or } \left[\frac{k}{k-1}\frac{p}{\rho}\right]_{p_0}^{p} = -g[Z - Z_0]$$

or $$\frac{k}{k-1}\left[\frac{p}{\rho} - \frac{p_0}{\rho_0}\right] = -g[Z - Z_0]$$

If datum line is taken at Z_0, where pressure, temperature and density are p_0, T_0 and ρ_0, then $Z_0 = 0$.

$$\therefore \qquad \frac{k}{k-1}\left[\frac{p}{\rho} - \frac{p_0}{\rho_0}\right] = -gZ \quad \text{or} \quad \frac{p}{\rho} - \frac{p_0}{\rho_0} = -gZ\,\frac{(k-1)}{k}$$

or $$\frac{p}{\rho} = \frac{p_0}{\rho_0} - gZ\,\frac{(k-1)}{k} = \frac{p_0}{\rho_0}\left[1 - \frac{k-1}{k}\,gZ\,\frac{\rho_0}{p_0}\right]$$

or $$\frac{p}{\rho}\times\frac{\rho_0}{p_0}=\left[1+\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right] \qquad \text{...(iii)}$$

But from equation (*i*), $$\frac{p}{\rho^k}=\frac{p_0}{\rho_0^k} \quad \text{or} \quad \left(\frac{\rho_0}{\rho}\right)^k=\frac{p_0}{p} \text{ or } \frac{\rho_0}{\rho}=\left(\frac{p_0}{p}\right)^{1/k}$$

Substituting the value of $\frac{\rho_0}{\rho}$ in equation (*iii*), we get

$$\frac{p}{p_0}\times\left(\frac{p_0}{p}\right)^{1/k}=\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right]$$

or $$\frac{p}{p_0}\times\left(\frac{p}{p_0}\right)^{-1/k}=\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right]$$

or $$\left(\frac{p}{p_0}\right)^{1-\frac{1}{k}}=\left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}=\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right]$$

$\therefore$ $$\frac{p}{p_0}=\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{\rho_0}\right]^{\frac{k}{k-1}}$$

$\therefore$ Pressure at a height Z from ground level is given by

$$p=p_0\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right]^{\frac{k}{k-1}} \qquad \text{...(2.19)}$$

In equation (2.19), p_0 = pressure at ground level, where $Z_0 = 0$

ρ_0 = density of air at ground level

Equation of state is $$\frac{p_0}{\rho_0}=RT_0 \text{ or } \frac{\rho_0}{p_0}=\frac{1}{RT_0}$$

Substituting the values of $\frac{\rho_0}{p_0}$ in equation (2.19), we get

$$p=p_0\left[1-\frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\frac{k}{k-1}} \qquad \text{...(2.20)}$$

2.8.3 Temperature at any Point in Compressible Fluid. For the adiabatic process, the temperature at any height in air is calculated as :

Equation of state at ground level and at a height Z from ground level is written as

$$\frac{p_0}{\rho_0}=RT_0 \text{ and } \frac{p}{\rho}=RT$$

Dividing these equations, we get

$$\left(\frac{p_0}{\rho_0}\right)\div\frac{p}{\rho}=\frac{RT_0}{RT}=\frac{T_0}{T} \quad \text{or} \quad \frac{p_0}{\rho_0}\times\frac{\rho}{p}=\frac{T_0}{T}$$

or $$\frac{T}{T_0}=\frac{\rho_0}{p_0}\times\frac{p}{\rho}=\frac{p}{p_0}\times\frac{\rho_0}{\rho} \qquad \text{...(i)}$$

But $\frac{p}{p_0}$ from equation (2.20) is given by

$$\frac{p}{p_0} = \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\frac{k}{k-1}}$$

Also for adiabatic process $\frac{p}{\rho^k} = \frac{p_0}{\rho_0^k}$ or $\left(\frac{\rho_0}{\rho}\right)^k = \frac{p_0}{p}$

or
$$\frac{\rho_0}{\rho} = \left(\frac{p_0}{p}\right)^{\frac{1}{k}} = \left(\frac{p}{p_0}\right)^{-\frac{1}{k}}$$

$$= \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\left(\frac{k}{k-1}\right)\times\left(-\frac{1}{k}\right)} = \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{-\frac{1}{k-1}}$$

Substituting the values of $\frac{p}{p_0}$ and $\frac{\rho_0}{\rho}$ in equation (*i*), we get

$$\frac{T}{T_0} = \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\frac{k}{k-1}} \times \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{-\frac{1}{k-1}}$$

$$= \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\frac{k}{k-1}-\frac{1}{k-1}} = \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]$$

$\therefore$
$$T = T_0\left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right] \quad ...(2.21)$$

2.8.4 Temperature Lapse-Rate (L). It is defined as the rate at which the temperature changes with elevation. To obtain an expression for the temperature lapse-rate, the temperature given by equation (2.21) is differentiated with respect to Z as

$$\frac{dT}{dZ} = \frac{d}{dZ}\left[T_0\left(1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right)\right]$$

where T_0, K, g and R are constant

$\therefore$
$$\frac{dT}{dZ} = -\frac{k-1}{k} \times \frac{g}{RT_0} \times T_0 = \frac{-g}{R}\left(\frac{k-1}{k}\right)$$

The temperature lapse-rate is denoted by L and hence

$$L = \frac{dT}{dZ} = \frac{-g}{R}\left(\frac{k-1}{k}\right) \quad ...(2.22)$$

In equation (2.22), if (*i*) $k = 1$ which means isothermal process, $\frac{dT}{dZ} = 0$, which means temperature is constant with height.

(*ii*) If $k > 1$, the lapse-rate is negative which means temperature decreases with the increase in height.

In atmosphere, the value of k varies with height and hence the value of temperature lapse-rate also varies. From the sea-level upto an elevation of about 11000 m (or 11 km), the temperature of air decreases uniformly at the rate of 0.0065°C/m. from 11000 m to 32000 m, the temperature remains constant at – 56.5°C and hence in this range lapse-rate is zero. Temperature rises again after 32000 m in air.

Problem 2.22 (SI Units) *If the atmosphere pressure at sea level is 10.143 N/cm², determine the pressure at a height of 2500 m assuming the pressure variation follows (i) Hydrostatic law, and (ii) isothermal law. The density of air is given as 1.208 kg/m³.*

Solution. Given :

Pressure at sea-level, $p_0 = 10.143 \text{ N/cm}^2 = 10.143 \times 10^4 \text{ N/m}^2$

Height, $Z = 2500$ m

Density of air, $\rho_0 = 1.208 \text{ kg/m}^3$

(*i*) **Pressure by hydrostatic law.** For hydrostatic law, ρ is assumed constant and hence p is given by equation $\dfrac{dp}{dZ} = -\rho g$

Integrating, we get
$$\int_{p_0}^{p} dp = \int -\rho g \, dZ = -\rho g \int_{Z_0}^{Z} dZ$$

or
$$p - p_0 = -\rho g \, [Z - Z_0]$$

For datum line at sea-level, $Z_0 = 0$

$\therefore$
$$p - p_0 = -\rho g Z \quad \text{or} \quad p = p_0 - \rho g Z$$
$$= 10.143 \times 10^4 - 1.208 \times 9.81 \times 2500 \quad [\because \rho = \rho_0 = 1.208]$$
$$= 101430 - 29626 = 71804 \frac{\text{N}}{\text{m}^2} \text{ or } \frac{71804}{10^4} \text{ N/cm}^2$$
$$= \mathbf{7.18 \text{ N/cm}^2}. \textbf{ Ans.}$$

(*ii*) **Pressure by Isothermal Law.** Pressure at any height Z by isothermal law is given by equation (2.18) as
$$p = p_0 \, e^{-gZ/RT}$$
$$= 10.143 \times 10^4 \, e^{-\frac{gZ \times \rho_0}{p_0}} \qquad \left[\because \frac{p_0}{\rho_0} = RT \text{ and } \rho_0 g = w_0\right]$$
$$= 10.143 \times 10^4 \, e^{-\frac{Z\rho_0 \times g}{p_0}}$$
$$= 10.143 \times 10^4 \, e^{(-2500 \times 1.208 \times 9.81)/10.143 \times 10^4}$$
$$= 101430 \times e^{-.292} = 101430 \times \frac{1}{1.3391} = 75743 \text{ N/m}^2$$
$$= \frac{75743}{10^4} \text{ N/cm}^2 = \mathbf{7.574 \text{ N/cm}^2}. \textbf{ Ans.}$$

Problem 2.23 *The barometric pressure at sea level is 760 mm of mercury while that on a mountain top is 735 mm. If the density of air is assumed constant at 1.2 kg/m³, what is the elevation of the mountain top?*

Solution. Given :

Pressure* at sea, $p_0 = 760$ mm of Hg
$$= \frac{760}{1000} \times 13.6 \times 1000 \times 9.81 \text{ N/m}^2 = 101396 \text{ N/m}^2$$

* Here pressure head (Z) is given as 760 mm of Hg. Hence $(p/\rho g) = 760$ mm of Hg. The density (ρ) for mercury $= 13.6 \times 1000 \text{ kg/m}^3$. Hence pressure ($p$) will be equal to $\rho \times g \times Z$ *i.e.*, $13.6 \times 1000 \times 9.81 \times \dfrac{760}{1000} \text{ N/m}^2$.

Pressure at mountain, $p = 735$ mm of Hg

$$= \frac{735}{1000} \times 13.6 \times 1000 \times 9.81 = 98060 \text{ N/m}^2$$

Density of air, $\rho = 1.2$ kg/m^3

Let h = Height of the mountain from sea-level.

We know that as the elevation above the sea-level increases, the atmospheric pressure decreases. Here the density of air is given constant, hence the pressure at any height 'h' above the sea-level is given by the equation,

$$p = p_0 - \rho \times g \times h$$

or

$$h = \frac{p_0 - p}{\rho \times g} = \frac{101396 - 98060}{1.2 \times 9.81} = \mathbf{283.33 \text{ m.}} \text{ Ans.}$$

Problem 2.24 *Calculate the pressure at a height of 7500 m above sea level if the atmospheric pressure is 10.143 N/cm^2 and temperature is 15°C at the sea-level, assuming (i) air is incompressible, (ii) pressure variation follows isothermal law, and (iii) pressure variation follows adiabatic law. Take the density of air at the sea-level as equal to 1.285 kg/m^3. Neglect variation of g with altitude.*

Solution. Given :

Height above sea-level, $Z = 7500$ m

Pressure at sea-level, $p_0 = 10.143$ N/cm^2 $= 10.143 \times 10^4$ N/m^2

Temperature at sea-level, $t_0 = 15°$C

$\therefore$ $T_0 = 273 + 15 = 288°$K

Density of air, $\rho = \rho_0 = 1.285$ kg/m^3

(*i*) *Pressure when air is incompressible :*

$$\frac{dp}{dZ} = -\rho g$$

$\therefore$

$$\int_{p_0}^{p} dp = -\int_{Z_0}^{Z} \rho g dz \quad \text{or} \quad p - p_0 = -\rho g[Z - Z_0]$$

or

$$p = p_0 - \rho g Z \qquad \{\because \ Z_0 = \text{datum line} = 0\}$$

$$= 10.143 \times 10^4 - 1.285 \times 9.81 \times 7500$$

$$= 101430 - 94543 = 6887 \text{ N/m}^2 = \mathbf{0.688 \ \frac{N}{cm^2}}. \text{ Ans.}$$

(*ii*) *Pressure variation follows isothermal law :*

Using equation (2.18), we have $p = p_0 e^{-gZ/RT}$

$$= p_0 \, e^{-gZ\rho_0/p_0} \qquad \left\{\because \ \frac{p_0}{\rho_0} = RT \ \therefore \ \frac{\rho_0}{p_0} = \frac{1}{RT}\right\}$$

$$= 101430 \, e^{-gZ\rho_0/p_0} = 101430 \, e^{-7500 \times 1.285 \times 9.81/101430}$$

$$= 101430 \, e^{-.9320} = 101430 \times .39376$$

$$= \mathbf{39939 \text{ N/m}^2 \text{ or } 3.993 \text{ N/cm}^2}. \text{ Ans.}$$

(*iii*) *Pressure variation follows adiabatic law :* $[k = 1.4]$

Using equation (2.19), we have $p = p_0 \left[1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0}\right]^{k/(k-1)}$, where $\rho_0 = 1.285$ kg/m^3

$$\therefore \qquad p = 101430\left[1-\frac{(1.4-1.0)}{1.4}\times 9.81\times\frac{(7500\times 1.285)}{101430}\right]^{\frac{1.4}{1.4-1.0}}$$

$$= 101430\ [1-.2662]^{1.4/.4} = 101430 \times (.7337)^{3.5}$$

$$= \mathbf{34310\ N/m^2\ or\ 3.431\ \frac{N}{cm^2}}.\ \textbf{Ans.}$$

Problem 2.25 *Calculate the pressure and density of air at a height of 4000 m from sea-level where pressure and temperature of the air are 10.143 N/cm² and 15°C respectively. The temperature lapse rate is given as 0.0065°C/m. Take density of air at sea-level equal to 1.285 kg/m³.*

Solution. Given :

Height, $Z = 4000$ m

Pressure at sea-level, $p_0 = 10.143\ \text{N/cm}^2 = 10.143 \times 10^4 = 101430\ \frac{\text{N}}{\text{m}^2}$

Temperature at sea-level, $t_0 = 15°\text{C}$

$\therefore$ $T_0 = 273 + 15 = 288°\text{K}$

Temperature lapse-rate, $L = \frac{dT}{dZ} = -0.0065°\text{K/m}$

$\rho_0 = 1.285\ \text{kg/m}^3$

Using equation (2.22), we have $L = \frac{dT}{dZ} = -\frac{g}{R}\left(\frac{k-1}{k}\right)$

or
$$-0.0065 = -\frac{9.81}{R}\left(\frac{k-1}{k}\right), \text{ where } R = \frac{p_0}{\rho_0 T_0} = \frac{101430}{1.285\times 288} = 274.09$$

$$\therefore \qquad -0.0065 = \frac{-9.81}{274.09}\times\left(\frac{k-1}{k}\right)$$

$$\therefore \qquad \frac{k-1}{k} = \frac{0.0065\times 274.09}{9.81} = 0.1815$$

$$\therefore \qquad k[1-.1815] = 1$$

$$\therefore \qquad k = \frac{1}{1-.1815} = \frac{1}{.8184} = 1.222$$

This means that the value of power index $k = 1.222$.

(*i*) **Pressure** at 4000 m height is given by equation (2.19) as

$$p = p_0\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right]^{\frac{k}{k-1}}, \text{ where } k = 1.222 \text{ and } \rho_0 = 1.285$$

$$\therefore \qquad p = 101430\left[1-\left(\frac{1.222-1.0}{1.222}\right)\times 9.81\times\frac{4000\times 1.285}{101430}\right]^{\frac{1.222}{1.222-1.0}}$$

$$= 101430\ [1-0.09]^{5.50} = 101430 \times .595$$

$$= 60350\ \text{N/m}^2 = \mathbf{6.035\ \frac{N}{cm^2}}.\ \textbf{Ans.}$$

(*ii*) **Density.** Using equation of state, we get

$$\frac{p}{\rho} = RT$$

where p = Pressure at 4000 m height
ρ = Density at 4000 m height
T = Temperature at 4000 m height

Now T is calculated from temperature lapse-rate as

$$t \text{ at } 4000 \text{ m} = t_0 + \frac{dT}{dZ} \times 4000 = 15 - .0065 \times 4000 = 15 - 26 = -11°\text{C}$$

$$\therefore \qquad T = 273 + t = 273 - 11 = 262°\text{K}$$

$\therefore$ Density is given by $\rho = \dfrac{p}{RT} = \dfrac{60350}{274.09 \times 262} \text{ kg/m}^3 = \mathbf{0.84 \text{ kg/m}^3}$. **Ans.**

Problem 2.26 *An aeroplane is flying at an altitude of 5000 m. Calculate the pressure around the aeroplane, given the lapse-rate in the atmosphere as 0.0065°K/m. Neglect variation of g with altitude. Take pressure and temperature at ground level as 10.143 N/cm² and 15°C and density of air as 1.285 kg/cm³.*

Solution. Given :

Height, $Z = 5000$ m

Lapse-rate, $L = \dfrac{dT}{dZ} = -.0065°\text{K/m}$

Pressure at ground level, $p_0 = 10.143 \times 10^4 \text{ N/m}^2$

$t_0 = 15°\text{C}$

$\therefore$ $T_0 = 273 + 15 = 288°\text{K}$

Density, $\rho_0 = 1.285 \text{ kg/m}^3$

$\therefore$ Temperature at 5000 m height $= T_0 + \dfrac{dT}{dZ} \times \text{Height} = 288 - .0065 \times 5000$

$$= 288 - 32.5 = 255.5°\text{K}$$

First find the value of power index k as

From equation (2.22), we have $L = \dfrac{dT}{dZ} = -\dfrac{g}{R}\left(\dfrac{k-1}{k}\right)$

or

$$-.0065 = -\frac{9.81}{R}\left(\frac{k-1}{k}\right)$$

where $R = \dfrac{p_0}{\rho_0 T_0} = \dfrac{101430}{1.285 \times 288} = 274.09$

$$\therefore \qquad -.0065 = -\frac{9.81}{274.09}\left(\frac{k-1}{k}\right)$$

$$\therefore \qquad k = 1.222$$

The pressure is given by equation (2.19) as

$$p = p_0\left[1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0}\right]^{\left(\frac{k}{k-1}\right)}$$

$$= 101430\left[1-\left(\frac{1.222-1.0}{1.222}\right)\times 9.81\times\frac{5000\times 1.285}{101430}\right]^{\frac{1.222}{1.222-1.0}}$$

$$= 101430\left[1-\frac{.222}{1.222}\times 9.81\times\frac{5000\times 1.285}{101430}\right]^{\frac{1.222}{.222}}$$

$= 101430\,[1 - 0.11288]^{5.50} = 101430 \times 0.5175 = 52490$ N/m^3

= 5.249 N/cm^2. Ans.

HIGHLIGHTS

1. The pressure at any point in a fluid is defined as the force per unit area.
2. The Pascal's law states that intensity of pressure for a fluid at rest is equal in all directions.
3. Pressure variation at a point in a fluid at rest is given by the hydrostatic law which states that the rate of increase of pressure in the vertically downward direction is equal to the specific weight of the fluid, $\frac{dp}{dZ} = w = \rho \times g$.
4. The pressure at any point in a incompressible fluid (liquid) is equal to the product of density of fluid at that point, acceleration due to gravity and vertical height from free surface of fluid, $p = \rho \times g \times Z$.
5. Absolute pressure is the pressure in which absolute vacuum pressure is taken as datum while gauge pressure is the pressure in which the atmospheric pressure is taken as datum,
$$p_{\text{abs.}} = p_{\text{atm}} + p_{\text{gauge.}}$$
6. Manometer is a device used for measuring pressure at a point in a fluid.
7. Manometers are classified as (*a*) Simple manometers and (*b*) Differential manometers.
8. Simple manometers are used for measuring pressure at a point while differential manometers are used for measuring the difference of pressures between the two points in a pipe, or two different pipes.
9. A single column manometer (or micrometer) is used for measuring small pressures, where accuracy is required.
10. The pressure at a point in static compressible fluid is obtained by combining two equations, *i.e.*, equation of state for a gas and equation given by hydrostatic law.
11. The pressure at a height Z in a static compressible fluid (gas) under going isothermal compression $\left(\frac{p}{\rho} = \text{const.}\right)$
$$p = p_0\, e^{-gZ/RT}$$
where p_0 = Absolute pressure at sea-level or at ground level
Z = Height from sea or ground level
R = Gas constant
T = Absolute temperature.
12. The pressure and temperature at a height Z in a static compressible fluid (gas) undergoing adiabatic compression (p/ρ^k = const.)
$$p = p_0\left[1-\frac{k-1}{k}gZ\frac{\rho_0}{p_0}\right]^{\frac{k}{k-1}} = p_0\left[1-\frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\frac{k}{k-1}}$$

and temperature, $$T = T_0 \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]$$

where p_0, T_0 are pressure and temperature at sea-level $k = 1.4$ for air.

13. The rate at which the temperature changes with elevation is known as Temperature Lapse-Rate. It is given by

$$L = \frac{-g}{R}\left(\frac{k-1}{k}\right)$$

if (*i*) $k = 1$, temperature is zero.

(*ii*) $k > 1$, temperature decreases with the increase of height.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define pressure. Obtain an expression for the pressure intensity at a point in a fluid.
2. State and prove the Pascal's law.
3. What do you understand by Hydrostatic Law ?
4. Differentiate between : (*i*) Absolute and gauge pressure, (*ii*) Simple manometer and differential manometer, and (*iii*) Piezometer and pressure gauges.
5. What do you mean by vacuum pressure ?
6. What is a manometer ? How are they classified ?
7. What do you mean by single column manometers ? How are they used for the measurement of pressure ?
8. What is the difference between U-tube differential manometers and inverted U-tube differential manometers ? Where are they used ?
9. Distinguish between manometers and mechanical gauges. What are the different types of mechanical pressure gauges ?
10. Derive an expression for the pressure at a height Z from sea-level for a static air when the compression of the air is assumed isothermal. The pressure and temperature at sea-levels are p_0 and T_0 respectively.
11. Prove that the pressure and temperature for an adiabatic process at a height Z from sea-level for a static air are :

$$p_0 = p_0 \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]^{\frac{k}{k-1}} \text{ and } T = T_0 \left[1 - \frac{k-1}{k}\frac{gZ}{RT_0}\right]$$

where p_0 and T_0 are the pressure and temperature at sea-level.

12. What do you understand by the term, 'Temperature Lapse-Rate'? Obtain an expression for the temperature Lapse-Rate.
13. What is hydrostatic pressure distribution? Give one example where pressure distribution is non-hydrostatic.
14. Explain briefly the working principle of Bourdon Pressure Gauge with a neat sketch.

(J.N.T.U., Hyderabad, S 2002)

(B) NUMERICAL PROBLEMS

1. A hydraulic press has a ram of 30 cm diameter and a plunger of 5 cm diameter. Find the weight lifted by the hydraulic press when the force applied at the plunger is 400 N. [**Ans.** 14.4 kN]
2. A hydraulic press has a ram of 20 cm diameter and a plunger of 4 cm diameter. It is used for lifting a weight of 20 kN. Find the force required at the plunger. [**Ans.** 800 N]

3. Calculate the pressure due to a column of 0.4 m of (*a*) water, (*b*) an oil of sp. gr. 0.9, and (*c*) mercury of sp. gr. 13.6. Take density of water, $\rho = 1000 \frac{kg}{m^3}$. [**Ans.** (*a*) 0.3924 N/cm^2, (*b*) 0.353 N/cm^2, (*c*) 5.33 N/cm^2]
4. The pressure intensity at a point in a fluid is given 4.9 N/cm^2. Find the corresponding height of fluid when it is : (*a*) water, and (*b*) an oil of sp. gr. 0.8. [**Ans.** (*a*) 5 m of water, (*b*) 6.25 m of oil]
5. An oil of sp. gr. 0.8 is contained in a vessel. At a point the height of oil is 20 m. Find the corresponding height of water at that point. [**Ans.** 16 m]
6. An open tank contains water upto a depth of 1.5 m and above it an oil of sp. gr. 0.8 for a depth of 2 m. Find the pressure intensity : (*i*) at the interface of the two liquids, and (*ii*) at the bottom of the tank. [**Ans.** (*i*) 1.57 N/cm^2, (*ii*) 3.04 N/cm^2]
7. The diameters of a small piston and a large piston of a hydraulic jack are 2 cm and 10 cm respectively. A force of 60 N is applied on the small piston. Find the load lifted by the large piston, when : (*a*) the pistons are at the same level, and (*b*) small piston is 20 cm above the large piston. The density of the liquid in the jack is given as $1000 \frac{kg}{m^2}$. [**Ans.** (*a*) 1500 N, (*b*) 1520.5 N]
8. Determine the gauge and absolute pressure at a point which is 2.0 m below the free surface of water. Take atmospheric pressure as 10.1043 N/cm^2. [**Ans.** 1.962 N/cm^2 (gauge), 12.066 N/cm^2 (abs.)]
9. A simple manometer is used to measure the pressure of oil (sp. gr. = 0.8) flowing in a pipe line. Its right limb is open to the atmosphere and left limb is connected to the pipe. The centre of the pipe is 9 cm below the level of mercury (sp. gr. 13.6) in the right limb. If the difference of mercury level in the two limbs is 15 cm, determine the absolute pressure of the oil in the pipe in N/cm^2. [**Ans.** 12.058 N/cm^2]
10. A simple manometer (U-tube) containing mercury is connected to a pipe in which an oil of sp. gr. 0.8 is flowing. The pressure in the pipe is vacuum. The other end of the manometer is open to the atmosphere. Find the vacuum, pressure in pipe, if the difference of mercury level in the two limbs is 20 cm and height of oil in the left limb from the centre of the pipe is 15 cm below. [**Ans.** – 27.86 N/cm^2]
11. A single column vertical manometer (*i.e.,* micrometer) is connected to a pipe containing oil of sp. gr. 0.9. The area of the reservoir is 80 times the area of the manometer tube. The reservoir contains mercury of sp. gr. 13.6. The level of mercury in the reservoir is at a height of 30 cm below the centre of the pipe and difference of mercury levels in the reservoir and right limb is 50 cm. Find the pressure in the pipe. [**Ans.** 6.474 N/cm^2]
12. A pipe contains an oil of sp. gr. 0.8. A differential manometer connected at the two points A and B of the pipe shows a difference in mercury level as 20 cm. Find the difference of pressure at the two points. [**Ans.** 25113.6 N/m^2]
13. A U-tube differential manometer connects two pressure pipes *A* and *B*. Pipe *A* contains carbon tetrachloride having a specific gravity 1.594 under a pressure of 11.772 N/cm^2 and pipe *B* contains oil of sp. gr. 0.8 under a pressure of 11.772 N/cm^2. The pipe *A* lies 2.5 m above pipe *B*. Find the difference of pressure measured by mercury as fluid filling U-tube. [**Ans.** 31.36 cm of mercury]
14. A differential manometer is connected at the two points *A* and *B* as shown in Fig. 2.25. At *B* air pressure is 7.848 N/cm^2 (abs.), find the absolute pressure at *A*. [**Ans.** 6.91 N/cm^2]

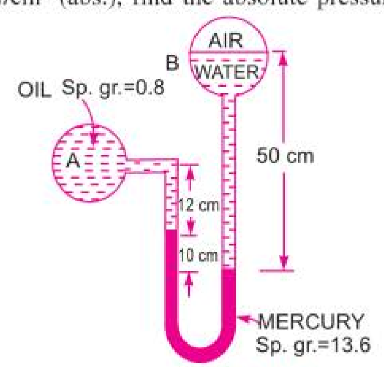


Fig. 2.25

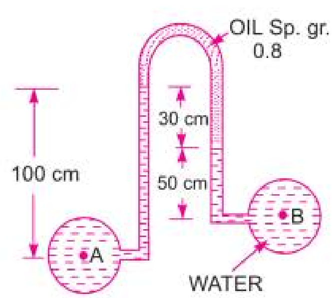


Fig. 2.26

15. An inverted differential manometer containing an oil of sp. gr. 0.9 is connected to find the difference of pressures at two points of a pipe containing water. If the manometer reading is 40 cm, find the difference of pressures. [**Ans.** 392.4 N/m^2]

16. In above Fig. 2.26 shows an inverted differential manometer connected to two pipes A and B containing water. The fluid in manometer is oil of sp. gr. 0.8. For the manometer readings shown in the figure, find the difference of pressure head between A and B. [**Ans.** 0.26 m of water]

17. If the atmospheric pressure at sea-level is 10.143 N/cm^2, determine the pressure at a height of 2000 m assuming that the pressure variation follows : (*i*) Hydrostatic law, and (*ii*) Isothermal law. The density of air is given as 1.208 kg/m^3. [**Ans.** (*i*) 7.77 N/cm^2, (*ii*) 8.03 N/cm^3]

18. Calculate the pressure at a height of 8000 m above sea-level if the atmospheric pressure is 101.3 kN/m^2 and temperature is 15°C at the sea-level assuming (*i*) air is incompressible, (*ii*) pressure variation follows adiabatic law, and (*iii*) pressure variation follows isothermal law. Take the density of air at the sea-level as equal to 1.285 kg/m^3. Neglect variation of g with altitude.
[**Ans.** (*i*) 607.5 N/m^2, (*ii*) 31.5 kN/m^2 (*iii*) 37.45 kN/m^2]

19. Calculate the pressure and density of air at a height of 3000 m above sea-level where pressure and temperature of the air are 10.143 N/cm^2 and 15°C respectively. The temperature lapse-rate is given as 0.0065° K/m. Take density of air at sea-level equal to 1.285 kg/m^3. [**Ans.** 6.896 N/cm^2, 0.937 kg/m^3]

20. An aeroplane is flying at an altitude of 4000 m. Calculate the pressure around the aeroplane, given the lapse-rate in the atmosphere as 0.0065°K/m. Neglect variation of g with altitude. Take pressure and temperature at ground level as 10.143 N/cm^2 and 15°C respectively. The density of air at ground level is given as 1.285 kg/m^3. [**Ans.** 6.038 N/cm^2]

21. The atmospheric pressure at the sea-level is 101.3 kN/m^2 and the temperature is 15°C. Calculate the pressure 8000 m above sea-level, assuming (*i*) air is incompressible, (*ii*) isothermal variation of pressure and density, and (*iii*) adiabatic variation of pressure and density. Assume density of air at sea-level as 1.285 kg/m^3. Neglect variation of 'g' with altitude.
[**Ans.** (*i*) 501.3 N/m^2, (*ii*) 37.45 kN/m^2, (*iii*) 31.5 kN/m^2]

22. An oil of sp. gr. is 0.8 under a pressure of 137.2 kN/m^2
(*i*) What is the pressure head expressed in metre of water ?
(*ii*) What is the pressure head expressed in metre of oil ? [**Ans.** (*i*) 14 m, (*ii*) 17.5 m]

23. The atmospheric pressure at the sea-level is 101.3 kN/m^2 and temperature is 15°C. Calculate the pressure 8000 m above sea-level, assuming : (*i*) isothermal variation of pressure and density, and (*ii*) adiabatic variation of pressure and density. Assume density of air at sea-level as 1.285 kg/m^3. Neglect variation of 'g' with altitude.
Derive the formula that you may use. [**Ans.** (*i*) 37.45 kN/m^2, (*ii*) 31.5 kN/m^2]

24. What are the gauge pressure and absolute pressure at a point 4 m below the free surface of a liquid of specific gravity 1.53, if atmospheric pressure is equivalent to 750 mm of mercury.
[**Ans.** 60037 N/m^2 and 160099 N/m^2]

25. Find the gauge pressure and absolute pressure in N/m^2 at a point 4 m below the free surface of a liquid of sp. gr. 1.2, if the atmospheric pressure is equivalent to 750 mm of mercury.
[**Ans.** 47088 N/m^2 ; 147150 N/m^2]

26. A tank contains a liquid of specific gravity 0.8. Find the absolute pressure and gauge pressure at a point, which is 2 m below the free surface of the liquid. The atmospheric pressure head is equivalent to 760 mm of mercury. [**Ans.** 117092 N/m^2 ; 15696 N/m^2]

3

CHAPTER

HYDROSTATIC FORCES ON SURFACES

▶ 3.1 INTRODUCTION

This chapter deals with the fluids (*i.e.,* liquids and gases) at rest. This means that there will be no relative motion between adjacent or neighbouring fluid layers. The velocity gradient, which is equal to the change of velocity between two adjacent fluid layers divided by the distance between the layers, will be zero or $\frac{du}{dy} = 0$. The shear stress which is equal to $\mu \frac{\partial u}{\partial y}$ will also be zero. Then the forces acting on the fluid particles will be :

1. due to pressure of fluid normal to the surface,
2. due to gravity (or self-weight of fluid particles).

▶ 3.2 TOTAL PRESSURE AND CENTRE OF PRESSURE

Total pressure is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface.

Centre of pressure is defined as the point of application of the total pressure on the surface. There are four cases of submerged surfaces on which the total pressure force and centre of pressure is to be determined. The submerged surfaces may be :

1. Vertical plane surface,
2. Horizontal plane surface,
3. Inclined plane surface, and
4. Curved surface.

▶ 3.3 VERTICAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane vertical surface of arbitrary shape immersed in a liquid as shown in Fig. 3.1.

Let A = Total area of the surface

$\bar{h}$ = Distance of C.G. of the area from free surface of liquid

G = Centre of gravity of plane surface

P = Centre of pressure

h^* = Distance of centre of pressure from free surface of liquid.

(*a*) **Total Pressure (F).** The total pressure on the surface may be determined by dividing the entire surface into a number of small parallel strips. The force on small strip is then calculated and the total pressure force on the whole area is calculated by integrating the force on small strip.

Consider a strip of thickness dh and width b at a depth of h from free surface of liquid as shown in Fig. 3.1

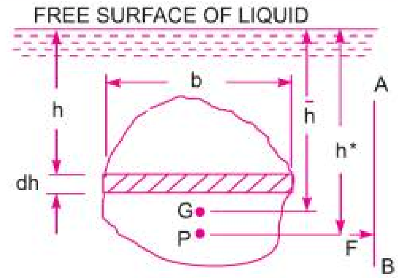


Fig. 3.1

Pressure intensity on the strip, $p = \rho g h$

(See equation 2.5)

Area of the strip, $dA = b \times dh$

Total pressure force on strip, $dF = p \times \text{Area}$

$= \rho g h \times b \times dh$

$\therefore$ Total pressure force on the whole surface,

$$F = \int dF = \int \rho g h \times b \times dh = \rho g \int b \times h \times dh$$

But $\int b \times h \times dh = \int h \times dA$

= Moment of surface area about the free surface of liquid

= Area of surface × Distance of C.G. from free surface

$= A \times \bar{h}$

$$\therefore \quad F = \rho g A \bar{h} \qquad \text{...(3.1)}$$

For water the value of $\rho = 1000 \text{ kg/m}^3$ and $g = 9.81 \text{ m/s}^2$. The force will be in Newton.

(*b*) **Centre of Pressure (h*).** Centre of pressure is calculated by using the "Principle of Moments", which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis.

The resultant force F is acting at P, at a distance h^* from free surface of the liquid as shown in Fig. 3.1. Hence moment of the force F about free surface of the liquid $= F \times h^*$ ...(3.2)

Moment of force dF, acting on a strip about free surface of liquid

$= dF \times h \qquad \{\because \; dF = \rho g h \times b \times dh\}$

$= \rho g h \times b \times dh \times h$

Sum of moments of all such forces about free surface of liquid

$$= \int \rho g h \times b \times dh \times h = \rho g \int b \times h \times h dh$$

$$= \rho g \int b h^2 \, dh = \rho g \int h^2 \, dA \qquad (\because \; b dh = dA)$$

But $\int h^2 dA = \int b h^2 dh$

= Moment of Inertia of the surface about free surface of liquid

$= I_0$

$\therefore$ Sum of moments about free surface

$$= \rho g I_0 \qquad \text{...(3.3)}$$

Equating (3.2) and (3.3), we get

$$F \times h^* = \rho g I_0$$

But $$F = \rho g A\bar{h}$$

$\therefore$ $$\rho g A\bar{h} \times h^* = \rho g I_0$$

or $$h^* = \frac{\rho g I_0}{\rho g A\bar{h}} = \frac{I_0}{A\bar{h}} \qquad ...(3.4)$$

By the theorem of parallel axis, we have

$$I_0 = I_G + A \times \overline{h^2}$$

where I_G = Moment of Inertia of area about an axis passing through the C.G. of the area and parallel to the free surface of liquid.

Substituting I_0 in equation (3.4), we get

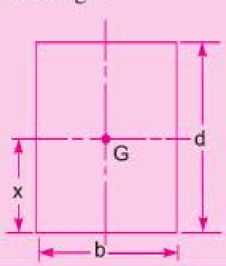
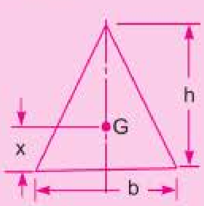
$$h^* = \frac{I_G + A\overline{h^2}}{A\bar{h}} = \frac{I_G}{A\bar{h}} + \bar{h} \qquad ...(3.5)$$

In equation (3.5), $\bar{h}$ is the distance of C.G. of the area of the vertical surface from free surface of the liquid. Hence from equation (3.5), it is clear that :

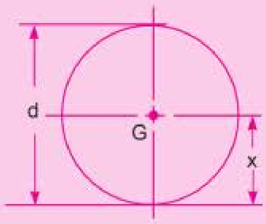
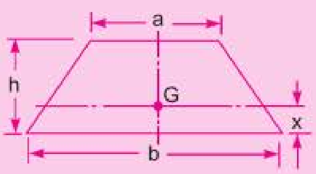
(*i*) Centre of pressure (*i.e.*, h^*) lies below the centre of gravity of the vertical surface.

(*ii*) The distance of centre of pressure from free surface of liquid is independent of the density of the liquid.

Table 3.1 The moments of inertia and other geometric properties of some important plane surfaces

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*	*Moment of inertia about base (I_0)*
1. Rectangle 	$x = \dfrac{d}{2}$	bd	$\dfrac{bd^3}{12}$	$\dfrac{bd^3}{3}$
2. Triangle 	$x = \dfrac{h}{3}$	$\dfrac{bh}{2}$	$\dfrac{bh^3}{36}$	$\dfrac{bh^3}{12}$

Contd...

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*	*Moment of inertia about base (I_0)*
3. Circle 	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	—
4. Trapezium 	$x = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$	$\frac{(a+b)}{2} \times h$	$\left(\frac{a^2+4ab+b^2}{36(a+b)}\right) \times h^3$	—

Problem 3.1 *A rectangular plane surface is 2 m wide and 3 m deep. It lies in vertical plane in water. Determine the total pressure and position of centre of pressure on the plane surface when its upper edge is horizontal and (a) coincides with water surface, (b) 2.5 m below the free water surface.*

Solution. Given :

Width of plane surface, $b = 2$ m

Depth of plane surface, $d = 3$ m

(*a*) **Upper edge coincides with water surface (Fig. 3.2).** Total pressure is given by equation (3.1) as

$$F = \rho g A \bar{h}$$

where $\rho = 1000$ kg/m^3, $g = 9.81$ m/s^2

$$A = 3 \times 2 = 6 \text{ m}^2,\ \bar{h} = \frac{1}{2}(3) = 1.5 \text{ m}.$$

$$\therefore \quad F = 1000 \times 9.81 \times 6 \times 1.5$$

$$= \mathbf{88290\ N.}\ \textbf{Ans.}$$

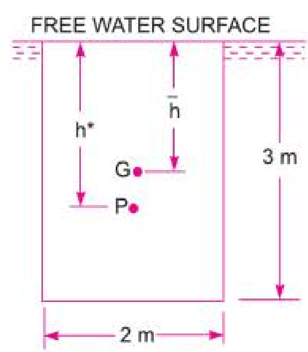


Fig. 3.2

Depth of centre of pressure is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where I_G = M.O.I. about C.G. of the area of surface

$$= \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

$$\therefore \qquad h^* = \frac{4.5}{6 \times 1.5} + 1.5 = 0.5 + 1.5 = \mathbf{2.0\ m.}\ \textbf{Ans.}$$

(*b*) **Upper edge is 2.5 m below water surface (Fig. 3.3).** Total pressure (F) is given by (3.1)

$$F = \rho g A \bar{h}$$

where $\bar{h}$ = Distance of C.G. from free surface of water

$$= 2.5 + \frac{3}{2} = 4.0 \text{ m}$$

$$\therefore \qquad F = 1000 \times 9.81 \times 6 \times 4.0$$
$$= \mathbf{235440\ N.}\ \textbf{Ans.}$$

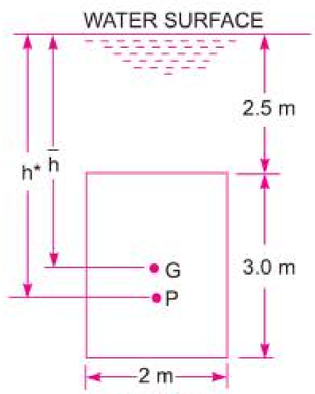


Fig. 3.3

Centre of pressure is given by $h^* = \dfrac{I_G}{A\bar{h}} + \bar{h}$

where $I_G = 4.5$, $A = 6.0$, $\bar{h} = 4.0$

$$\therefore \qquad h^* = \frac{4.5}{6.0 \times 4.0} + 4.0$$
$$= 0.1875 + 4.0 = 4.1875 = \mathbf{4.1875\ m.}\ \textbf{Ans.}$$

Problem 3.2 *Determine the total pressure on a circular plate of diameter 1.5 m which is placed vertically in water in such a way that the centre of the plate is 3 m below the free surface of water. Find the position of centre of pressure also.*

Solution. Given : Dia. of plate, $d = 1.5$ m

$$\therefore \text{ Area,} \qquad A = \frac{\pi}{4}(1.5)^2 = 1.767 \text{ m}^2$$

$$\bar{h} = 3.0 \text{ m}$$

Total pressure is given by equation (3.1),

$$F = \rho g A \bar{h}$$
$$= 1000 \times 9.81 \times 1.767 \times 3.0 \text{ N}$$
$$= \mathbf{52002.81\ N.}\ \textbf{Ans.}$$

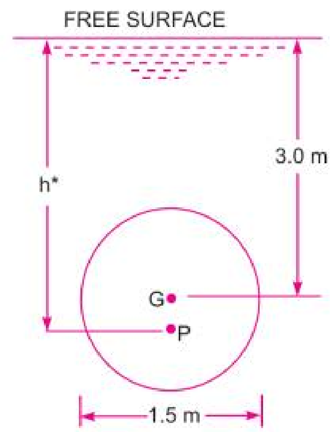


Fig. 3.4

Position of centre of pressure (h^*) is given by equation (3.5),

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where $I_G = \dfrac{\pi d^4}{64} = \dfrac{\pi \times 1.5^4}{64} = 0.2485 \text{ m}^4$

$$\therefore \qquad h^* = \frac{0.2485}{1.767 \times 3.0} + 3.0 = 0.0468 + 3.0$$
$$= \mathbf{3.0468\ m.}\ \textbf{Ans.}$$

Problem 3.3 *A rectangular sluice gate is situated on the vertical wall of a lock. The vertical side of the sluice is 'd' metres in length and depth of centroid of the area is 'p' m below the water surface. Prove that the depth of pressure is equal to* $\left(p+\dfrac{d^2}{12p}\right)$.

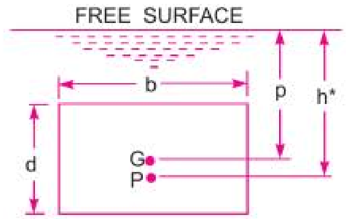


Fig. 3.5

Solution. Given :

Depth of vertical gate $= d$ m

Let the width of gate $= b$ m

$\therefore$ Area, $A = b \times d\ \text{m}^2$

Depth of C.G. from free surface

$$\bar{h} = p \text{ m.}$$

Let h^* is the depth of centre of pressure from free surface, which is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}, \text{ where } I_G = \frac{bd^3}{12}$$

$$\therefore \quad h^* = \left(\frac{bd^3}{12}\Big/ b\times d\times p\right) + p = \mathbf{\frac{d^2}{12p} + p} \quad \text{or} \quad \mathbf{p + \frac{d^2}{12p}}. \textbf{ Ans.}$$

Problem 3.4 *A circular opening, 3 m diameter, in a vertical side of a tank is closed by a disc of 3 m diameter which can rotate about a horizontal diameter. Calculate :*

(i) the force on the disc, and

(ii) the torque required to maintain the disc in equilibrium in the vertical position when the head of water above the horizontal diameter is 4 m.

Solution. Given :

Dia. of opening, $d = 3$ m

$\therefore$ Area, $A = \dfrac{\pi}{4} \times 3^2 = 7.0685\ \text{m}^2$

Depth of C.G., $\bar{h} = 4$ m

(*i*) Force on the disc is given by equation (3.1) as

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 7.0685 \times 4.0$$

$$= \mathbf{277368\ N = 277.368\ kN.} \textbf{ Ans.}$$

(*ii*) To find the torque required to maintain the disc in equilibrium, first calculate the point of application of force acting on the disc, *i.e.*, centre of pressure of the force F. The depth of centre of pressure (h^*) is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h} = \frac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2 \times 4.0} + 4.0 \qquad \left\{\because I_G = \frac{\pi}{64} d^4\right\}$$

$$= \frac{d^2}{16 \times 4.0} + 4.0 = \frac{3^2}{16 \times 4.0} + 4.0 = 0.14 + 4.0 = 4.14 \text{ m}$$

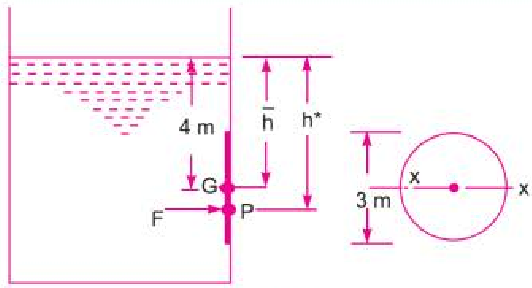


Fig. 3.6

The force F is acting at a distance of 4.14 m from free surface. Moment of this force about horizontal diameter X-X

$$= F \times \left(h^* - \bar{h}\right) = 277368\ (4.14 - 4.0) = \mathbf{38831\ Nm.}\ \textbf{Ans.}$$

Hence a torque of 38831 Nm must be applied on the disc in the clockwise direction.

Problem 3.5 *A pipe line which is 4 m in diameter contains a gate valve. The pressure at the centre of the pipe is 19.6 N/cm². If the pipe is filled with oil of sp. gr. 0.87, find the force exerted by the oil upon the gate and position of centre of pressure.*

Solution. Given :

Dia. of pipe, $d = 4$ m

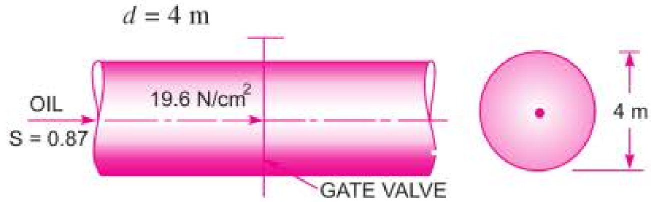


Fig. 3.7

$\therefore$ Area, $A = \dfrac{\pi}{4} \times 4^2 = 4\pi\ \text{m}^2$

Sp. gr. of oil, $S = 0.87$

$\therefore$ Density of oil, $\rho_0 = 0.87 \times 1000 = 870\ \text{kg/m}^3$

$\therefore$ Weight density of oil, $w_0 = \rho_0 \times g = 870 \times 9.81\ \text{N/m}^3$

Pressure at the centre of pipe, $p = 19.6\ \text{N/cm}^2 = 19.6 \times 10^4\ \text{N/m}^2$

$\therefore$ Pressure head at the centre $= \dfrac{p}{w_0} = \dfrac{19.6 \times 10^4}{870 \times 9.81} = 22.988$ m

$\therefore$ The height of equivalent free oil surface from the centre of pipe = 22.988 m.

The depth of C.G. of the gate valve from free oil surface $\bar{h} = 22.988$ m.

(*i*) Now the force exerted by the oil on the gate is given by

$$F = \rho g A \bar{h}$$

where ρ = density of oil = 870 kg/m³

$$F = 870 \times 9.81 \times 4\pi \times 22.988 = \mathbf{2465500\ N = 2.465\ MN.}\ \textbf{Ans.}$$

(*ii*) Position of centre of pressure (h^*) is given by (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h} = \frac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2 \times \bar{h}} + \bar{h} = \frac{d^2}{16\,\bar{h}} + \bar{h} = \frac{4^2}{16 \times 22.988} + 22.988$$

$$= 0.043 + 22.988 = \mathbf{23.031\ m.}\ \textbf{Ans.}$$

Or centre of pressure is below the centre of the pipe by a distance of 0.043 m. **Ans.**

Problem 3.6 *Determine the total pressure and centre of pressure on an isosceles triangular plate of base 4 m and altitude 4 m when it is immersed vertically in an oil of sp. gr. 0.9. The base of the plate coincides with the free surface of oil.*

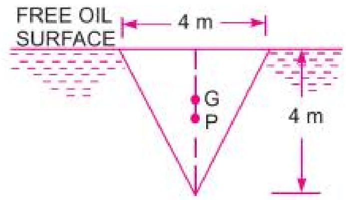


Fig. 3.8

Solution. Given :

Base of plate, $b = 4$ m

Height of plate, $h = 4$ m

$\therefore$ Area, $A = \frac{b \times h}{2} = \frac{4 \times 4}{2} = 8.0\ m^2$

Sp. gr. of oil, $S = 0.9$

$\therefore$ Density of oil, $\rho = 900\ kg/m^3$.

The distance of C.G. from free surface of oil,

$$\bar{h} = \frac{1}{3} \times h = \frac{1}{3} \times 4 = 1.33\ \text{m}.$$

Total pressure (F) is given by $F = \rho g A \bar{h}$

$$= 900 \times 9.81 \times 8.0 \times 1.33\ \text{N} = \mathbf{9597.6.\ N.}\ \textbf{Ans.}$$

Centre of pressure (h^*) from free surface of oil is given by

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where I_G = M.O.I. of triangular section about its C.G.

$$= \frac{bh^3}{36} = \frac{4 \times 4^3}{36} = 7.11\ m^4$$

$$\therefore \quad h^* = \frac{7.11}{8.0 \times 1.33} + 1.33 = 0.6667 + 1.33 = \mathbf{1.99\ m.}\ \textbf{Ans.}$$

Problem 3.7 *A vertical sluice gate is used to cover an opening in a dam. The opening is 2 m wide and 1.2 m high. On the upstream of the gate, the liquid of sp. gr. 1.45, lies upto a height of 1.5 m above the top of the gate, whereas on the downstream side the water is available upto a height touching the top of the gate. Find the resultant force acting on the gate and position of centre of pressure. Find also the force acting horizontally at the top of the gate which is capable of opening it. Assume that the gate is hinged at the bottom.*

Solution. Given :

Width of gate, $b = 2$ m

Depth of gate, $d = 1.2$ m

$\therefore$ Area, $A = b \times d = 2 \times 1.2 = 2.4\ m^2$

Sp. gr. of liquid $= 1.45$

$\therefore$ Density of liquid, $\rho_1 = 1.45 \times 1000 = 1450 \text{ kg/m}^3$

Let F_1 = Force exerted by the fluid of sp. gr. 1.45 on gate

F_2 = Force exerted by water on the gate.

The force F_1 is given by $F_1 = \rho_1 g \times A \times \bar{h}_1$

where $\rho_1 = 1.45 \times 1000 = 1450 \text{ kg/m}^2$

$\bar{h}_1$ = Depth of C.G. of gate from free surface of liquid

$$= 1.5 + \frac{1.2}{2} = 2.1 \text{ m.}$$

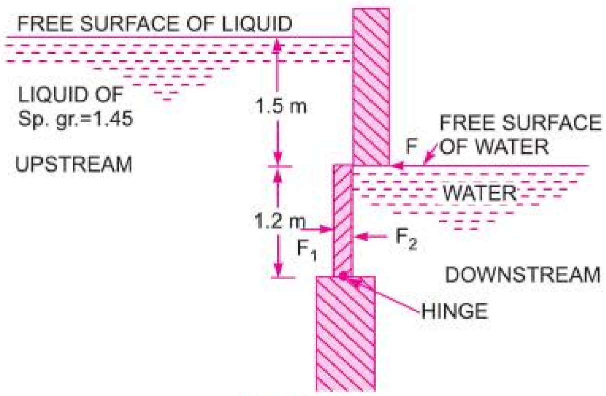


Fig. 3.9

$\therefore$ $F_1 = 1450 \times 9.81 \times 2.4 \times 2.1 = 71691 \text{ N}$

Similarly, $F_2 = \rho_2 g.A\bar{h}_2$

where $\rho_2 = 1{,}000 \text{ kg/m}^3$

$\bar{h}_2$ = Depth of C.G. of gate from free surface of water

$$= \frac{1}{2} \times 1.2 = 0.6 \text{ m}$$

$\therefore$ $F_2 = 1000 \times 9.81 \times 2.4 \times 0.6 = 14126 \text{ N}$

(*i*) **Resultant force on the gate** $= F_1 - F_2 = 71691 - 14126 =$ **57565 N. Ans.**

(*ii*) **Position of centre of pressure of resultant force.** The force F_1 will be acting at a depth of h_1^* from free surface of liquid, given by the relation

$$h_1^* = \frac{I_G}{A\bar{h}_1} + \bar{h}_1$$

where $I_G = \frac{bd^3}{12} = \frac{2 \times 1.2^3}{12} = 0.288 \text{ m}^4$

$$\therefore \quad h_1^* = \frac{.288}{2.4 \times 2.1} + 2.1 = 0.0571 + 2.1 = 2.1571 \text{ m}$$

$\therefore$ Distance of F_1 from hinge

$$= (1.5 + 1.2) - h_1^* = 2.7 - 2.1571 = 0.5429 \text{ m}$$

The force F_2 will be acting at a depth of h_2^* from free surface of water and is given by

$$h_2^* = \frac{I_G}{A\bar{h}_2} + \bar{h}_2$$

where $I_G = 0.288 \text{ m}^4$, $\bar{h}_2 = 0.6$ m, $A = 2.4 \text{ m}^2$,

$$h_2^* = \frac{0.288}{2.4 \times 0.6} + 0.6 = 0.2 + 0.6 = 0.8 \text{ m}$$

Distance of F_2 from hinge $= 1.2 - 0.8 = 0.4$ m

The resultant force 57565 N will be acting at a distance given by

$$= \frac{71691 \times .5429 - 14126 \times 0.4}{57565}$$

$$= \frac{38921 - 5650.4}{57565} \text{ m above hinge}$$

= **0.578 m above the hinge. Ans.**

(*iii*) **Force at the top of gate which is capable of opening the gate.** Let F is the force required on the top of the gate to open it as shown in Fig. 3.9. Taking the moments of F, F_1 and F_2 about the hinge, we get

$$F \times 1.2 + F_2 \times 0.4 = F_1 \times .5429$$

or

$$F = \frac{F_1 \times .5429 - F_2 \times 0.4}{1.2}$$

$$= \frac{71691 \times .5429 - 14126 \times 0.4}{1.2} = \frac{38921 - 5650.4}{1.2}$$

= **27725.5 N. Ans.**

Problem 3.8 *A caisson for closing the entrance to a dry dock is of trapezoidal form 16 m wide at the top and 10 m wide at the bottom and 6 m deep. Find the total pressure and centre of pressure on the caisson if the water on the outside is just level with the top and dock is empty.*

Solution. Given :

Width at top $= 16$ m

Width at bottom $= 10$ m

Depth, $d = 6$ m

Area of trapezoidal $ABCD$,

$$A = \frac{(BC + AD)}{2} \times d$$

$$= \frac{(10 + 16)}{2} \times 6 = 78 \text{ m}^2$$

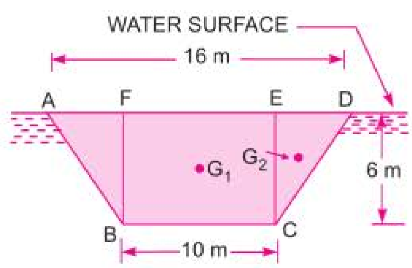


Fig. 3.10

Depth of C.G. of trapezoidal area $ABCD$ from free surface of water,

$$\bar{h} = \frac{10 \times 6 \times 3 + \frac{(16 - 10)}{2} \times 6 \times \frac{1}{3} \times 6}{78}$$

$$= \frac{180 + 36}{78} = 2.769 \text{ m from water surface.}$$

(*i*) **Total Pressure (F).** Total pressure, F is given by

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 78 \times 2.769 \text{ N}$$
$$= 2118783 \text{ N} = \mathbf{2.118783 \text{ MN.}} \text{ Ans.}$$

(*ii*) **Centre of Pressure (h*).** Centre of pressure is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where I_G = M.O.I. of trapezoidal *ABCD* about its C.G.

Let I_{G_1} = M.O.I. of rectangle *FBCE* about its C.G.

I_{G_2} = M.O.I. of two Δs *ABF* and *ECD* about its C.G.

Then $$I_{G_1} = \frac{bd^3}{12} = \frac{10 \times 6^3}{12} = 180 \text{ m}^4$$

I_{G_1} is the M.O.I. of the rectangle about the axis passing through G_1.

$\therefore$ M.O.I. of the rectangle about the axis passing through the C.G. of the trapezoidal I_{G_1} + Area of rectangle $\times x_1^2$

where x_1 is distance between the C.G. of rectangle and C.G. of trapezoidal

$$= (3.0 - 2.769) = 0.231 \text{ m}$$

$\therefore$ M.O.I. of *FBCE* passing through C.G. of trapezoidal

$$= 180 + 10 \times 6 \times (0.231)^2 = 180 + 3.20 = 183.20 \text{ m}^4$$

Now I_{G_2} = M.O.I. of ΔABD in Fig. 3.11 about $G_2 = \dfrac{bd^3}{36}$

$$= \frac{(16-10) \times 6^3}{36} = 36 \text{ m}^4$$

The distance between the C.G. of triangle and C.G. of trapezoidal

$$= (2.769 - 2.0) = 0.769$$

$\therefore$ M.O.I. of the two Δs about an axis passing through C.G. of trapezoidal

$$= I_{G_2} + \text{Area of triangles} \times (.769)^2$$
$$= 36.0 + \frac{6 \times 6}{2} \times (.769)^2$$
$$= 36.0 + 10.64 = 46.64$$

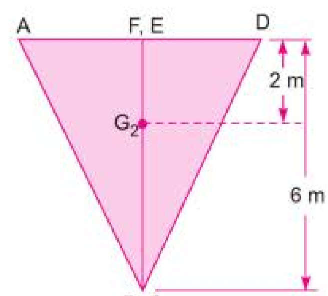


Fig. 3.11

$\therefore$ I_G = M.O.I. of trapezoidal about its C.G.
= M.O.I. of rectangle about the C.G. of trapezoidal
+ M.O.I. of triangles about the C.G. of the trapezoidal
$= 183.20 + 46.64 = 229.84 \text{ m}^4$

$\therefore$ $$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where $A = 78$, $\bar{h} = 2.769$

$$h^* = \frac{229.84}{78 \times 2.769} + 2.769 = 1.064 + 2.769 = \mathbf{3.833 \text{ m.}} \text{ Ans.}$$

Alternate Method

The distance of the C.G. of the trapezoidal channel from surface *AD* is given by (refer to Table 3.1 on page 71)

$$x = \frac{(2a+b)}{(a+b)} \times \frac{h}{3}$$

$$= \frac{(2 \times 10 + 16)}{(10+16)} \times \frac{6}{3} \qquad (\because \quad a = 10, b = 16 \text{ and } h = 6)$$

$$= \frac{36}{26} \times 2 = 2.769 \text{ m}$$

This is also equal to the distance of the C.G. of the trapezoidal from free surface of water.

$$\bar{h} = 2.769 \text{ m}$$

$\therefore$ Total pressure, $F = \rho g A \bar{h}$ $(\because \quad A = 78)$

$= 1000 \times 9.81 \times 78 \times 2.769$ N = **2118783 N. Ans.**

Centre of Pressure, $(h^*) = \dfrac{I_G}{A\bar{h}} + \bar{h}$

Now I_G from Table 3.1 is given by,

$$I_G = \frac{(a^2 + 4ab + b^2)}{36(a+b)} \times h^3 = \frac{(10^2 + 4 \times 10 \times 16 + 16^2)}{36(10+16)} \times 6^3$$

$$= \frac{(100 + 640 + 256)}{36 \times 26} \times 216 = 229.846 \text{ m}^4$$

$\therefore$ $h^* = \dfrac{229.846}{78 \times 2.769} + 2.769$ $(\because \quad A = 78 \text{ m}^2)$

= **3.833 m. Ans.**

Problem 3.9 *A trapezoidal channel 2 m wide at the bottom and 1 m deep has side slopes 1 : 1. Determine :*

(i) the total pressure, and

(ii) the centre of pressure on the vertical gate closing the channel when it is full of water.

Solution. Given :

Width at bottom = 2 m

Depth, $d = 1$ m

Side slopes = 1 : 1

$\therefore$ Top width, $AD = 2 + 1 + 1 = 4$ m

Area of rectangle *FBEC*, $A_1 = 2 \times 1 = 2 \text{ m}^2$

Area of two triangles *ABF* and *ECD*, $A_2 = \dfrac{(4-2)}{2} \times 1 = 1 \text{ m}^2$

$\therefore$ Area of trapezoidal *ABCD*, $A = A_1 + A_2 = 2 + 1 = 3 \text{ m}^2$

Depth of C.G. of rectangle *FBEC* from water surface,

$$\bar{h}_1 = \frac{1}{2} = 0.5 \text{ m}$$

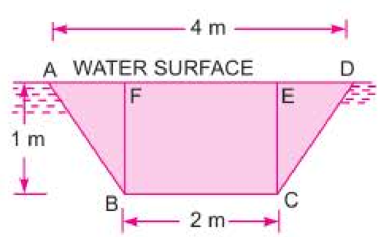


Fig. 3.12

Depth of C.G. of two triangles *ABF* and *ECD* from water surface,

$$\bar{h}_2 = \frac{1}{3} \times 1 = \frac{1}{3} \text{ m}$$

$\therefore$ Depth of C.G. of trapezoidal *ABCD* from free surface of water

$$\bar{h} = \frac{A_1 \times \bar{h}_1 + A_2 \times \bar{h}_2}{(A_1 + A_2)} = \frac{2 \times 0.5 + 1 \times 0.33333}{(2+1)} = .44444$$

(*i*) **Total Pressure (F).** Total pressure F is given by

$$F = \rho g A\bar{h}$$

$$= 1000 \times 9.81 \times 3.0 \times 0.44444 = \mathbf{13079.9\ N.\ Ans.}$$

(*ii*) **Centre of Pressure (h*).** M.O.I. of rectangle *FBCE* about its C.G.,

$$I_{G_1} = \frac{bd^3}{12} = \frac{2 \times 1^3}{12} = \frac{1}{6} \text{ m}^4$$

M.O.I. of *FBCE* about an axis passing through the C.G. of trapezoidal

or $$I_{G_1}{}^* = I_{G_1} + A_1 \times [\text{Distance between C.G. of rectangle and C.G. of trapezoidal}]^2$$

$$= \frac{1}{6} + 2 \times \left[\bar{h}_1 - \bar{h}\right]^2$$

$$= \frac{1}{6} + 2 \times [0.5 - .4444]^2 = .1666 + .006182 = 0.1727$$

M.O.I. of the two triangles *ABF* and *ECD* about their C.G.,

$$I_{G_2} = \frac{bd^3}{36} = \frac{(1+1) \times 1^3}{36} = \frac{2}{36} = \frac{1}{18} \text{ m}^4.$$

M.O.I. of the two triangles about the C.G. of trapezoidal,

$$I_{G_2}{}^* = I_{G_1} + A_2 \times [\text{Distance between C.G. of triangles and C.G. of trapezoidal}]^2$$

$$= \frac{1}{18} + 1 \times \left[\bar{h} - \bar{h}_2\right]^2 = \frac{1}{18} + 1 \times \left[.4444 - \frac{1}{3}\right]^2$$

$$= \frac{1}{18} + (.1111)^2 = 0.0555 + (.1111)^2$$

$$= .0555 + 0.01234 = 0.06789 \text{ m}^4$$

$\therefore$ M.O.I. of the trapezoidal about its C.G.

$$I_G = I_{G_1}{}^* + I_{G_2}{}^* = .1727 + .06789 = 0.24059 \text{ m}^4$$

Then centre of pressure (h^*) on the vertical trapezoidal,

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h} = \frac{0.24059}{3 \times .4444} + .4444 = 0.18046 + .4444 = 0.6248$$

$$\simeq \mathbf{0.625\ m.\ Ans.}$$

Alternate Method

The distance of the C.G. of the trapezoidal channel from surface AD is given by (refer to Table 3.1 on page 71).

$$x = \frac{(2a+b)}{(a+b)} \times \frac{h}{3} = \frac{(2 \times 2 + 4)}{(2+4)} \times \frac{1}{3} \qquad (\because \quad a = 2, b = 4 \text{ and } h = 1)$$

$$= 0.444 \text{ m}$$

$$\therefore \qquad \bar{h} = x = 0.444 \text{ m}$$

$\therefore$ Total pressure, $\quad F = \rho g A \bar{h} = 1000 \times 9.81 \times 3.0 \times .444 \qquad (\because \quad A = 3.0)$

$= \mathbf{13079\ N.}$ **Ans.**

Centre of pressure, $\quad h^* = \dfrac{I_G}{A\bar{h}} + \bar{h}$

where I_G from Table 3.1 is given by

$$I_G = \frac{(a^2 + 4ab + b^2)}{36(a+b)} \times h^3 = \frac{(2^2 + 4 \times 2 \times 4 + 4^2)}{36(2+4)} \times 1^3 = \frac{52}{36 \times 6} = 0.2407 \text{ m}^4$$

$$\therefore \qquad h^* = \frac{0.2407}{3.0 \times .444} + .444 = \mathbf{0.625\ m.}\ \textbf{Ans.}$$

Problem 3.10 *A square aperture in the vertical side of a tank has one diagonal vertical and is completely covered by a plane plate hinged along one of the upper sides of the aperture. The diagonals of the aperture are 2 m long and the tank contains a liquid of specific gravity 1.15. The centre of aperture is 1.5 m below the free surface. Calculate the thrust exerted on the plate by the liquid and position of its centre of pressure.*

Solution. Given : Diagonals of aperture, $AC = BD = 2$ m

$\therefore$ Area of square aperture, A = Area of ΔACB + Area of ΔACD

$$= \frac{AC \times BO}{2} + \frac{AC \times OD}{2} = \frac{2 \times 1}{2} + \frac{2 \times 1}{2} = 1 + 1 = 2.0 \text{ m}^2$$

Sp. gr. of liquid $= 1.15$

$\therefore$ Density of liquid, $\quad \rho = 1.15 \times 1000 = 1150 \text{ kg/m}^3$

Depth of centre of aperture from free surface,

$$\bar{h} = 1.5 \text{ m.}$$

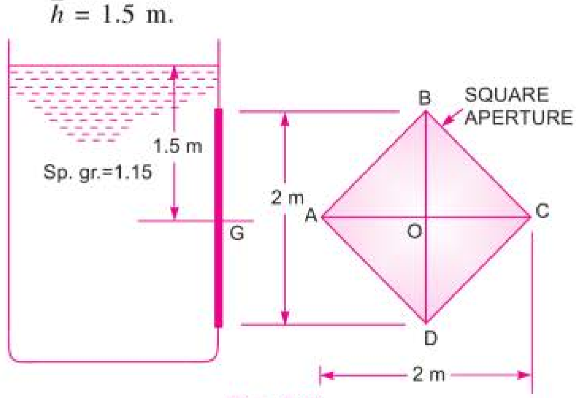


Fig. 3.13

(*i*) The thrust on the plate is given by

$$F = \rho g A\bar{h} = 1150 \times 9.81 \times 2 \times 1.5 = \mathbf{33844.5.}\ \textbf{Ans.}$$

(*ii*) Centre of pressure (h^*) is given by

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where I_G = M.O.I. of *ABCD* about diagonal *AC*

= M.O.I. of triangle *ABC* about *AC* + M.O.I. of triangle *ACD* about *AC*

$$= \frac{AC \times OB^3}{12} + \frac{AC \times OD^3}{12} \qquad \left(\because \text{ M.O.I. of a triangle about its base} = \frac{bh^3}{12}\right)$$

$$= \frac{2 \times 1^3}{12} + \frac{2 \times 1^3}{12} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}\text{ m}^4$$

$$\therefore \qquad h^* = \frac{\frac{1}{3}}{2 \times 1.5} + 1.5 = \frac{1}{3 \times 2 \times 1.5} + 1.5 = \mathbf{1.611\ m.}\ \textbf{Ans.}$$

Problem 3.11 *A tank contains water upto a height of 0.5 m above the base. An immiscible liquid of sp. gr. 0.8 is filled on the top of water upto 1 m height. Calculate :*

(i) total pressure on one side of the tank,

(ii) the position of centre of pressure for one side of the tank, which is 2 m wide.

Solution. Given :

Depth of water = 0.5 m

Depth of liquid = 1 m

Sp. gr. of liquid = 0.8

Density of liquid, $\rho_1 = 0.8 \times 1000 = 800$ kg/m^3

Density of water, $\rho_2 = 1000$ kg/m^3

Width of tank = 2 m

(*i*) **Total pressure on one side** is calculated by drawing pressure diagram, which is shown in Fig. 3.14.

Intensity of pressure on top, $p_A = 0$

Intensity of pressure on *D* (or *DE*), $p_D = \rho_1 g . h_1$

$= 800 \times 9.81 \times 1.0 = 7848$ N/m^2

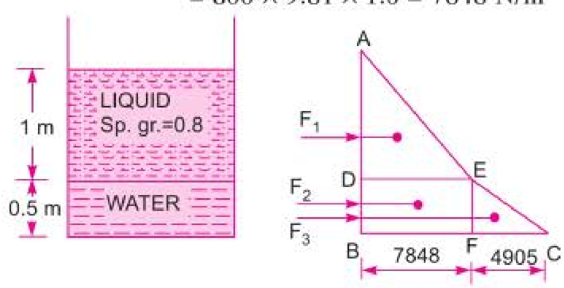


Fig. 3.14

Intensity of pressure on base (or *BC*), $p_B = \rho_1 g h_1 + \rho_2 g \times 0.5$

$= 7848 + 1000 \times 9.81 \times 0.5 = 7848 + 4905 = 12753$ N/m^2

Now force F_1 = Area of ΔADE × Width of tank

$$= \frac{1}{2} \times AD \times DE \times 2.0 = \frac{1}{2} \times 1 \times 7848 \times 2.0 = 7848\text{ N}$$

Force F_2 = Area of rectangle $DBFE$ × Width of tank

$= 0.5 \times 7848 \times 2 = 7848$ N

F_3 = Area of ΔEFC × Width of tank

$$= \frac{1}{2} \times EF \times FC \times 2.0 = \frac{1}{2} \times 0.5 \times 4905 \times 2.0 = 2452.5 \text{ N}$$

∴ Total pressure, $F = F_1 + F_2 + F_3$

$= 7848 + 7848 + 2452.5 =$ **18148.5 N. Ans.**

(*ii*) **Centre of Pressure (h*).** Taking the moments of all force about A, we get

$$F \times h^* = F_1 \times \frac{2}{3} AD + F_2 (AD + \frac{1}{2} BD) + F_3[AD + \frac{2}{3} BD]$$

$$18148.5 \times h^* = 7848 \times \frac{2}{3} \times 1 + 7848 \left(1.0 + \frac{0.5}{2}\right) + 2452.5 \left(1.0 + \frac{2}{3} \times .5\right)$$

$$= 5232 + 9810 + 3270 = 18312$$

∴ $$h^* = \frac{18312}{18148.5} = \textbf{1.009 m from top. Ans.}$$

Problem 3.12 *A cubical tank has sides of 1.5 m. It contains water for the lower 0.6 m depth. The upper remaining part is filled with oil of specific gravity 0.9. Calculate for one vertical side of the tank:*

(*a*) *total pressure, and*

(*b*) *position of centre of pressure.*

Solution. Given :

Cubical tank of sides 1.5 m means the dimensions of the tank are 1.5 m × 1.5 m × 1.5 m.

Depth of water = 0.6 m

Depth of liquid = 1.5 – 0.6 = 0.9 m

Sp. gr. of liquid = 0.9

Density of liquid, $\rho_1 = 0.9 \times 1000 = 900$ kg/m^3

Density of water, $\rho_2 = 1000$ kg/m^3

(*a*) **Total pressure** on one vertical side is calculated by drawing pressure diagram, which is shown in Fig. 3.15.

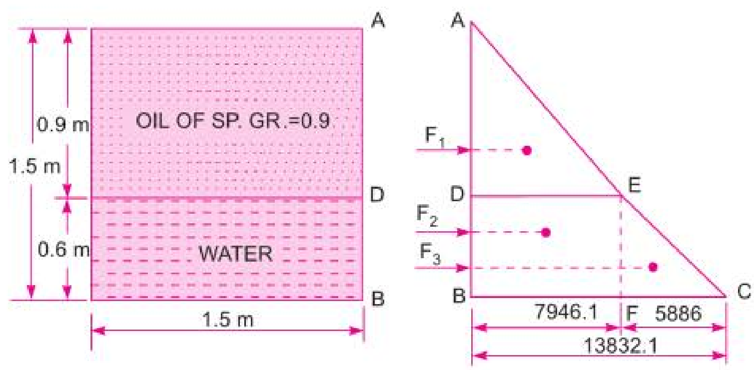


Fig. 3.15

Intensity of pressure at A, $p_A = 0$

Intensity of pressure at $D, p_D = \rho_1 g \times h = 900 \times 9.81 \times 0.9 = 7946.1$ N/m^2

Intensity of pressure at B, $p_B = \rho_1 g h_1 + \rho_2 g h_2 = 900 \times 9.81 \times 0.9 + 1000 \times 9.81 \times 0.6$

$= 7946.1 + 5886 = 13832.1$ N/m^2

Hence in pressure diagram :

$$DE = 7946.1 \text{ N/m}^2,\ BC = 13832.1 \text{ N/m}^2,\ FC = 5886 \text{ N/m}^2$$

The pressure diagram is split into triangle ADE, rectangle $BDEF$ and triangle EFC. The total pressure force consists of the following components :

(*i*) Force F_1 = Area of triangle ADE × Width of tank

$$= \left(\tfrac{1}{2} \times AD \times DE\right) \times 1.5 \qquad (\because \text{ Width} = 1.5 \text{ m})$$

$$= \left(\tfrac{1}{2} \times 0.9 \times 7946.1\right) \times 1.5 \text{ N} = 5363.6 \text{ N}$$

This force will be acting at the C.G. of the triangle ADE, *i.e.*, at a distance of $\frac{2}{3} \times 0.9 = 0.6$ m below A

(*ii*) Force F_2 = Area of rectangle $BDEF$ × Width of tank

$= (BD \times DE) \times 1.5 = (0.6 \times 7946.1) \times 1.5 = 7151.5$

This force will be acting at the C.G. of the rectangle $BDEF$ *i.e.*, at a distance of $0.9 + \frac{0.6}{2} = 1.2$ m below A.

(*iii*) Force F_3 = Area of triangle EFC × Width of tank

$$= \left(\tfrac{1}{2} \times EF \times FC\right) \times 1.5 = \left(\tfrac{1}{2} \times 0.6 \times 5886\right) \times 1.5 = 2648.7 \text{ N}$$

This force will be acting at the C.G. of the triangle EFC, *i.e.*, at a distance of $0.9 + \frac{2}{3} \times 0.6 = 1.30$ m below A.

∴ Total pressure force on one vertical face of the tank,

$$F = F_1 + F_2 + F_3$$

$$= 5363.6 + 7151.5 + 2648.7 = \mathbf{15163.8 \text{ N.}}$$ **Ans.**

(*b*) **Position of centre of pressure**

Let the total force F is acting at a depth of h^* from the free surface of liquid, *i.e.*, from A.

Taking the moments of all forces about A, we get

$$F \times h^* = F_1 \times 0.6 + F_2 \times 1.2 + F_3 \times 1.3$$

or

$$h^* = \frac{F_1 \times 0.6 + F_2 \times 1.2 + F_3 \times 1.3}{F}$$

$$= \frac{5363.6 \times 0.6 + 7151.5 \times 1.2 + 2648.7 \times 1.3}{15163.8}$$

= 1.005 m from A. Ans.

▶ 3.4 HORIZONTAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane horizontal surface immersed in a static fluid. As every point of the surface is at the same depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface and equal to, $p = \rho g h$, where h is depth of surface.

Let A = Total area of surface

Then total force, F, on the surface

$$= p \times \text{Area} = \rho g \times h \times A = \rho g A \bar{h}$$

where $\bar{h}$ = Depth of C.G. from free surface of liquid = h

also h^* = Depth of centre of pressure from free surface = h.

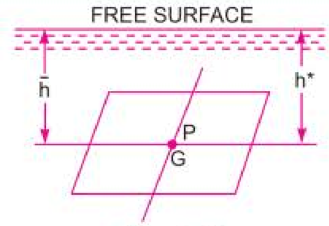


Fig. 3.16

Problem 3.13 *Fig. 3.17 shows a tank full of water. Find :*

(*i*) *Total pressure on the bottom of tank.*

(*ii*) *Weight of water in the tank.*

(*iii*) *Hydrostatic paradox between the results of (i) and (ii). Width of tank is 2 m.*

Solution. Given :

Depth of water on bottom of tank

$$h_1 = 3 + 0.6 = 3.6 \text{ m}$$

Width of tank $= 2$ m

Length of tank at bottom $= 4$ m

$\therefore$ Area at the bottom, $A = 4 \times 2 = 8 \text{ m}^2$

(*i*) Total pressure F, on the bottom is

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 8 \times 3.6$$

$$= \mathbf{282528\ N.}\ \textbf{Ans.}$$

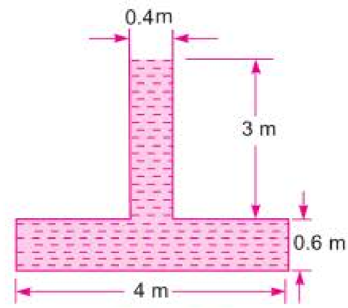


Fig. 3.17

(*ii*) Weight of water in tank = $\rho g \times$ Volume of tank

$$= 1000 \times 9.81 \times [3 \times 0.4 \times 2 + 4 \times .6 \times 2]$$

$$= 1000 \times 9.81\ [2.4 + 4.8] = \mathbf{70632\ N.}\ \textbf{Ans.}$$

(*iii*) From the results of (*i*) and (*ii*), it is observed that the total weight of water in the tank is much less than the total pressure at the bottom of the tank. This is known as Hydrostatic paradox.

▶ 3.5 INCLINED PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane surface of arbitrary shape immersed in a liquid in such a way that the plane of the surface makes an angle θ with the free surface of the liquid as shown in Fig. 3.18.

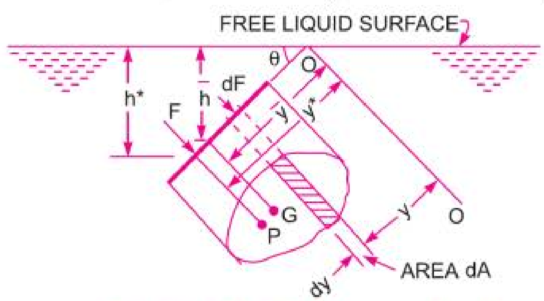


Fig. 3.18 *Inclined immersed surface.*

Let A = Total area of inclined surface

$\bar{h}$ = Depth of C.G. of inclined area from free surface

h^* = Distance of centre of pressure from free surface of liquid

θ = Angle made by the plane of the surface with free liquid surface.

Let the plane of the surface, if produced meet the free liquid surface at O. Then O-O is the axis perpendicular to the plane of the surface.

Let $\bar{y}$ = distance of the C.G. of the inclined surface from O-O

y^* = distance of the centre of pressure from O-O.

Consider a small strip of area dA at a depth 'h' from free surface and at a distance y from the axis O-O as shown in Fig. 3.18.

Pressure intensity on the strip, $p = \rho gh$

$\therefore$ Pressure force, dF, on the strip, $dF = p \times$ Area of strip $= \rho gh \times dA$

Total pressure force on the whole area, $F = \int dF = \int \rho ghdA$

But from Fig. 3.18,
$$\frac{h}{y} = \frac{\bar{h}}{\bar{y}} = \frac{h^*}{y^*} = \sin\theta$$

$$\therefore \quad h = y \sin\theta$$

$$\therefore \quad F = \int \rho g \times y \times \sin\theta \times dA = \rho g \sin\theta \int ydA$$

But
$$\int ydA = A\bar{y}$$

where $\bar{y}$ = Distance of C.G. from axis O-O

$$\therefore \quad F = \rho g \sin\theta \; \bar{y} \times A$$

$$= \rho g A\bar{h} \qquad (\because \; \bar{h} = \bar{y}\sin\theta) \quad \text{...(3.6)}$$

Centre of Pressure (h*)

Pressure force on the strip, $dF = \rho ghdA$

$= \rho gy \sin\theta \; dA$ $\qquad [h = y \sin\theta]$

Moment of the force, dF, about axis O-O

$= dF \times y = \rho gy \sin\theta \; dA \times y = \rho g \sin\theta \; y^2 dA$

Sum of moments of all such forces about O-O

$$= \int \rho g \sin\theta \; y^2 \; dA = \rho g \sin\theta \int y^2 \; dA$$

But $\int y^2 \; dA$ = M.O.I. of the surface about O-O $= I_0$

$\therefore$ Sum of moments of all forces about O-$O = \rho g \sin\theta \; I_0$ ...(3.7)

Moment of the total force, F, about O-O is also given by

$= F \times y^*$ ...(3.8)

where y^* = Distance of centre of pressure from O-O.

Equating the two values given by equations (3.7) and (3.8)

$$F \times y^* = \rho g \sin\theta \; I_0$$

or
$$y^* = \frac{\rho g \sin\theta \; I_0}{F} \quad \text{...(3.9)}$$

Now
$$y^* = \frac{h^*}{\sin\theta}, \; F = \rho g A\bar{h}$$

and I_0 by the theorem of parallel axis $= I_G + A\bar{y}^2$.

Substituting these values in equation (3.9), we get

$$\frac{h^*}{\sin\theta} = \frac{\rho g \sin\theta}{\rho g A\bar{h}}\left[I_G + A\bar{y}^2\right]$$

$$\therefore \qquad h^* = \frac{\sin^2\theta}{A\bar{h}}\,[I_G + A\bar{y}^2]$$

But
$$\frac{\bar{h}}{\bar{y}} = \sin\theta \quad \text{or} \quad \bar{y} = \frac{\bar{h}}{\sin\theta}$$

$$\therefore \qquad h^* = \frac{\sin^2\theta}{A\bar{h}}\left[I_G + A\times\frac{\bar{h}^2}{\sin^2\theta}\right]$$

or
$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h} \qquad \text{...(3.10)}$$

If $\theta = 90°$, equation (3.10) becomes same as equation (3.5) which is applicable to vertically plane submerged surfaces.

In equation (3.10), I_G = M.O.I. of inclined surfaces about an axis passing through G and parallel to O-O.

Problem 3.14 (*a*) *A rectangular plane surface 2 m wide and 3 m deep lies in water in such a way that its plane makes an angle of 30° with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge is 1.5 m below the free water surface.*

Solution. Given :

Width of plane surface, $b = 2$ m

Depth, $d = 3$ m

Angle, $\theta = 30°$

Distance of upper edge from free water surface = 1.5 m

(*i*) **Total pressure** force is given by equation (3.6) as

$$F = \rho g A\bar{h}$$

where $\rho = 1000$ kg/m^3

$A = b \times d = 3 \times 2 = 6$ m^2

$\therefore \quad \bar{h}$ = Depth of C.G. from free water surface

$= 1.5 + 1.5 \sin 30°$

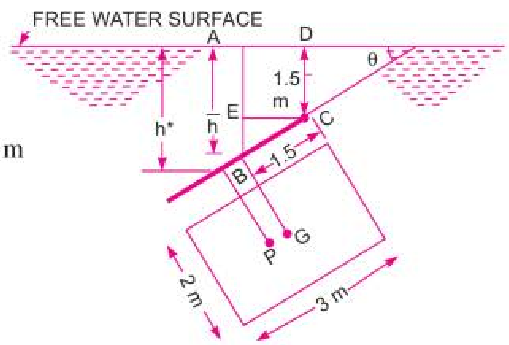


Fig. 3.19

$\{\because \quad \bar{h} = AE + EB = 1.5 + BC \sin 30° = 1.5 + 1.5 \sin 30°\}$

$= 1.5 + 1.5 \times \frac{1}{2} = 2.25$ m

$$\therefore \qquad F = 1000 \times 9.81 \times 6 \times 2.25 = \mathbf{132435\ N.}\ \textbf{Ans.}$$

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}, \quad \text{where } I_G = \frac{bd^3}{12} = \frac{2\times 3^3}{12} = 4.5 \text{ m}^4$$

$$\therefore \qquad h^* = \frac{4.5\times\sin^2 30°}{6\times 2.25} + 2.25 = \frac{4.5\times\frac{1}{4}}{6\times 2.25} + 2.25$$

$$= 0.0833 + 2.25 = \mathbf{2.3333\ m.}\ \textbf{Ans.}$$

Problem 3.14 (*b*) *A rectangular plane surface 3 m wide and 4 m deep lies in water in such a way that its plane makes an angle of 30° with the free surface of water. Determine the total pressure force and position of centre of pressure, when the upper edge is 2 m below the free surface.*

Solution. Given :

$$b = 3 \text{ m}, \; d = 4 \text{ m}, \; \theta = 30°$$

Distance of upper edge from free surface of water = 2 m

(*i*) **Total pressure force** is given by equation (3.6) as

$$F = \rho g A \bar{h}$$

where $\rho = 1000 \text{ kg/m}^3$,

$A = b \times d = 3 \times 4 = 12 \text{ m}^2$

and $\bar{h}$ = Depth of C.G. of plate from free water surface

$= 2 + \text{BE} = 2 + \text{BC} \sin\theta$

$= 2 + 2 \sin 30° = 2 + 2 \times \dfrac{1}{2} = 3 \text{ m}$

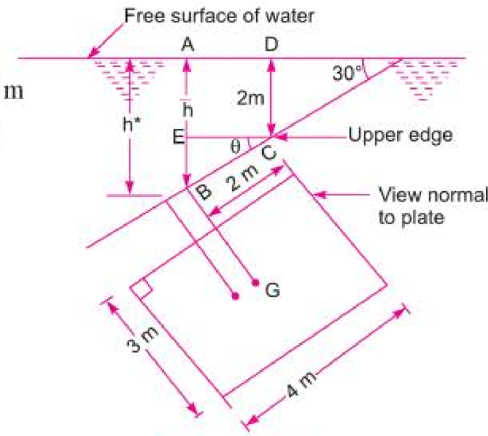


Fig. 3.19 (*a*)

∴ $F = 1000 \times 9.81 \times 12 \times 3 = 353167 \text{ N} =$ **353.167 kN. Ans.**

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have $h^* = \dfrac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}$

where $I_G = \dfrac{bd^3}{12} = \dfrac{3 \times 4^3}{12} = 16 \text{ m}^4$

∴ $h^* = \dfrac{16 \times \sin^2 30°}{12 \times 3} + 3 = \dfrac{16 \times \frac{1}{4}}{36} + 3 =$ **3.111 m. Ans.**

Problem 3.15 (*a*) *A circular plate 3.0 m diameter is immersed in water in such a way that its greatest and least depth below the free surface are 4 m and 1.5 m respectively. Determine the total pressure on one face of the plate and position of the centre of pressure.*

Solution. Given :

Dia. of plate, $d = 3.0 \text{ m}$

∴ Area, $A = \dfrac{\pi}{4} d^2 = \dfrac{\pi}{4} (3.0)^2 = 7.0685 \text{ m}^2$

Distance $DC = 1.5 \text{ m}, \; BE = 4 \text{ m}$

Distance of C.G. from free surface

$= \bar{h} = CD + GC \sin\theta = 1.5 + 1.5 \sin\theta$

But $\sin\theta = \dfrac{AB}{BC} = \dfrac{BE - AE}{BC} = \dfrac{4.0 - DC}{3.0} = \dfrac{4.0 - 1.5}{3.0}$

$= \dfrac{2.5}{3.0} = 0.8333$

∴ $\bar{h} = 1.5 + 1.5 \times .8333 = 1.5 + 1.249 = 2.749 \text{ m}$

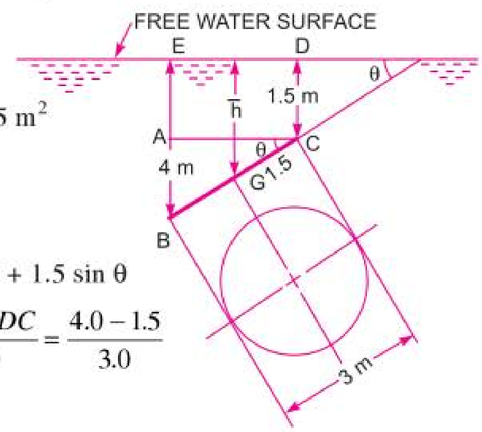


Fig. 3.20

(*i*) **Total pressure (F)**

$$F = \rho g A \bar{h}$$
$$= 1000 \times 9.81 \times 7.0685 \times 2.749 = \mathbf{190621\ N.}\ \textbf{Ans.}$$

(*ii*) **Centre of pressure (h*)**

Using equation (3.10), we have $h^* = \dfrac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}$

where $I_G = \dfrac{\pi}{64} d^4 = \dfrac{\pi}{64} (3)^4 = 3.976\ m^4$

$$h^* = \frac{3.976 \times (.8333) \times .8333}{7.0685 \times 2.749} + 2.749 = 0.1420 + 2.749$$
$$= \mathbf{2.891\ m.}\ \textbf{Ans.}$$

Problem 3.15 (*b*) *If in the above problem, the given circular plate is having a concentric circular hole of diameter 1.5 m, then calculate the total pressure and position of the centre of pressure on one face of the plate.*

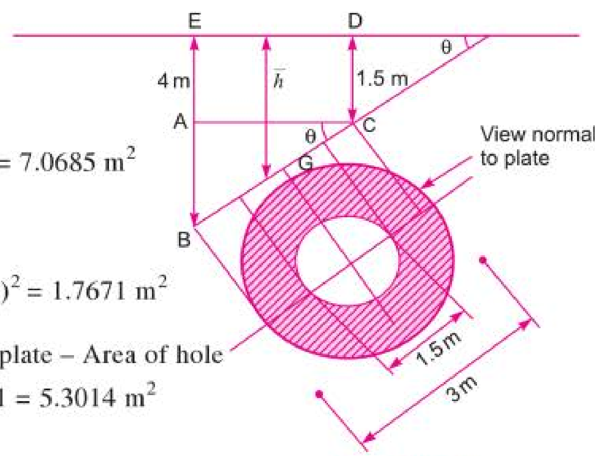


Fig. 3.20 (*a*)

Solution. Given : [Refer to Fig. 3.20 (*a*)]

Dia. of plate, $d = 3.0$ m

$\therefore$ Area of solid plate $= \dfrac{\pi}{4} d^2 = \dfrac{\pi}{4} (3)^2 = 7.0685\ m^2$

Dia. of hole in the plate, $d_0 = 1.5$ m

$\therefore$ Area of hole $= \dfrac{\pi}{4} {d_0}^2 = \dfrac{\pi}{4} (1.5)^2 = 1.7671\ m^2$

$\therefore$ Area of the given plate, A = Area of solid plate – Area of hole

$= 7.0685 - 1.7671 = 5.3014\ m^2$

Distance $CD = 1.5$, $BE = 4$ m

Distance of C.G. from the free surface,

$$\bar{h} = CD + GC \sin\theta$$
$$= 1.5 + 1.5 \sin\theta$$

But $\sin\theta = \dfrac{AB}{BC} = \dfrac{BE - AE}{BC} = \dfrac{4 - 1.5}{3} = \dfrac{2.5}{3}$

$$\therefore \quad \bar{h} = 1.5 + 1.5 \times \frac{2.5}{3} = 1.5 + 1.25 = 2.75\ m$$

(*i*) **Total pressure force (F)**

$$F = \rho g A \bar{h}$$
$$= 1000 \times 9.81 \times 5.3014 \times 2.75$$
$$= 143018\ N = \mathbf{143.018\ kN.}\ \textbf{Ans.}$$

(*ii*) **Position of centre of pressure (h*)**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A\bar{h}} + \bar{h}$$

where $I_G = \frac{\pi}{64}[d^4 - d_0^4] = \frac{\pi}{64}[3^4 - 1.5^4] \text{ m}^4$

$$A = \frac{\pi}{4}[d^2 - d_0^2] = \frac{\pi}{4}[3^2 - 1.5^2] \text{ m}^2$$

$$\sin \theta = \frac{2.5}{3} \text{ and } \bar{h} = 2.75$$

$$\therefore \qquad h^* = \frac{\frac{\pi}{64}\left[3^4 - 1.5^4\right] \times \left(\frac{2.5}{3}\right)^2}{\frac{\pi}{4}\left[3^2 - 1.5^2\right] \times 2.75} + 2.75$$

$$= \frac{\frac{1}{16}\left[3^2 + 1.5^2\right] \times \left(\frac{2.5}{3}\right)^2}{2.75} + 2.75 = \frac{1 \times 11.25 \times 6.25}{16 \times 2.75 \times 9} + 2.75$$

$$= 0.177 + 2.75 = \mathbf{2.927 \text{ m.}} \textbf{ Ans.}$$

Problem 3.16 *A circular plate 3 metre diameter is submerged in water as shown in Fig. 3.21. Its greatest and least depths are below the surfaces being 2 metre and 1 metre respectively. Find : (i) the total pressure on front face of the plate, and (ii) the position of centre of pressure.*

Solution. Given :

Dia. of plate, $d = 3.0$ m

$\therefore$ Area, $A = \frac{\pi}{4}(3.0)^2 = 7.0685 \text{ m}^2$

Distance, $DC = 1$ m, $BE = 2$ m

In ΔABC, $\sin \theta = \frac{AB}{AC} = \frac{BE - AE}{BC} = \frac{BE - DC}{BC} = \frac{2.0 - 1.0}{3.0} = \frac{1}{3}$

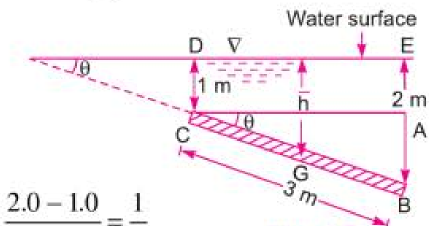


Fig. 3.21

The centre of gravity of the plate is at the middle of BC, *i.e.*, at a distance 1.5 m from C.

The distance of centre of gravity from the free surface of the water is given by

$$\bar{h} = CD + CG \sin \theta = 1.0 + 1.5 \times \frac{1}{3} \qquad (\because \sin \theta = \tfrac{1}{3})$$

$$= 1.5 \text{ m.}$$

(*i*) Total pressure on the front face of the plate is given by

$$F = \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times 7.0685 \times 1.5 = \mathbf{104013 \text{ N.}} \textbf{ Ans.}$$

(*ii*) Let the distance of the centre of pressure from the free surface of the water be h^*. Then using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A\bar{h}} + \bar{h}$$

where $\quad I_G = \frac{\pi}{64} d^4 = \frac{\pi}{64} (3)^4, A = \frac{\pi}{4} d^2, \bar{h} = 1.5$ m and $\sin\theta = \frac{1}{3}$

Substituting the values, we get

$$h^* = \frac{\frac{\pi}{64} d^4 \times \left(\frac{1}{3}\right)^2}{\frac{\pi}{4} d^2 \times 1.5} + 1.5 = \frac{d^2}{16} \times \frac{1}{9 \times 1.5} + 1.5$$

$$= \frac{3^2}{16 \times 9 \times 1.5} + 1.5 = .0416 + 1.5 = \mathbf{1.5416\ m.}\ \textbf{Ans.}$$

Problem 3.17 *A rectangular gate 5 m × 2 m is hinged at its base and inclined at 60° to the horizontal as shown in Fig. 3.22. To keep the gate in a stable position, a counter weight of 5000 kgf is attached at the upper end of the gate as shown in figure. Find the depth of water at which the gate begins to fall. Neglect the weight of the gate and friction at the hinge and pulley.*

Solution. Given :

Length of gate $= 5$ m

Width of gate $= 2$ m

$\theta = 60°$

Weight, $W = 5000$ kgf
$= 5000 \times 9.81$ N
$= 49050$ N $\quad (\because 1 \text{ kgf} = 9.81 \text{ N})$

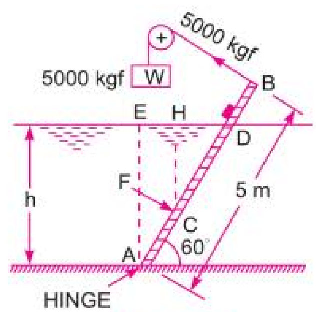


Fig. 3.22

As the pulley is frictionless, the force acting at $B = 49050$ N. First find the total force F acting on the gate AB for a given depth of water.

From figure, $\quad AD = \frac{AE}{\sin\theta} = \frac{h}{\sin 60°} = \frac{5}{\sqrt{3}/2} = \frac{2h}{\sqrt{3}}$

$\therefore$ Area of gate immersed in water, $A = AD \times \text{Width} \times \frac{2h}{\sqrt{3}} \times 2 = \frac{4h}{\sqrt{3}}\ \text{m}^2$

Also depth of the C.G. of the immersed area $= \bar{h} = \frac{h}{2} = 0.5\ h$

$\therefore$ Total force F is given by $F = \rho g A \bar{h} = 1000 \times 9.81 \times \frac{4h}{\sqrt{3}} \times \frac{h}{\sqrt{2}} = \frac{19620}{\sqrt{3}} h^2\text{N}$

The centre of pressure of the immersed surface, h^* is given by

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}$$

where $\quad I_G =$ M.O.I. of the immersed area

$$= \frac{b \times (AD)^3}{12} = \frac{2}{12} \times \left(\frac{2h}{\sqrt{3}}\right)^3 \qquad \left\{\because AD = \frac{2h}{\sqrt{3}}\right\}$$

$$= \frac{16h^3}{12 \times 3 \times \sqrt{3}} = \frac{4h^3}{9 \times \sqrt{3}}\ \text{m}^4$$

$$\therefore \qquad h^* = \frac{4h^3}{9\times\sqrt{3}} \times \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\frac{4h}{\sqrt{3}}\times\frac{h}{2}} + \frac{h}{2} = \frac{3h^3}{18h^2} + \frac{h}{2} = \frac{h}{6} + \frac{h}{2} = \frac{h+3h}{6} = \frac{2h}{3}$$

Now in the ΔCHD, $\qquad CH = h^* = \frac{2h}{3}$, $\angle CDH = 60°$

$$\therefore \qquad \frac{CH}{CD} = \sin 60°$$

$$\therefore \qquad CD = \frac{CH}{\sin 60°} = \frac{h^*}{\sin 60°} = \frac{2h}{3\times\frac{\sqrt{3}}{2}} = \frac{4h}{3\times\sqrt{3}}$$

$$\therefore \qquad AC = AD - CD = \frac{2h}{\sqrt{3}} - \frac{4h}{3\sqrt{3}} = \frac{6h-4h}{3\sqrt{3}} = \frac{2h}{3\sqrt{3}} \text{ m}$$

Taking the moments about hinge, we get

$$49050 \times 5.0 = F \times AC = \frac{19620}{\sqrt{3}}\, h^2 \times \frac{2h}{3\sqrt{3}}$$

or

$$.245250 = \frac{39240\, h^3}{3\times 3}$$

$$\therefore \qquad h^3 = \frac{9\times 245250}{39240} = 56.25$$

$$\therefore \qquad h = (56.25)^{1/3} = \mathbf{3.83\ m.}\ \textbf{Ans.}$$

Problem 3.18 *An inclined rectangular sluice gate AB, 1.2 m by 5 m size as shown in Fig. 3.23 is installed to control the discharge of water. The end A is hinged. Determine the force normal to the gate applied at B to open it.*

Solution. Given :

$$A = \text{Area of gate} = 1.2 \times 5.0 = 6.0 \text{ m}^2$$

Depth of C.G. of the gate from free surface of the water $= \bar{h}$

$$= DG = BC - BE$$
$$= 5.0 - BG \sin 45°$$
$$= 5.0 - 0.6 \times \frac{1}{\sqrt{2}} = 4.576 \text{ m}$$

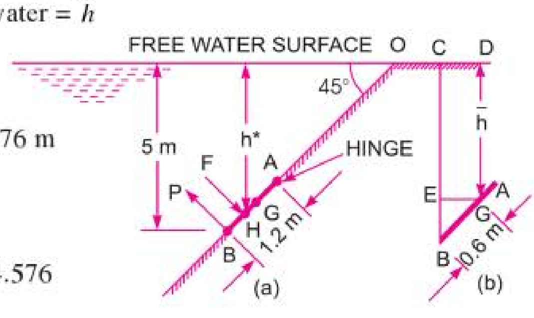


Fig. 3.23

The total pressure force (F) acting on the gate,

$$F = \rho g A \bar{h}$$
$$= 1000 \times 9.81 \times 6.0 \times 4.576$$
$$= 269343 \text{ N}$$

This force is acting at H, where the depth of H from free surface is given by

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}$$

where $\quad I_G$ = M.O.I. of gate $= \dfrac{bd^3}{12} = \dfrac{5.0 \times 1.2^3}{12} = 0.72$ m

$\therefore$ Depth of centre of pressure $h^* = \dfrac{0.72 \times \sin^2 45°}{6 \times 4.576} + 4.576 = .013 + 4.576 = 4.589$ m

But from Fig. 3.23 (*a*), $\qquad \dfrac{h^*}{OH} = \sin 45°$

$\therefore$ Distance, $\qquad OH = \dfrac{h^*}{\sin 45°} = \dfrac{4.589}{\frac{1}{\sqrt{2}}} = 4.589 \times \sqrt{2} = 6.489$ m

Distance, $\qquad BO = \dfrac{5}{\sin 45°} = 5 \times \sqrt{2} = 7.071$ m

Distance, $\qquad BH = BO - OH = 7.071 - 6.489 = 0.582$ m

$\therefore$ Distance $\qquad AH = AB - BH = 1.2 - 0.582 = 0.618$ m

Taking the moments about the hinge A

$$P \times AB = F \times (AH)$$

where P is the force normal to the gate applied at B

$\therefore \qquad P \times 1.2 = 269343 \times 0.618$

$\therefore \qquad P = \dfrac{269343 \times 0.618}{1.2} =$ **138708 N. Ans.**

Problem 3.19 *A gate supporting water is shown in Fig. 3.24. Find the height h of the water so that the gate tips about the hinge. Take the width of the gate as unity.*

Solution. Given : $\qquad \theta = 60°$

Distance, $\qquad AC = \dfrac{h}{\sin 60°} = \dfrac{2h}{\sqrt{3}}$

where h = Depth of water.

The gate will start tipping about hinge B if the resultant pressure force acts at B. If the resultant pressure force passes through a point which is lying from B to C anywhere on the gate, the gate will tip over the hinge. Hence limiting case is when the resultant force passes through B. But the resultant force passes through the centre of pressure. Hence for the given position, point B becomes the centre of pressure. Hence depth of centre of pressure,

$$h^* = (h - 3) \text{ m}$$

But h^* is also given by $\qquad = \dfrac{I_G \sin^2 \theta}{Ah} + \bar{h}$

Taking width of gate unity. Then

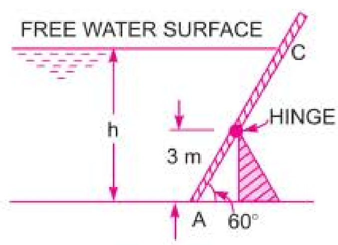


Fig. 3.24

Area, $\qquad A = AC \times 1 = \dfrac{2h}{\sqrt{3}} \times 1 \; ; \; \bar{h} = \dfrac{h}{2}$

$$I_G = \frac{bd^3}{12} = \frac{1 \times AC^3}{12} = \frac{1 \times \left(\frac{2h}{\sqrt{3}}\right)^3}{12} = \frac{8h^3}{12 \times 3 \times \sqrt{3}} = \frac{2h^3}{9 \times \sqrt{3}}$$

$$\therefore \qquad h^* = \frac{2h^3}{9\times\sqrt{3}} \times \frac{\sin^2 60^\circ}{\frac{2h}{\sqrt{3}} \times \frac{h}{2}} + \frac{h}{2} = \frac{2h^3 \times \frac{3}{4}}{9h^2} + \frac{h}{2} = \frac{h}{6} + \frac{h}{2} = \frac{2h}{3}$$

Equating the two values of h^*,

$$h - 3 = \frac{2h}{3} \quad \text{or} \quad h - \frac{2h}{3} = 3 \quad \text{or} \quad \frac{h}{3} = 3$$

$$\therefore \qquad h = 3 \times 3 = 9 \text{ m}$$

$\therefore$ Height of water for tipping the gate = **9 m. Ans.**

Problem 3.20 *A rectangular sluice gate AB, 2 m wide and 3 m long is hinged at A as shown in Fig. 3.25. It is kept closed by a weight fixed to the gate. The total weight of the gate and weight fixed to the gate is 343350 N. Find the height of the water 'h' which will just cause the gate to open. The centre of gravity of the weight and gate is at* **G**.

Solution. Given :

Width of gate, $b = 2$ m ; Length of gate $L = 3$ m

$\therefore$ Area, $A = 2 \times 3 = 6 \text{ m}^2$

Weight of gate and $W = 343350$ N

Angle of inclination, $\theta = 45^\circ$

Let h is the required height of water.

Depth of C.G. of the gate and weight = $\bar{h}$

From Fig. 3.25 (*a*),

$$\bar{h} = h - ED = h - (AD - AE)$$

$$= h - (AB \sin\theta - EG \tan\theta) \qquad \left\{\because \tan\theta = \frac{AE}{EG} \therefore AE = EG\tan\theta\right\}$$

$$= h - (3 \sin 45^\circ - 0.6 \tan 45^\circ)$$

$$= h - (2.121 - 0.6) = (h - 1.521) \text{ m}$$

The total pressure force, F is given by

$$F = \rho g A\bar{h} = 1000 \times 9.81 \times 6 \times (h - 1.521)$$

$$= 58860\,(h - 1.521) \text{ N.}$$

The total force F is acting at the centre of pressure as shown in Fig. 3.25 (*b*) at H. The depth of H from free surface is given by h^* which is equal to

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}, \text{ where } I_G = \frac{bd^3}{12} = \frac{2\times 3^3}{12} = \frac{54}{12} = 4.5 \text{ m}^4$$

$$\therefore \qquad h^* = \frac{4.5 \times \sin^2 45^\circ}{6\times(h-1.521)} + (h - 1.521) = \frac{0.375}{(h-1.521)} + (h - 1.521) \text{ m}$$

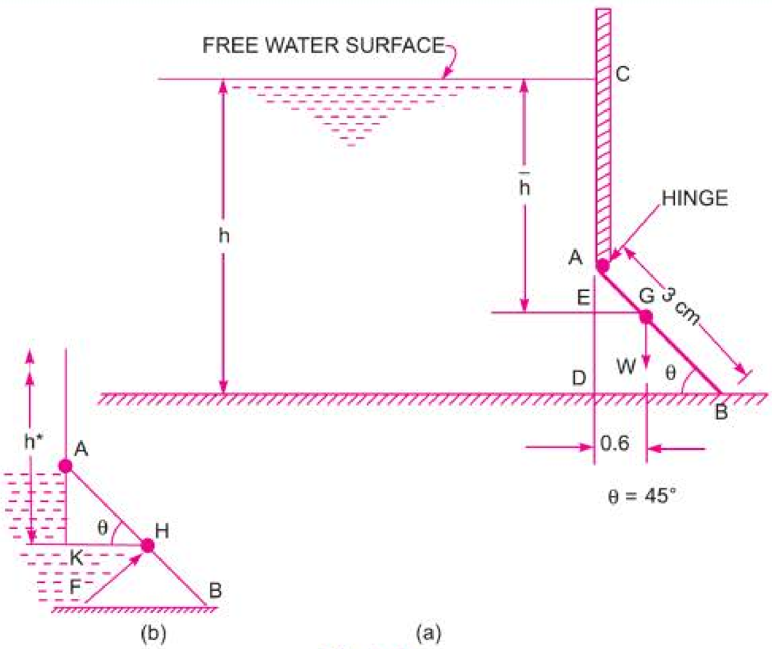


Fig. 3.25

Now taking moments about hinge A, we get

$$343350 \times EG = F \times AH$$

or
$$343350 \times 0.6 = F \times \frac{AK}{\sin 45°}$$

$$\left[\text{From } \Delta AKH, \text{Fig. 3.25 } (b)\ AK = AH \sin \theta = AH \sin 45° \therefore AH = \frac{AK}{\sin 45°}\right]$$

$$= \frac{58860\,(h - 1.521) \times AK}{\sin 45°}$$

$\therefore$
$$AK = \frac{343350 \times 0.6 \times \sin 45°}{58860\,(h - 1.521)} = \frac{0.3535 \times 7}{(h - 1.521)} \qquad ...(i)$$

But
$$AK = h^* - AC = \frac{.375}{(h - 1.521)} + (h - 1.521) - AC \qquad ...(ii)$$

But
$$AC = CD - AD = h - AB \sin 45° = h - 3 \times \sin 45° = h - 2.121$$

$\therefore$ Substituting this value in (*ii*), we get

$$AK = \frac{.375}{h - 1.521} + (h - 1.521) - (h - 2.121)$$

$$= \frac{.375}{h - 1.521} + 2.121 - 1.521 = \frac{.375}{h - 1.521} + 0.6 \qquad ...(iii)$$

Equating the two values of AK from (*i*) and (*iii*)

$$\frac{0.3535 \times 7}{h - 1.521} = \frac{0.375}{h - 1.521} + 0.6$$

or $$0.3535 \times 7 = 0.375 + 0.6\,(h - 1.521) = 0.375 + 0.6\,h - 0.6 \times 1.521$$

or $$0.6h = 2.4745 - .375 + 0.6 \times 1.521 = 2.0995 + 0.9126 = 3.0121$$

$\therefore$ $$h = \frac{3.0121}{0.6} = \mathbf{5.02\ m.}\ \textbf{Ans.}$$

Problem 3.21 *Find the total pressure and position of centre of pressure on a triangular plate of base 2 m and height 3 m which is immersed in water in such a way that the plane of the plate makes an angle of 60° with the free surface of the water. The base of the plate is parallel to water surface and at a depth of 2.5 m from water surface.*

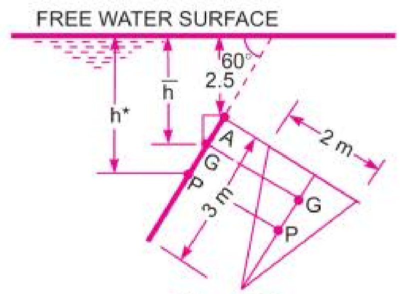


Fig. 3.26

Solution. Given :

Base of plate, $b = 2$ m

Height of plate, $h = 3$ m

$\therefore$ Area, $$A = \frac{b \times h}{2} = \frac{2 \times 3}{2} = 3\ \text{m}^2$$

Inclination, $\theta = 60°$

Depth of centre of gravity from free surface of water,

$$\bar{h} = 2.5 + AG \sin 60°$$

$$= 2.5 + \frac{1}{3} \times 3 \times \frac{\sqrt{3}}{2} \qquad \left\{\because AG = \frac{1}{3} \text{ of height of triangle}\right\}$$

$$= 2.5 + .866 \text{ m} = 3.366 \text{ m}$$

(*i*) **Total pressure force (F)**

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 3 \times 3.366 = \mathbf{99061.38\ N.}\ \textbf{Ans.}$$

(*ii*) **Centre of pressure (h*).** Depth of centre of pressure from free surface of water is given by

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}$$

where $$I_G = \frac{bh^3}{36} = \frac{2 \times 3^3}{36} = \frac{3}{2} = 1.5\ \text{m}^4$$

$\therefore$ $$h^* = \frac{1.5 \times \sin^2 60°}{3 \times 3.366} + 3.366 = 0.111 + 3.366 = \mathbf{3.477\ m.}\ \textbf{Ans.}$$

▶ 3.6 CURVED SURFACE SUB-MERGED IN LIQUID

Consider a curved surface *AB*, sub-merged in a static fluid as shown in Fig. 3.27. Let *dA* is the area of a small strip at a depth of *h* from water surface.

Then pressure intensity on the area *dA* is $= \rho g h$

and pressure force, $$dF = p \times \text{Area} = \rho g h \times dA \qquad ...(3.11)$$

This force *dF* acts normal to the surface.

Hence total pressure force on the curved surface should be

$$F = \int \rho g h\, dA \qquad ...(3.12)$$

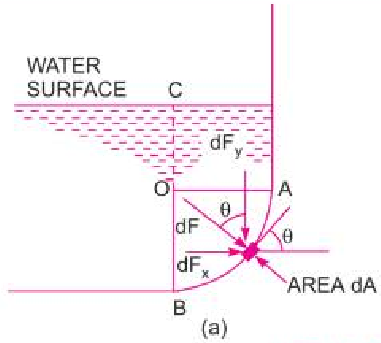


Fig. 3.27

But here as the direction of the forces on the small areas are not in the same direction, but varies from point to point. Hence integration of equation (3.11) for curved surface is impossible. The problem can, however, be solved by resolving the force dF in two components dF_x and dF_y in the x and y directions respectively. The total force in the x and y directions, *i.e.*, F_x and F_y are obtained by integrating dF_x and dF_y. Then total force on the curved surface is

$$F = \sqrt{F_x^2 + F_y^2} \qquad ...(3.13)$$

and inclination of resultant with horizontal is $\tan \phi = \dfrac{F_y}{F_x}$ ...(3.14)

Resolving the force dF given by equation (3.11) in x and y directions :

$$dF_x = dF \sin \theta = \rho g h dA \sin \theta \qquad \{\because dF = \rho g h dA\}$$

and

$$dF_y = dF \cos \theta = \rho g h dA \cos \theta$$

Total forces in the x and y direction are :

$$F_x = \int dF_x = \int \rho g h dA \sin \theta = \rho g \int h dA \sin \theta \qquad ...(3.15)$$

and

$$F_y = \int dF_y = \int \rho g h dA \cos \theta = \rho g \int h dA \cos \theta \qquad ...(3.16)$$

Fig. 3.27 (*b*) shows the enlarged area dA. From this figure, *i.e.*, ΔEFG,

$$EF = dA$$
$$FG = dA \sin \theta$$
$$EG = dA \cos \theta$$

Thus in equation (3.15), $dA \sin \theta = FG$ = Vertical projection of the area dA and hence the expression $\rho g \int h dA \sin \theta$ represents the total pressure force on the projected area of the curved surface on the vertical plane. Thus

F_x = Total pressure force on the projected area of the curved surface on vertical plane. ...(3.17)

Also $dA \cos \theta = EG$ = horizontal projection of dA and hence $h dA \cos \theta$ is the volume of the liquid contained in the elementary area dA upto free surface of the liquid. Thus $\int h dA \cos \theta$ is the total volume contained between the curved surface extended upto free surface.

Hence $\rho g \int h dA \cos \theta$ is the total weight supported by the curved surface. Thus

$$F_y = \rho g \int h dA \cos \theta$$

= weight of liquid supported by the curved surface upto free surface of liquid. ...(3.18)

In Fig. 3.28, the curved surface AB is not supporting any fluid. In such cases, F_y is equal to the weight of the imaginary liquid supported by AB upto free surface of liquid. The direction of F_y will be taken in upward direction.

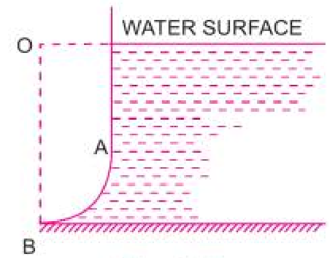


Fig. 3.28

Problem 3.22 *Compute the horizontal and vertical components of the total force acting on a curved surface AB, which is in the form of a quadrant of a circle of radius 2 m as shown in Fig. 3.29. Take the width of the gate as unity.*

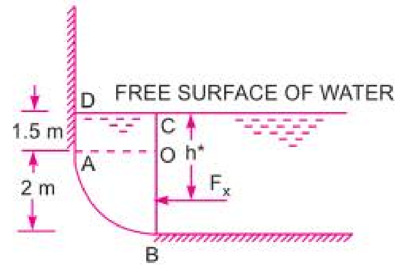


Fig. 3.29

Solution. Given :

Width of gate $= 1.0$ m

Radius of the gate $= 2.0$ m

$\therefore$ Distance $AO = OB = 2$ m

Horizontal force, F_x exerted by water on gate is given by equation (3.17) as

F_x = Total pressure force on the projected area of curved surface AB on vertical plane

= Total pressure force on OB

{projected area of curved surface on vertical plane $= OB \times 1$}

$$= \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times 2 \times 1 \times \left(1.5 + \frac{2}{2}\right)$$

{$\because$ Area of $OB = A = BO \times 1 = 2 \times 1 = 2$,

$\bar{h}$ = Depth of C.G. of OB from free surface $= 1.5 + \frac{2}{2}$ }

$$F_x = 9.81 \times 2000 \times 2.5 = \mathbf{49050\ N.}\ \textbf{Ans.}$$

The point of application of F_x is given by $h^* = \dfrac{I_G}{A\bar{h}} + \bar{h}$

where I_G = M.O.I. of OB about its C.G. $= \dfrac{bd^3}{12} = \dfrac{1 \times 2^3}{12} = \dfrac{2}{3}$ m^4

$$\therefore \qquad h^* = \frac{\frac{2}{3}}{2 \times 2.5} + 2.5 = \frac{1}{7.5} + 2.5 \text{ m}$$

$= 0.1333 + 2.5 = 2.633$ m from free surface.

Vertical force, F_y, exerted by water is given by equation (3.18)

F_y = Weight of water supported by AB upto free surface

= Weight of portion $DABOC$

= Weight of $DAOC$ + Weight of water AOB

$= \rho g$ [Volume of $DAOC$ + Volume of AOB]

$$= 1000 \times 9.81 \left[AD \times AO \times 1 + \frac{\pi}{4}(AO)^2 \times 1\right]$$

$$= 1000 \times 9.81 \left[1.5 \times 2.0 \times 1 + \frac{\pi}{4} \times 2^2 \times 1\right]$$

$$= 1000 \times 9.81\ [3.0 + \pi]\text{N} = \mathbf{60249.1\ N.}\ \textbf{Ans.}$$

Problem 3.23 *Fig. 3.30 shows a gate having a quadrant shape of radius 2 m. Find the resultant force due to water per metre length of the gate. Find also the angle at which the total force will act.*

Solution. Given :

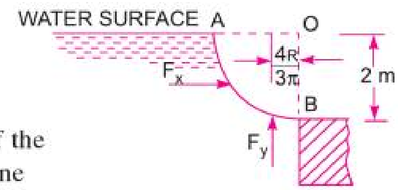


Fig. 3.30

Radius of gate = 2 m

Width of gate = 1 m

Horizontal Force

F_x = Force on the projected area of the curved surface on vertical plane

$= \text{Force on } BO = \rho g A\bar{h}$

where $A = \text{Area of } BO = 2 \times 1 = 2\ \text{m}^2$, $\bar{h} = \frac{1}{2} \times 2 = 1$ m ;

$$F_x = 1000 \times 9.81 \times 2 \times 1 = 19620\ \text{N}$$

This will act at a depth of $\frac{2}{3} \times 2 = \frac{4}{3}$ m from free surface of liquid,

Vertical Force, F_y

F_y = Weight of water (imagined) supported by AB

$= \rho g \times \text{Area of } AOB \times 1.0$

$$= 1000 \times 9.81 \times \frac{\pi}{4}(2)^4 \times 1.0 = 30819\ \text{N}$$

This will act at a distance of $\frac{4R}{3\pi} = \frac{4 \times 2.0}{3\pi} = 0.848$ m from OB.

$\therefore$ Resultant force, F is given by

$$F = \sqrt{F_x^2 + F_y^2}$$

$$= \sqrt{19620 + 30819} = \sqrt{384944400 + 949810761}$$

$$= \mathbf{36534.4\ N.}\ \textbf{Ans.}$$

The angle made by the resultant with horizontal is given by

$$\tan\theta = \frac{F_y}{F_x} = \frac{30819}{19620} = 1.5708$$

$$\therefore \qquad \theta = \tan^{-1} 1.5708 = \mathbf{57° 31'}.\ \textbf{Ans.}$$

Problem 3.24 *Find the magnitude and direction of the resultant force due to water acting on a roller gate of cylindrical form of 4.0 m diameter, when the gate is placed on the dam in such a way that water is just going to spill. Take the length of the gate as 8 m.*

Solution. Given :

Dia. of gate = 4 m

$\therefore$ Radius, $R = 2$ m

Length of gate, $l = 8$ m

Horizontal force, F_x acting on the gate is

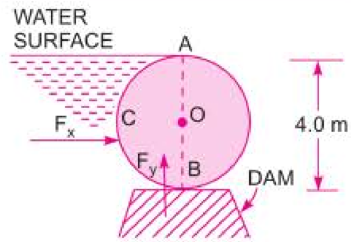


Fig. 3.31

$$F_x = \rho g A\bar{h} = \text{Force on projected area of curved surface } ACB \text{ on vertical plane}$$
$$= \text{Force on vertical area } AOB$$

where A = Area of $AOB = 4.0 \times 8.0 = 32.0 \text{ m}^2$

$\bar{h}$ = Depth of C.G. of $AOB = 4/2 = 2.0$ m

$\therefore$
$$F_x = 1000 \times 9.81 \times 32.0 \times 2.0 = 627840 \text{ N.}$$

Vertical force, F_y is given by

F_y = Weight of water enclosed or supported (actually or imaginary) by the curved surface ACB

$= \rho g \times$ Volume of portion ACB

$= \rho g \times$ Area of $ACB \times l$

$$= 1000 \times 9.81 \times \frac{\pi}{2}(R)^2 \times 8.0 = 9810 \times \frac{\pi}{2}(2)^2 \times 8.0 = 493104 \text{ N}$$

It will be acting in the upward direction.

$\therefore$ Resultant force, $F = \sqrt{F_x^2 + F_y^2} = \sqrt{627840 + 493104}$ = **798328 N. Ans.**

Direction of resultant force is given by $\tan\theta = \dfrac{F_y}{F_x} = \dfrac{493104}{627840} = 0.7853$

$\therefore$ $\theta = 31° \, 8'$. **Ans.**

Problem 3.25 *Find the horizontal and vertical component of water pressure acting on the face of a tainter gate of 90° sector of radius 4 m as shown in Fig. 3.32. Take width of gate unity.*

Solution. Given :

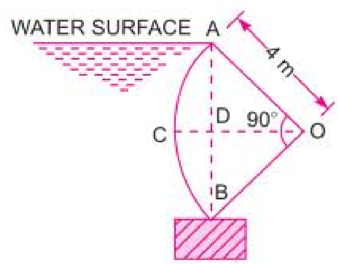


Fig. 3.32

Radius of gate, $R = 4$ m

Horizontal component of force acting on the gate is

F_x = Force on area of gate projected on vertical plane

= Force on area ADB

$= \rho g A\bar{h}$

where $A = AB \times$ Width of gate

$= 2 \times AD \times 1$ $\quad (\because AB = 2AD)$

$= 2 \times 4 \times \sin 45° = 8 \times .707 = 5.656 \text{ m}^2$ $\quad \{\because AD = 4 \sin 45°\}$

$$\bar{h} = \frac{AB}{2} = \frac{5.656}{2} = 2.828 \text{ m}$$

$\therefore$ $F_x = 1000 \times 9.81 \times 5.656 \times 2.828$ N = **156911 N. Ans.**

Vertical component

F_y = Weight of water supported or enclosed by the curved surface

= Weight of water in portion $ACBDA$

$= \rho g \times$ Area of $ACBDA \times$ Width of gate

$= 1000 \times 9.81 \times [\text{Area of sector } ACBOA - \text{Area of } \Delta ABO] \times 1$

$$= 9810 \times \left[\frac{\pi}{4}R^2 - \frac{AO \times BO}{2}\right] \qquad [\because \Delta AOB \text{ is a right angled}]$$

$$= 9810 \times \left[\frac{\pi}{4}4^2 - \frac{4 \times 4}{2}\right] = \mathbf{44796\ N.}\ \textbf{Ans.}$$

Problem 3.26 *Calculate the horizontal and vertical components of the water pressure exerted on a tainter gate of radius 8 m as shown in Fig. 3.33. Take width of gate unity.*

Solution. The horizontal component of water pressure is given by

$F_x = \rho g A\bar{h}$ = Force on the area projected on vertical plane
= Force on the vertical area of BD

where $A = BD \times$ Width of gate $= 4.0 \times 1 = 4.0$ m

$$\bar{h} = \frac{1}{2} \times 4 = 2 \text{ m}$$

$$F_x = 1000 \times 9.81 \times 4.0 \times 2.0 = \mathbf{78480\ N.}\ \textbf{Ans.}$$

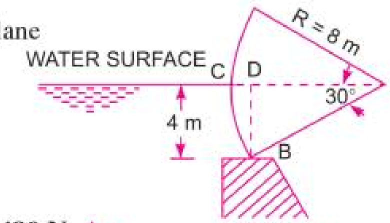


Fig. 3.33

Vertical component of the water pressure is given by

F_y = Weight of water supported or enclosed (imaginary) by curved surface CB
= Weight of water in the portion $CBDC$
$= \rho g \times$ [Area of portion $CBDC$] $\times$ Width of gate
$= \rho g \times$ [Area of sector CBO – Area of the triangle BOD] $\times$ 1

$$= 1000 \times 9.81 \times \left[\frac{30}{360} \times \pi R^2 - \frac{BD \times DO}{2}\right]$$

$$= 9810 \times \left[\frac{1}{12}\pi \times 8^2 - \frac{4.0 \times 8.8 \cos 30^\circ}{2}\right]$$

$$\{\because DO = BO \cos 30^\circ = 8 \times \cos 30^\circ\}$$

$$= 9810 \times [16.755 - 13.856] = \mathbf{28439\ N.}\ \textbf{Ans.}$$

Problem 3.27 *A cylindrical gate of 4 m diameter 2 m long has water on its both sides as shown in Fig. 3.34. Determine the magnitude, location and direction of the resultant force exerted by the water on the gate. Find also the least weight of the cylinder so that it may not be lifted away from the floor.*

Solution. Given :

Dia. of gate $= 4$ m

Radius $= 2$ m

(*i*) The forces acting on the left side of the cylinder are :

The horizontal component, F_{x_1}

where F_{x_1} = Force of water on area projected on vertical plane
= Force on area AOC
$= \rho g A\bar{h}$
$= 1000 \times 9.81 \times 8 \times 2$
$= 156960$ N

where $A = AC \times \text{Width} = 4 \times 2 = 8 \text{ m}^2$

$= \bar{h} = \frac{1}{2} \times 4 = 2$ m

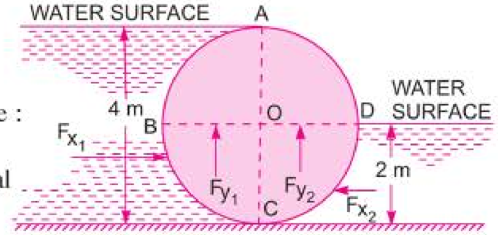


Fig. 3.34

F_{y_1} = weight of water enclosed by $ABCOA$

$$= 1000 \times 9.81 \times \left[\frac{\pi}{2} R^2\right] \times 2.0 = 9810 \times \frac{\pi}{2} \times 2^2 \times 2.0 = \mathbf{123276\ N.}$$

Right Side of the Cylinder

$$F_{x_2} = \rho g A_2 \bar{h}_2 = \text{Force on vertical area } CO$$

$$= 1000 \times 9.81 \times 2 \times 2 \times \frac{2}{2} \left\{ A_2 = CO \times 1 = 2 \times 1 = 2\ \text{m}^2, \bar{h}_2 = \frac{2}{2} = 1.0 \right\}$$

$$= 39240\ \text{N}$$

F_{y_2} = Weight of water enclosed by $DOCD$

$$= \rho g \times \left[\frac{\pi}{4} R^2\right] \times \text{Width of gate}$$

$$= 1000 \times 9.81 \times \frac{\pi}{4} \times 2^2 \times 2 = 61638\ \text{N}$$

∴ Resultant force in the direction of x,

$$F_x = F_{x_1} - F_{x_2} = 156960 - 39240 = 117720\ \text{N}$$

Resultant force in the direction of y,

$$F_y = F_{y_1} + F_{y_2} = 123276 + 61638 = 184914\ \text{N}$$

(*i*) **Resultant force,** F is given as

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(117720)^2 + (184914)^2} = \mathbf{219206\ N.}\ \textbf{Ans.}$$

(*ii*) **Direction of resultant force** is given by

$$\tan\theta = \frac{F_y}{F_x} = \frac{184914}{117720} = 1.5707$$

$$\therefore \quad \theta = \mathbf{57^\circ\ 31'}.\ \textbf{Ans.}$$

(*iii*) **Location of the resultant force**

Force, F_{x_1} acts at a distance of $\frac{2 \times 4}{3} = 2.67$ m from the top surface of water on left side, while F_{x_2} acts at a distance of $\frac{2}{3} \times 2 = 1.33$ m from free surface on the right side of the cylinder. The resultant force F_x in the direction of x will act at a distance of y from the bottom as

$$F_x \times y = F_{x_1} [4 - 2.67] - F_{x_2} [2 - 1.33]$$

or

$$117720 \times y = 156960 \times 1.33 - 39240 \times .67 = 208756.8 - 26290.8 = 182466$$

$$\therefore \quad y = \frac{182466}{117720} = 1.55 \text{ m from the bottom.}$$

Force F_{y_1} acts at a distance $\frac{4R}{3\pi}$ from AOC or at a distance $\frac{4 \times 2.0}{3\pi} = 0.8488$ m from AOC towards left of AOC.

Also F_{y_2} acts at a distance $\frac{4R}{3\pi} = 0.8488$ m from AOC towards the right of AOC. The resultant force F_y will act at a distance x from AOC which is given by

$$F_y \times x = F_{y_1} \times .8488 - F_{y_2} \times .8488$$

or $$184914 \times x = 123276 \times .8488 - 61638 \times .8488 = .8488\ [123276 - 61638] = 52318.4$$

$$\therefore \quad x = \frac{52318.4}{184914} = 0.2829 \text{ m from AOC.}$$

(*iv*) **Least weight of cylinder.** The resultant force in the upward direction is

$$F_y = 184914 \text{ N}$$

Thus the weight of cylinder should not be less than the upward force F_y. Hence least weight of cylinder should be at least.

$$= \mathbf{184914\ N.}$$ **Ans.**

Problem 3.28 *Fig. 3.35 shows the cross-section of a tank full of water under pressure. The length of the tank is 2 m. An empty cylinder lies along the length of the tank on one of its corner as shown. Find the horizontal and vertical components of the force acting on the curved surface ABC of the cylinder.*

Solution. Radius, $R = 1$ m

Length of tank, $l = 2$ m

Pressure, $p = 0.2 \text{ kgf/cm}^2 = 0.2 \times 9.81 \text{ N/cm}^2$
$= 1.962 \text{ N/cm}^2 = 1.962 \times 10^4 \text{ N/m}^2$

$\therefore$ Pressure head, $$h = \frac{p}{\rho g} = \frac{1.962 \times 10^4}{1000 \times 9.81} = 2 \text{ m}$$

$\therefore$ Free surface of water will be at a height of 2 m from the top of the tank.

$\therefore$ Fig. 3.36 shows the equivalent free surface of water.

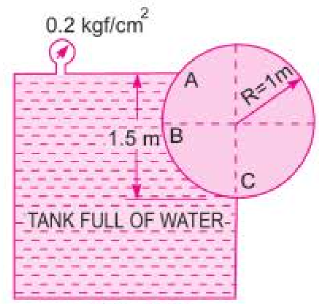


Fig. 3.35

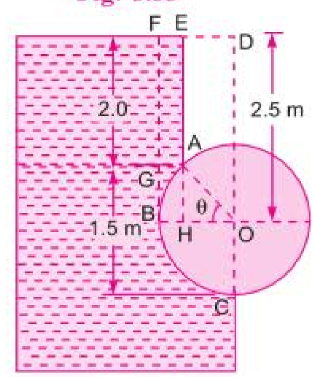


Fig. 3.36

(*i*) **Horizontal Component of Force**

$$F_x = \rho g A \bar{h}$$

where A = Area projected on vertical plane
$= 1.5 \times 2.0 = 3.0 \text{ m}^2$

$$\bar{h} = 2 + \frac{1.5}{2} = 2.75$$

$$\therefore \quad F_x = 1000 \times 9.81 \times 3.0 \times 2.75$$
$$= \mathbf{80932.5\ N.}$$ **Ans.**

(*ii*) **Vertical Component of Force**

F_y = Weight of water enclosed or supported actually or imaginary by curved surface ABC
= Weight of water in the portion *CODE ABC*
= Weight of water in *CODFBC* – Weight of water in *AEFB*

But weight of water in *CODFBC*
= Weight of water in [*COB* + *ODFBO*]

$$= \rho g \left[\frac{\pi R^2}{4} + BO \times OD\right] \times 2 = 1000 \times 9.81 \left[\frac{\pi}{4} \times 1^2 + 1.0 \times 2.5\right] \times 2$$

$= 64458.5$ N

Weight of water in $AEFB = \rho g$ [Area of $AEFB$] $\times 2.0$

$= 1000 \times 9.81$ [Area of $(AEFG + AGBH - AHB)$] $\times 2.0$

In ΔAHO, $\sin\theta = \dfrac{AH}{AO} = \dfrac{0.5}{1.0} = 0.5 \quad \therefore \quad \theta = 30°$

$BH = BO - HO = 1.0 - AO\cos\theta = 1.0 - 1 \times \cos 30° = 0.134$

Area, ABH = Area ABO – Area AHO

$$= \pi R^2 \times \frac{30}{360} - \frac{AH \times HO}{2.0} = \frac{\pi R^2}{12} - \frac{0.5 \times .866}{2} = 0.0453$$

$\therefore$ Weight of water in $AEFB$

$= 9810 \times [AE \times AG + AG \times AH - 0.0453] \times 2.0$

$= 9810 \times [2.0 \times .134 + .134 \times .5 - .0453] \times 2.0$

$= 9810 \times [.268 + .067 - .0453] \times 2.0 = 5684$ N

$\therefore$ $F_y = 64458.5 - 5684 = \mathbf{58774.5\ N.}$ **Ans.**

Problem 3.29 *Find the magnitude and direction of the resultant water pressure acting on a curved face of a dam which is shaped according to the relation* $y = \dfrac{x^2}{9}$ *as shown in Fig. 3.37. The height of the water retained by the dam is 10 m. Consider the width of the dam as unity.*

Solution. Equation of curve AB is

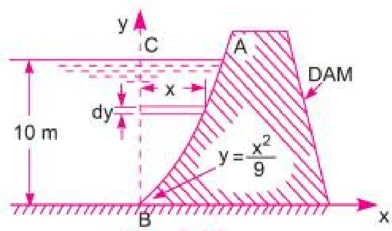


Fig. 3.37

$$y = \frac{x^2}{9} \quad \text{or} \quad x^2 = 9y$$

$\therefore$ $x = \sqrt{9y} = 3\sqrt{y}$

Height of water, $h = 10$ m

Width, $b = 1$ m

The horizontal component, F_x is given by

F_x = Pressure due to water on the curved area projected on vertical plane

= Pressure on area BC

$= \rho g A\bar{h}$

where $A = BC \times 1 = 10 \times 1\ \text{m}^2$, $\bar{h} = \frac{1}{2} \times 10 = 5$ m

$F_x = 1000 \times 9.81 \times 10 \times 5 = 490500$ N

Vertical component, F_y is given by

F_y = Weight of water supported by the curve AB

= Weight of water in the portion ABC

$= \rho g$[Area of ABC] $\times$ Width of dam

$$= \rho g \left[\int_0^{10} x \times dy\right] \times 1.0 \qquad \left\{\text{Area of strip} = xdy \quad \therefore \text{Area } ABC = \int_0^{10} xdy\right\}$$

$$= 1000 \times 9.81 \times \int_0^{10} 3\sqrt{y}\, dy \qquad \{\because x = 3\sqrt{y}\}$$

$$= 29430 \left[\frac{y^{3/2}}{3/2}\right]_0^{10} = 29430 \times \frac{2}{3}\left[y^{3/2}\right]_0^{10} = 19620\,[10^{3/2}]$$

$= 19620 \times 31.622 = 620439$ N

$\therefore$ Resultant water pressure on dam

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(490500)^2 + (620439)^2}$$

$$= 790907 \text{ N} = \mathbf{790.907 \text{ kN. Ans.}}$$

Direction of the resultant is given by

$$\tan\theta = \frac{F_y}{F_x} = \frac{620439}{490500} = 1.265$$

$\therefore$ $\theta = \mathbf{51° \ 40'}$. **Ans.**

Problem 3.30 *A dam has a parabolic shape $y = y_0\left(\frac{x}{x_0}\right)^2$ as shown in Fig. 3.38 below having $x_0 = 6$ m and $y_0 = 9$ m. The fluid is water with density = 1000 kg/m^3. Compute the horizontal, vertical and the resultant thrust exerted by water per metre length of the dam.*

Solution. Given :

Equation of the curve OA is

$$y = y_0\left(\frac{x}{x_0}\right)^2 = 9\left(\frac{x}{6}\right)^2 = 9 \times \frac{x^2}{36} = \frac{x^2}{4}$$

or $x^2 = 4y$

$\therefore$ $x = \sqrt{4y} = 2y^{1/2}$

Width of dam, $b = 1$ m.

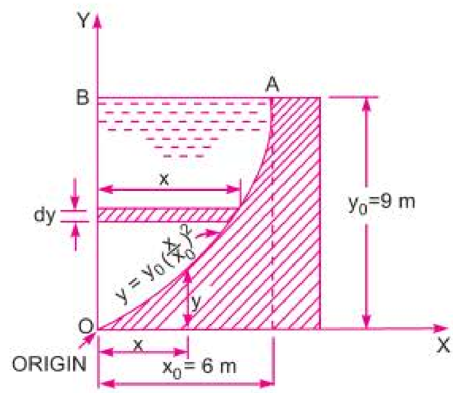


Fig. 3.38

(*i*) **Horizontal thrust exerted by water**

F_x = Force exerted by water on vertical surface OB, *i.e.*, the surface obtained by projecting the curved surface on vertical plane

$$= \rho g A\bar{h}$$

$$= 1000 \times 9.81 \times (9 \times 1) \times \frac{9}{2} = \mathbf{397305 \text{ N. Ans.}}$$

(*ii*) **Vertical thrust exerted by water**

F_y = Weight of water supported by curved surface OA upto free surface of water

= Weight of water in the portion ABO

= $\rho g \times$ Area of $OAB \times$ Width of dam

$$= 1000 \times 9.81 \times \left[\int_0^9 x \times dy\right] \times 1.0$$

$$= 1000 \times 9.81 \times \left[\int_0^9 2y^{1/2} \times dy\right] \times 1.0 \qquad (\because x = 2y^{1/2})$$

$$= 19620 \times \left[\frac{y^{3/2}}{(3/2)}\right]_0^9 = 19620 \times \frac{2}{3}\,[9^{3/2}]$$

$$= 19620 \times \frac{2}{3} \times 27 = \mathbf{353160 \text{ N. Ans.}}$$

(*iii*) **Resultant thrust exerted by water**

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{397305 + 353160} = \mathbf{531574\ N.}\ \textbf{Ans.}$$

Direction of resultant is given by

$$\tan\theta = \frac{F_y}{F_x} = \frac{353160}{397305} = 0.888$$

$$\theta = \tan^{-1} 0.888 = \mathbf{41.63°.}\ \textbf{Ans.}$$

Problem 3.31 *A cylinder 3 m in diameter and 4 m long retains water on one side. The cylinder is supported as shown in Fig. 3.39. Determine the horizontal reaction at A and the vertical reaction at B. The cylinder weighs 196.2 kN. Ignore friction.*

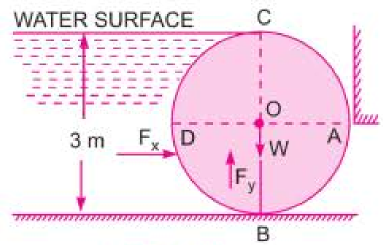


Fig. 3.39

Solution. Given :

Dia. of cylinder $= 3$ m

Length of cylinder $= 4$ m

Weight of cylinder, $W = 196.2$ kN $= 196200$ N

Horizontal force exerted by water

$$F_x = \text{Force on vertical area } BOC$$
$$= \rho g A\bar{h}$$

where $A = BOC \times l = 3 \times 4 = 12\ \text{m}^2,\ \bar{h} = \frac{1}{2} \times 3 = 1.5$ m

$$F_x = 1000 \times 9.81 \times 12 \times 1.5 = 176580\ \text{N}$$

The vertical force exerted by water

$$F_y = \text{Weight of water enclosed in } BDCOB$$
$$= \rho g \times \left(\frac{\pi}{2}R^2\right) \times l = 1000 \times 9.81 \times \frac{\pi}{2} \times (1.5)^2 \times 4 = 138684\ \text{N}$$

Force F_y is acting in the upward direction.

For the equilibrium of cylinder

Horizontal reaction at $A = F_x = 176580$ N

Vertical reaction at $B = \text{Weight of cylinder} - F_y$
$= 196200 - 138684 = \mathbf{57516\ N.}$ **Ans.**

▶ 3.7 TOTAL PRESSURE AND CENTRE OF PRESSURE ON LOCK GATES

Lock gates are the devices used for changing the water level in a canal or a river for navigation. Fig. 3.40 shows plan and elevation of a pair of lock gates. Let *AB* and *BC* be the two lock gates. Each gate is supported on two hinges fixed on their top and bottom at the ends *A* and *C*. In the closed position, the gates meet at *B*.

Let F = Resultant force due to water on the gate *AB* or *BC* acting are right angles to the gate

R = Reaction at the lower and upper hinge

P = Reaction at the common contact surface of the two gates and acting perpendicular to the contact surface.

Let the force *P* and *F* meet at *O*. Then the reaction *R* must pass through *O* as the gate *AB* is in the equilibrium under the action of three forces. Let θ is the inclination of the lock gate with the normal to the side of the lock.

In $\angle ABO$, $\angle OAB = \angle ABO = \theta$.

Resolving all forces along the gate AB and putting equal to zero, we get

$$R\cos\theta - P\cos\theta = 0 \text{ or } R = P \qquad \text{...(3.19)}$$

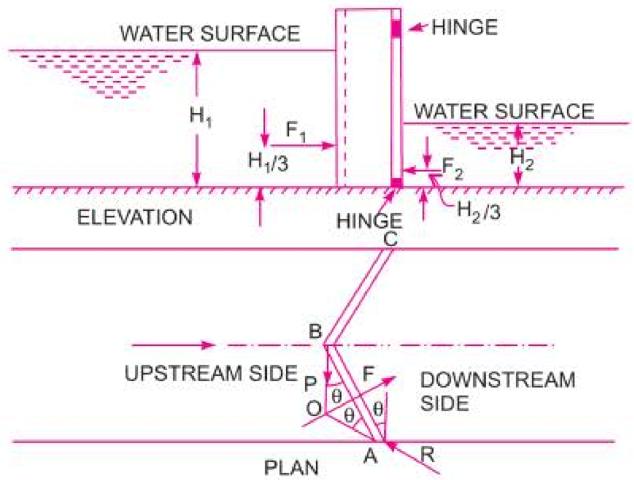


Fig. 3.40

Resolving forces normal to the gate AB

$$R\sin\theta + P\sin\theta - F = 0$$

or $$F = R\sin\theta + P\sin\theta = 2P\sin\theta \qquad \{\because R = P\}$$

$\therefore$ $$P = \frac{F}{2\sin\theta} \qquad \text{...(3.20)}$$

To calculate P and R

In equation (3.20), P can be calculated if F and θ are known. The value of θ is calculated from the angle between the lock gates. The angle between the two lock gate is equal to $180^\circ - 2\theta$. Hence θ can be calculated. The value of F is calculated as :

Let H_1 = Height of water on the upstream side

H_2 = Height of water on the downstream side

F_1 = Water pressure on the gate on upstream side

F_2 = Water pressure on the gate on downstream side of the gate

l = Width of gate

Now $$F_1 = \rho g A_1 \bar{h}_1$$

$$= \rho g \times H_1 \times l \times \frac{H_1}{2}$$

$$= \rho g l \frac{H_1^2}{2} \qquad \left[\because A_1 = H_1 \times l, \bar{h}_1 = \frac{H_1}{2}\right]$$

Similarly, $$F_2 = \rho g A_2 \bar{h}_2 = \rho g \times (H_2 \times l) \times \frac{H_2}{2} = \frac{\rho g l H_2^2}{2}$$

$\therefore$ Resultant force $$F = F_1 - F_2 = \frac{\rho g l H_1^2}{2} - \frac{\rho g l H_2^2}{2}$$

Substituting the value of θ and F in equation (3.20), the value of P and R can be calculated.

Reactions at the top and bottom hinges

Let R_t = Reaction of the top hinge

R_b = Reaction of the bottom hinge

Then $R = R_t + R_b$

The resultant water pressure F acts normal to the gate. Half of the value of F is resisted by the hinges of one lock gates and other half will be resisted by the hinges of other lock gate. Also F_1 acts at a distance of $\frac{H_1}{3}$ from bottom while F_2 acts at a distance of $\frac{H_2}{3}$ from bottom.

Taking moments about the lower hinge

$$R_t \times \sin\theta \times H = \frac{F_1}{2} \times \frac{H_1}{3} - \frac{F_2}{2} \times \frac{H_2}{3} \quad ...(i)$$

where H = Distance between two hinges

Resolving forces horizontally

$$R_t \sin\theta + R_b \sin\theta = \frac{F_1}{2} - \frac{F_2}{2} \quad ...(ii)$$

From equations (*i*) and (*ii*), we can find R_t and R_b.

Problem 3.32 *Each gate of a lock is 6 m high and is supported by two hinges placed on the top and bottom of the gate. When the gates are closed, they make an angle of 120°. The width of lock is 5 m. If the water levels are 4 m and 2 m on the upstream and downstream sides respectively, determine the magnitude of the forces on the hinges due to water pressure.*

Solution. Given :

Height of lock = 6 m

Width of lock = 5 m

Width of each lock gate = AB

or $l = \frac{AD}{\cos 30°} = \frac{2.5}{\cos 30°}$

$= 2.887$ m

Angle between gates = 120°

$\therefore \quad \theta = \frac{180° - 120°}{2} = \frac{60°}{2} = 30°$

Height of water on upstream side

$H_1 = 4$ m

and $H_2 = 2$ m

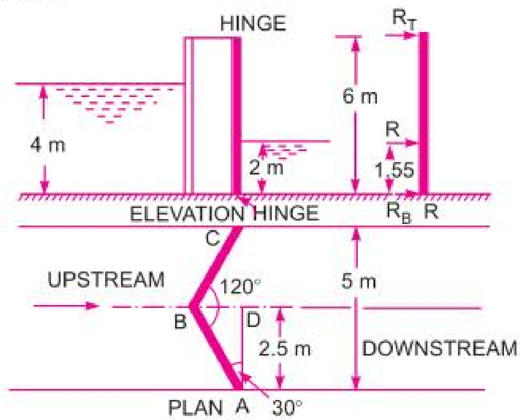


Fig. 3.41

$\therefore$ Total water pressure on upstream side

$$F_1 = \rho g A_1 \bar{h}_1, \text{ where } A_1 = H_1 \times l = 4.0 \times 2.887 \text{ m}^2$$

$$= 1000 \times 9.81 \times 4 \times 2.887 \times 2.0 \qquad \left\{\bar{h}_1 = \frac{H_1}{2} = \frac{4}{2} = 2.0 \text{ m}\right\}$$

$$= 226571 \text{ N}$$

Force F_1 will be acting at a distance of $\frac{H_1}{3} = \frac{4}{3} = 1.33$ m from bottom.

Similarly, total water pressure on the downstream side

$$F_2 = \rho g A_2 \bar{h}_2, \text{ where } A_2 = H_2 \times l = 2 \times 2.887 \text{ m}^2$$

$$= 1000 \times 9.81 \times 2 \times 2.887 \times 1.0 \qquad \bar{h}_2 = \frac{H_2}{2} = \frac{2}{2} = 1.0 \text{ m}$$

$= 56643$ N

F_2 will act at a distance of $\frac{H_2}{3} = \frac{2}{3} = 0.67$ m from bottom,

Resultant water pressure on each gate

$$F = F_1 - F_2 = 226571 - 56643 = 169928 \text{ N.}$$

Let x is height of F from the bottom, then taking moments of F_1, F_2 and F about the bottom, we have

$$F \times x = F_1 \times 1.33 - F_2 \times 0.67$$

or
$$169928 \times x = 226571 \times 1.33 - 56643 \times 0.67$$

$\therefore$
$$x = \frac{226571 \times 1.33 - 56643 \times 0.67}{169928} = \frac{301339 - 37950}{169928} = 1.55 \text{ m}$$

From equation (3.20), $P = \frac{F}{2 \sin \theta} = \frac{169928}{2 \sin 30} = 169928$ N.

From equation (3.19), $R = P = 169928$ N.

If R_T and R_B are the reactions at the top and bottom hinges, then $R_T + R_B = R = 169928$ N.

Taking movements of hinge reactions R_T, R_B and R about the bottom hinges, we have

$$R_T \times 6.0 + R_B \times 0 = R \times 1.55$$

$\therefore$
$$R_T = \frac{169928 \times 1.55}{6.0} = 43898 \text{ N}$$

$\therefore$
$$R_B = R - R_T = 169928 - 43898 = \mathbf{126030 \text{ N.}} \text{ Ans.}$$

Problem 3.33 *The end gates ABC of a lock are 9 m high and when closed include an angle of 120°. The width of the lock is 10 m. Each gate is supported by two hinges located at 1 m and 6 m above the bottom of the lock. The depths of water on the two sides are 8 m and 4 m respectively. Find:*

(i) Resultant water force on each gate,

(ii) Reaction between the gates AB and BC, and

(iii) Force on each hinge, considering the reaction of the gate acting in the same horizontal plane as resultant water pressure.

Solution. Given :

Height of gate $= 9$ m

Inclination of gate $= 120°$

$\therefore$
$$\theta = \frac{180° - 120°}{2} = 30°$$

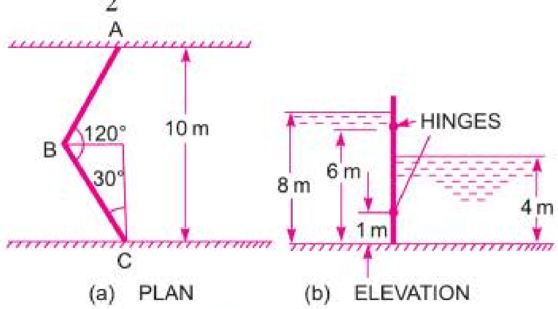


Fig. 3.42

Width of lock $= 10$ m

$\therefore$ Width of each lock $= \dfrac{5}{\cos 30^\circ}$ or $l = 5.773$ m

Depth of water on upstream side, $H_1 = 8$ m

Depth of water on downstream side, $H_2 = 4$ m

(*i*) **Water pressure on upstream side**

$$F_1 = \rho g A_1 \bar{h}_1$$

where $A_1 = l \times H_1 = 5.773 \times 8 = 46.184$ m, $\bar{h}_1 = \dfrac{H_1}{2} = \dfrac{8}{2} = 4.0$ m

$$F_1 = 1000 \times 9.81 \times 46.184 \times 4.0 = 1812260 \text{ N} = 1812.26 \text{ kN}$$

Water pressure on downstream side,

$$F_2 = \rho g A_2 \bar{h}_2$$

where $A_2 = l \times H_2 = 5.773 \times 4 = 23.092$ m, $\bar{h}_2 = \dfrac{4}{2} = 2.0$

$$F_2 = 1000 \times 9.81 \times 23.092 \times 2.0 = 453065 \text{ N} = 453.065 \text{ kN}$$

$\therefore$ Resultant water pressure

$$= F_1 - F_2 = 1812.26 - 453.065 = 1359.195 \text{ kN}$$

(*ii*) **Reaction between the gates AB and BC.** The reaction (P) between the gates AB and BC is given by equation (3.20) as

$$F = \frac{F}{2\sin\theta} = \frac{1359.195}{2 \times \sin 30^\circ} = \textbf{1359.195 kN. Ans.}$$

(*iii*) **Force on each hinge.** If R_T and R_B are the reactions at the top and bottom hinges then

$$R_T + R_B = R$$

But from equation (3.19), $R = P = 1359.195$

$\therefore$ $R_T + R_B = 1359.195$

The force F_1 is acting at $\dfrac{H_1}{3} = \dfrac{8}{3} = 2.67$ m from bottom and F_2 at $\dfrac{H_2}{3} = \dfrac{4}{3} = 1.33$ m from bottom.

The resultant force F will act at a distance x from bottom is given by

$$F \times x = F_1 \times 2.67 - F_2 \times 1.33$$

or

$$x = \frac{F_1 \times 2.67 - F_2 \times 1.33}{F} = \frac{1812.26 \times 2.67 - 453.065 \times 1.33}{1359.195}$$

$$= \frac{4838.734 - 602.576}{1359.195} = 3.116 = 3.11 \text{ m}$$

Hence R is also acting at a distance 3.11 m from bottom.

Taking moments of R_T and R about the bottom hinge

$$R_T \times [6.0 - 1.0] = R \times (x - 1.0)$$

$$\therefore \quad R_T = \frac{R \times (x - 1.0)}{5.0} = \frac{1359.195 \times 2.11}{5.0} = 573.58 \text{ N}$$

$$\therefore \quad R_B = R - R_T = 1359.195 - 573.58$$

$$= \textbf{785.615 kN. Ans.}$$

▶ 3.8 PRESSURE DISTRIBUTION IN A LIQUID SUBJECTED TO CONSTANT HORIZONTAL/VERTICAL ACCELERATION

In chapters 2 and 3, the containers which contains liquids, are assumed to be at rest. Hence the liquids are also at rest. They are in static equilibrium with respect to containers. But if the container containing a liquid is made to move with a constant acceleration, the liquid particles initially will move relative to each other and after some time, there will not be any relative motion between the liquid particles and boundaries of the container. The liquid will take up a new position under the effect of acceleration imparted to its container. The liquid will come to rest in this new position relative to the container. The entire fluid mass moves as a single unit. Since the liquid after attaining a new position is in static condition relative to the container, the laws of hydrostatic can be applied to determine the liquid pressure. As there is no relative motion between the liquid particles, hence the shear stresses and shear forces between liquid particles will be zero. The pressure will be normal to the surface in contact with the liquid.

The following are the important cases under consideration :

(*i*) Liquid containers subject to constant horizontal acceleration.

(*ii*) Liquid containers subject to constant vertical acceleration.

3.8.1 Liquid Containers Subject to Constant Horizontal Acceleration. Fig. 3.43 (*a*) shows a tank containing a liquid upto a certain depth. The tank is stationary and free surface of liquid is horizontal. Let this tank is moving with a constant acceleration '*a*' in the horizontal direction towards right as shown in Fig. 3.43 (*b*). The initial free surface of liquid which was horizontal, now takes the shape as shown in Fig. 3.43 (*b*). Now *AB* represents the new free surface of the liquid. Thus the free surface of liquid due to horizontal acceleration will become a downward sloping inclined plane, with the liquid rising at the back end, the liquid falling at the front end. The equation for the free liquid surface can be derived by considering the equilibrium of a fluid element C lying on the free surface. The forces acting on the element *C* are :

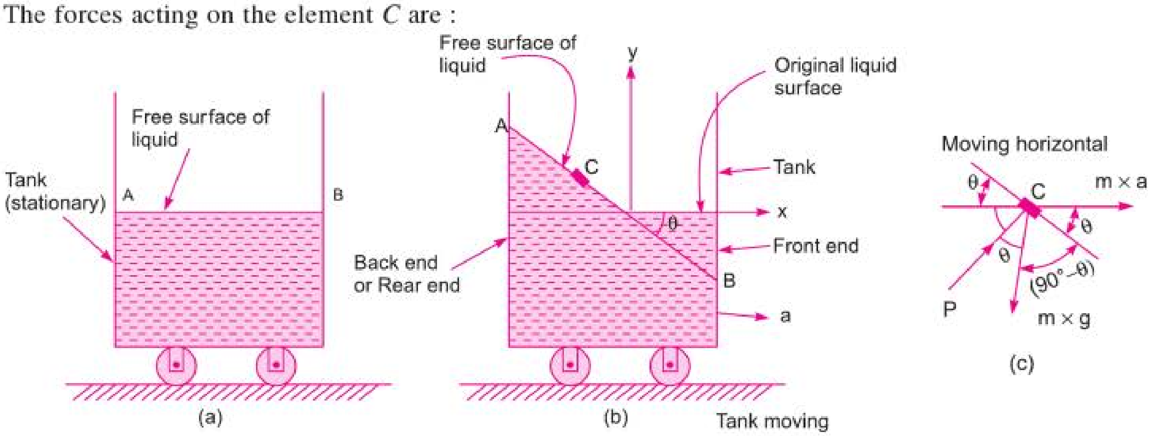


Fig. 3.43

(*i*) the pressure force *P* exerted by the surrounding fluid on the element *C*. This force is normal to the free surface.

(*ii*) the weight of the fluid element *i.e.*, $m \times g$ acting vertically downward.

(*iii*) accelerating force *i.e.*, $m \times a$ acting in horizontal direction.

Resolving the forces horizontally, we get

$$P \sin \theta + m \times a = 0$$

or $$P \sin \theta = -ma \qquad ...(i)$$

Resolving the forces vertically, we get

$$P \cos \theta - mg = 0$$

or $$P \cos \theta = m \times g \qquad ...(ii)$$

Dividing (*i*) by (*ii*), we get

$$\tan \theta = -\frac{a}{g} \quad \left(\text{or } \frac{a}{g} \text{ Numerically}\right) \qquad ...(3.20A)$$

The above equation, gives the slope of the free surface of the liquid which is contained in a tank which is subjected to horizontal constant acceleration. The term (a/g) is a constant and hence $\tan \theta$ will be constant. The –ve sign shows that the free surface of liquid is sloping downwards. Hence the free surface is a straight plane inclined down at an angle θ along the direction of acceleration.

Now let us find the expression for the pressure at any point D in the liquid mass subjected to horizontal acceleration. Let the point D is at a depth of 'h' from the free surface. Consider an elementary prism DE of height 'h' and cross-sectional area dA as shown in Fig. 3.44.

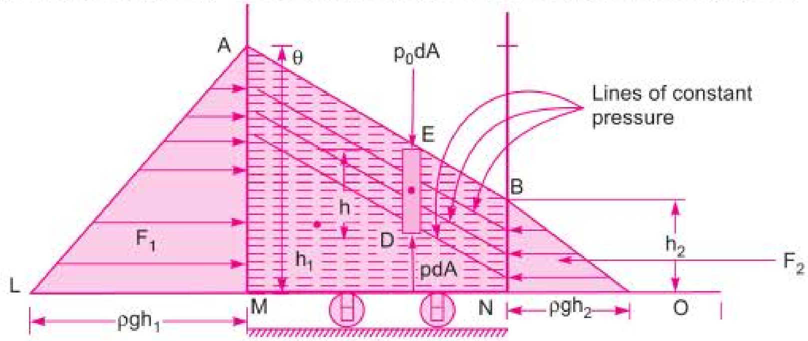


Fig. 3.44

Consider the equilibrium of the elementary prism DE.

The forces acting on this prism DE in the vertical direction are :

(*i*) the atmospheric pressure force $(p_0 \times dA)$ at the top end of the prism acting downwards,

(*ii*) the weight of the element $(\rho \times g \times h \times dA)$ at the C.G. of the element acting in the downward direction, and

(*iii*) the pressure force $(p \times dA)$ at the bottom end of the prism acting upwards.

Since there is no vertical acceleration given to the tank, hence net force acting vertically should be zero.

$$\therefore \quad p \times dA - p_0 \times dA - \rho g h \, dA = 0$$

or $$p - p_0 - \rho g h = 0 \quad \text{or} \quad p = p_0 + \rho g h$$

or $$p - p_0 = \rho g h$$

or gauge pressure at point D is given by

$$p = \rho g h$$

or pressure head at point D, $$\frac{p}{\rho g} = h.$$

From the above equation, it is clear that pressure head at any point in a liquid subjected to a constant horizontal acceleration is equal to the height of the liquid column above that point. Therefore the pressure distribution in a liquid subjected to a constant horizontal acceleration is same as hydrostatic pressure distribution. The planes of constant pressure are therefore, parallel to the inclined surface as shown in Fig. 3.44. This figure also shows the variation of pressure on the rear and front end of the tank.

If h_1 = Depth of liquid at the rear end of the tank

h_2 = Depth of liquid at the front end of the tank

F_1 = Total pressure force exerted by liquid on the rear side of the tank

F_2 = Total pressure force exerted by liquid on the front side of the tank,

then F_1 = (Area of triangle AML) × Width

$$= (\tfrac{1}{2} \times LM \times AM \times b) = \tfrac{1}{2} \times \rho g h_1 \times h_1 \times b = \frac{\rho g . b . h_1^2}{2}$$

and F_2 = (Area of triangle BNO) × Width

$$= (\tfrac{1}{2} \times BN \times NO) = \tfrac{1}{2} \times h_2 \times \rho g h_2 \times b = \frac{\rho g . b . h_2^2}{2}$$

where b = Width of tank perpendicular to the plane of the paper.

The values of F_1 and F_2 can also be obtained as [Refer to Fig. 3.44 (a)]

$$F_1 = \rho \times g \times A_1 \times \bar{h}_1, \text{ where } A_1 = h_1 \times b \text{ and } \bar{h}_1 = \frac{h_1}{2}$$

$$= \rho \times g \times (h_1 \times b) \times \frac{h_1}{2} = \frac{1}{2} \rho g . b . h_1^2$$

and

$$F_2 = \rho \times g \times A_2 \times \bar{h}_2, \text{ where } A_2 = h_2 \times b \text{ and } \bar{h}_2 = \frac{h_2}{2}$$

$$= \rho \times g \times (h_2 \times b) \times \frac{h_2}{2}$$

$$= \frac{1}{2} \rho g . b \times h_2^2.$$

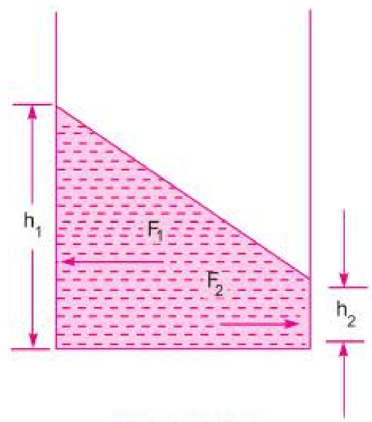


Fig. 3.44(a)

It can also be proved that the difference of these two forces (*i.e.*, $F_1 - F_2$) is equal to the force required to accelerate the mass of the liquid contained in the tank *i.e.*,

$$F_1 - F_2 = M \times a$$

where M = Total mass of the liquid contained in the tank

a = Horizontal constant acceleration.

Note : (i) If a tank completely filled with liquid and open at the top is subjected to a constant horizontal acceleration, then some of the liquid will spill out from the tank and new free surface with its slope given by equation $\tan \theta = -\frac{a}{g}$ will be developed.

(ii) If a tank partly filled with liquid and open at the top is subjected to a constant horizontal acceleration, spilling of the liquid may take place depending upon the magnitude of the acceleration.

(iii) If a tank completely filled with liquid and closed at the top is subjected to a constant horizontal acceleration, then the liquid would not spill out from the tank and also there will be no adjustment in the surface elevation of the liquid. But the equation $\tan \theta = -\frac{a}{g}$ is applicable for this case also.

(iv) The example for a tank with liquid subjected to a constant horizontal acceleration, is a fuel tank on an airplane during take off.

Problem 3.34 *A rectangular tank is moving horizontally in the direction of its length with a constant acceleration of 2.4 m/s². The length, width and depth of the tank are 6 m, 2.5 m and 2 m respectively. If the depth of water in the tank is 1 m and tank is open at the top then calculate :*

(*i*) *the angle of the water surface to the horizontal,*

(*ii*) *the maximum and minimum pressure intensities at the bottom,*

(*iii*) *the total force due to water acting on each end of the tank.*

Solution. Given :

Constant acceleration, $a = 2.4 \text{ m/s}^2$.

Length = 6 m ; Width = 2.5 m and depth = 2 m.

Depth of water in tank, $h = 1$ m

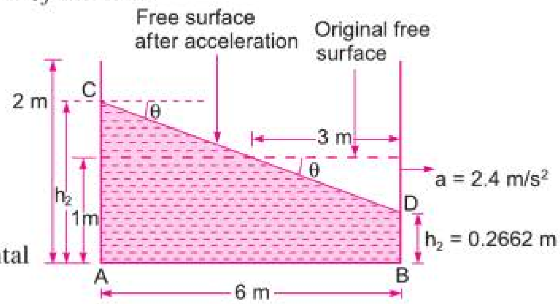


Fig. 3.45

(*i*) **The angle of the water surface to the horizontal**

Let θ = the angle of water surface to the horizontal

Using equation (3.20), we get

$$\tan\theta = -\frac{a}{g} = -\frac{2.4}{9.81} = -0.2446$$

(the –ve sign shows that the free surface of water is sloping downward as shown in Fig. 3.45)

$\therefore$ $\tan\theta = 0.2446$ (slope downward)

$\therefore$ $\theta = \tan^{-1} 0.2446 = \mathbf{13.7446°}$ or $\mathbf{13° \; 44.6'}$. **Ans.**

(*ii*) **The maximum and minimum pressure intensities at the bottom of the tank**

From the Fig. 3.45,

Depth of water at the front end,

$$h_1 = 1 - 3\tan\theta = 1 - 3 \times 0.2446 = 0.2662 \text{ m}$$

Depth of water at the rear end,

$$h_2 = 1 + 3\tan\theta = 1 + 3 \times 0.2446 = 1.7338 \text{ m}$$

The pressure intensity will be maximum at the bottom, where depth of water is maximum.

Now the maximum pressure intensity at the bottom will be at point *A* and it is given by,

$$p_{max} = \rho \times g \times h_2$$
$$= 1000 \times 9.81 \times 1.7338 \text{ N/m}^2 = \mathbf{17008.5 \text{ N/m}^2}. \textbf{ Ans.}$$

The minimum pressure intensity at the bottom will be at point B and it is given by

$$p_{min} = \rho \times g \times h_1$$
$$= 1000 \times 9.81 \times 0.2662 = \mathbf{2611.4 \text{ N/m}^2}. \textbf{ Ans.}$$

(*iii*) **The total force due to water acting on each end of the tank**

Let F_1 = total force acting on the front side (*i.e.*, on face *BD*)

F_2 = total force acting on the rear side (*i.e.*, on face *AC*)

Then $F_1 = \rho g A_1 \bar{h}_1$, where $A_1 = BD \times$ width of tank $= h_1 \times 2.5 = 0.2662 \times 2.5$

and $\bar{h}_1 = \dfrac{BD}{2} = \dfrac{h_1}{2} = \dfrac{0.2662}{2} = 0.1331$ m

$$= 1000 \times 9.81 \times (0.2662 \times 2.5) \times 0.1331$$
$$= \mathbf{868.95 \text{ N}}. \textbf{ Ans.}$$

and $F_2 = \rho.g.A_2.\bar{h}_2$, where $A_2 = AB \times$ width of tank $= h_2 \times 2.5 = 1.7338 \times 2.5$

$$\bar{h}_2 = \frac{AB}{2} = \frac{h_2}{2} = \frac{1.7338}{2} = 0.8669 \text{ m}$$

$$= 1000 \times 9.81 \times (1.7338 \times 2.5) \times 0.8669$$

$$= \mathbf{36861.8 \text{ N.}}$$ **Ans.**

$\therefore$ Resultant force $= F_1 - F_2$

$$= 36861.8 \text{ N} - 868.95$$

$$= 35992.88 \text{ N}$$

Note. The difference of the forces acting on the two ends of the tank is equal to the force necessary to accelerate the liquid mass. This can be proved as shown below :

Consider the control volume of the liquid *i.e.*, control volume is *ACDBA* as shown in Fig. 3.46. The net force acting on the control volume in the horizontal direction must be equal to the product of mass of the liquid in control volume and acceleration of the liquid.

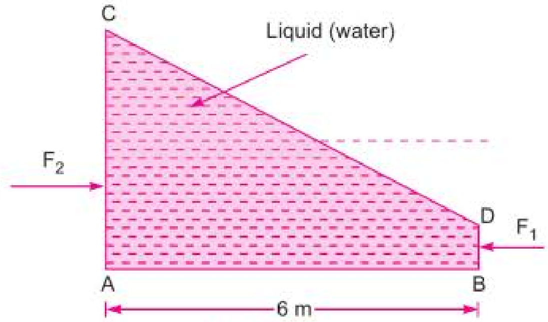


Fig. 3.46

$$\therefore \quad (F_1 - F_2) = M \times a$$

$$= (\rho \times \text{volume of control volume}) \times a$$

$$= (1000 \times \text{Area of } ABDCE \times \text{width}) \times 2.4$$

$$= \left[1000 \times \left(\frac{AC + BD}{2}\right) \times AB \times \text{width}\right] \times 2.4$$

$$\left[\because \text{ Area of trapezium} = \left(\frac{AC + BD}{2}\right) \times AB\right]$$

$$= 1000 \times \left(\frac{1.7338 + 0.2662}{2}\right) \times 6 \times 2.5 \times 2.4$$

$$= 36000 \text{ N}$$

($\because AC = h_2 = 1.7338$ m, $BD = h_1 = 0.2662$ m, and $AB = 6$ m, width = 2.5 m)

The above force is nearly the same as the difference of the forces acting on the two ends of the tank. (*i.e.*, $35992.88 \simeq 36000$).

Problem 3.35 *The rectangular tank of the above problem contains water to a depth of 1.5 m. Find the horizontal acceleration which may be imparted to the tank in the direction of its length so that*

(*i*) *the spilling of water from the tank is just on the verge of taking place,*

(*ii*) *the front bottom corner of the tank is just exposed,*

(*iii*) *the bottom of the tank is exposed upto its mid-point.*

Also calculate the total forces exerted by the water on each end of the tank in each case. Also prove that the difference between these forces is equal to the force necessary to accelerate the mass of water tank.

Solution. Given :

Dimensions of the tank from previous problem,

$$L = 6 \text{ m, width } (b) = 2.5 \text{ m and depth} = 2 \text{ m}$$

Depth of water in tank, $h = 1.5$ m

Horizontal acceleration imparted to the tank

(*i*) (*a*) *When the spilling of water from the tank is just on the verge of taking place*

Let a = required horizontal acceleration

When the spilling of water from the tank is just on the verge of taking place, the water would rise upto the rear top corner of the tank as shown in Fig. 3.47 (*a*)

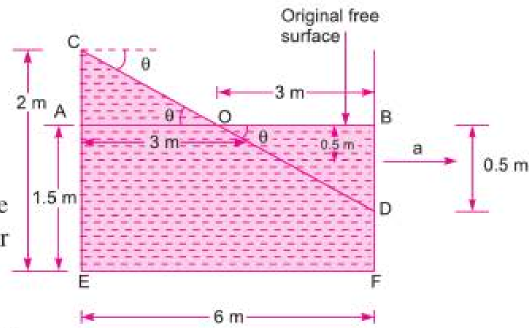


Fig. 3.47 (*a*) *Spilling of water is just on the verge of taking place.*

$$\therefore \qquad \tan\theta = \frac{AC}{AO} = \frac{(2-1.5)}{3} = \frac{0.5}{3} = 0.1667$$

But from equation (3.20) $\tan\theta = \dfrac{a}{g}$ (Numerically)

$$\therefore \qquad a = g \times \tan\theta = 9.81 \times 0.1667 = \mathbf{1.635\ m/s^2}.\ \textbf{Ans.}$$

(*b*) *Total forces exerted by water on each end of the tank*

The force exerted by water on the end *CE* of the tank is

$$F_1 = \rho g A_1 \bar{h}_1, \text{ where } A_1 = CE \times \text{width of the tank} = 2 \times 2.5$$

$$\bar{h}_1 = \frac{CE}{2} = \frac{2}{2} = 1 \text{ m}$$

$$= 1000 \times 9.81 \times (2 \times 2.5) \times 1$$

$$= \mathbf{49050\ N.}\ \textbf{Ans.}$$

The force exerted by water on the end *FD* of the tank is

$$F_2 = \rho g A_2 \times \bar{h}_2, \text{ where } A_2 = FD \times \text{width} = 1 \times 2.5$$

$$(\because AC = BD = 0.5 \text{ m}, \quad \therefore \quad FD = BF - BD = 1.5 - 0.5 = 1)$$

$$= 1000 \times 9.81 \times (1 \times 2.5) \times 0.5 \qquad \bar{h}_2 = \frac{FD}{2} = \frac{1}{2} = 0.5 \text{ m}$$

$$= \mathbf{12262.5\ N.}\ \textbf{Ans.}$$

(*c*) *Difference of the forces is equal to the force necessary to accelerate the mass of water in the tank*

Difference of the forces $= F_1 - F_2$

$$= 49050 - 12262.5 = 36787.5 \text{ N}$$

Volume of water in the tank before acceleration is imparted to it $= L \times b \times$ depth of water

$$= 6 \times 2.5 \times 1.5 = 22.5 \text{ m}^3.$$

The force necessary to accelerate the mass of water in the tank

= Mass of water in tank × Acceleration

$= (\rho \times \text{volume of water}) \times 1.635$ $\qquad (\because a = 1.635 \text{ m/s}^2)$

$= 1000 \times 22.5 \times 1.635$ [There is no spilling of water and volume of water $= 22.5$ m^3]

$= 36787.5$ N

Hence the difference between the forces on the two ends of the tank is equal to the force necessary to accelerate the mass of water in the tank.

Volume of water in the tank can also be calculated as volume = $\left(\frac{CE + FD}{2}\right) \times EF \times$ Width [Refer to Fig. 3.47 (*a*)]

$$= \left(\frac{2+1}{2}\right) \times 6 \times 2.5 = 22.5 \text{ m}^3.$$

(*ii*) (*a*) *Horizontal acceleration when the front bottom corner of the tank is just exposed*

Refer to Fig. 3.47 (*b*). In this case the free surface of water in the tank will be along *CD*.

Let a = required horizontal acceleration.

In this case, $\tan\theta = \frac{CE}{ED} = \frac{2}{6} = \frac{1}{3}$

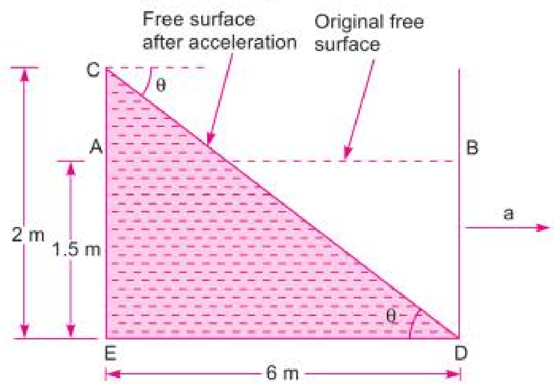


Fig. 3.47 (*b*)

But from equation (3.17),

$$\tan\theta = \frac{a}{g} \text{ (Numerically)}$$

$\therefore$ $a = g \times \tan\theta = 9.81 \times \frac{1}{3} = \mathbf{3.27\ m/s^2}$. **Ans.**

(*b*) *Total forces exerted by water on each end of the tank*

The force exerted by water on the end *CE* of the tank is

$$F_1 = \rho g \times A_1 \times \bar{h}_1$$

where $A_1 = CE \times \text{width} = 2 \times 2.5 = 5 \text{ m}^2$

$\bar{h}_1 = \frac{CE}{2} = \frac{2}{2} = 1$ m $\quad = 1000 \times 9.81 \times 5 \times 1$

= 49050 N. Ans.

The force exerted by water on the end *BD* of the tank is zero as there is no water against the face *BD*

$\therefore$ $F_2 = 0$

$\therefore$ Difference of forces = 49050 – 0 = 49050 N

(*c*) *Difference of forces is equal to the force necessary to accelerate the mass of water in the tank.*

Volume of water in the tank = Area of *CED* × Width of tank

$$= \left(\frac{CE \times ED}{2}\right) \times 2.5 \qquad (\because \text{ Width of tank} = 2.5 \text{ m})$$

$$= \frac{2 \times 6}{2} \times 2.5 = 15 \text{ m}^3$$

$\therefore$ Force necessary to accelerate the mass of water in the tank

= Mass of water in tank × Acceleration

= (1000 × Volume of water) × 3.27

= 1000 × 15 × 3.27 = 49050 N

Difference of two forces is also = 49050 N

Hence difference between the forces on the two ends of the tank is equal to the force necessary to accelerate the mass of water in the tank.

(*iii*) (*a*) *Horizontal acceleration when the bottom of the tank is exposed upto its mid-point*

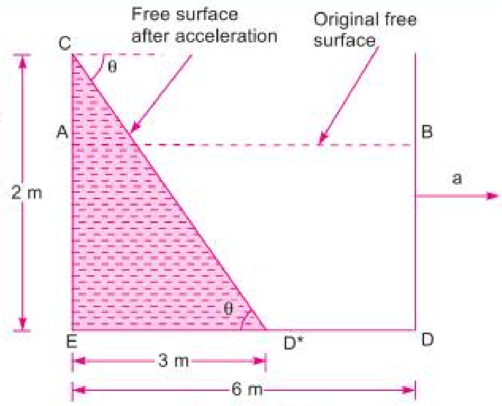


Fig. 3.47 (*c*)

Refer to Fig. 3.47 (*c*). In this case the free surface of water in the tank will be along CD^*, where D^* is the mid-point of ED.

Let a = required horizontal acceleration from Fig. 3.47 (*c*), it is clear that

$$\tan\theta = \frac{CE}{ED^*} = \frac{2}{3}$$

But from equation (3.20) numerically

$$\tan\theta = \frac{a}{g}$$

$$\therefore \qquad a = g \times \tan\theta = 9.81 \times \frac{2}{3} = \mathbf{6.54\ m/s^2}.\ \textbf{Ans.}$$

(*b*) *Total forces exerted by water on each end of the tank*

The force exerted by water on the end CE of the tank is

$$F_1 = \rho \times g \times A_1 \times \bar{h}_1$$

where $A_1 = CE \times \text{Width} = 2 \times 2.5 = 5\ \text{m}^2$

$$\bar{h}_1 = \frac{CE}{2} = \frac{2}{2} = 1\ \text{m}$$

$$= 1000 \times 9.81 \times 5 \times 1$$

$$= \mathbf{49050\ N.}\ \textbf{Ans.}$$

The force exerted by water on the end BD is zero as there is no water against the face BD.

$$\therefore \qquad F_2 = 0$$

$\therefore$ Difference of the forces $= F_1 - F_2 = 49050 - 0 = 49050$ N

(*c*) *Difference of the two forces is equal to the force necessary to accelerate the mass of water remaining in the tank*

Volume of water in the tank = Area CED^* × Width of tank

$$= \frac{CE \times ED^*}{2} \times 2.5 = \frac{2 \times 3}{2} \times 2.5 = 7.5\ \text{m}^3$$

Force necessary to accelerate the mass of water in the tank

= Mass of water × Acceleration

$= \rho \times$ Volume of water $\times 6.54$ $\qquad (\because a = 6.54\ \text{m/s}^2)$

$= 1000 \times 7.5 \times 6.54$

$= 49050$ N

This is the same force as the difference of the two forces on the two ends of the tank.

Problem 3.36 *A rectangular tank of length 6 m, width 2.5 m and height 2 m is completely filled with water when at rest. The tank is open at the top. The tank is subjected to a horizontal constant linear acceleration of 2.4 m/s*^{2} *in the direction of its length. Find the volume of water spilled from the tank.*

Solution. Given :

$$L = 6 \text{ m}, b = 2.5 \text{ m and height}, H = 2 \text{ m}$$

Horizontal acceleration, $a = 2.4 \text{ m/s}^2$.

The slope of the free surface of water after the tank is subjected to linear constant acceleration is given by equation (3.20) as

$$\tan\theta = \frac{a}{g} \text{ (Numerically)}$$

$$= \frac{2.4}{9.81} = 0.2446$$

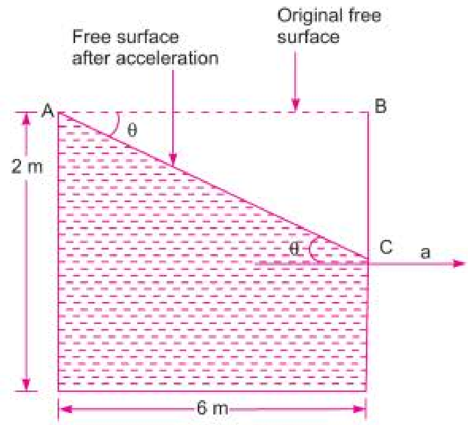


Fig. 3.48

From Fig. 3.48,

$$\tan\theta = \frac{BC}{AB}$$

$\therefore$ $$BC = AB \times \tan\theta$$

$$= 6 \times 0.2446$$

$(\because AB = \text{Length} = 6 \text{ m} ; \tan\theta = 0.2446)$

$$= 1.4676 \text{ m}$$

$\therefore$ Volume of water spilled = Area of ABC × Width of tank

$$= \left(\tfrac{1}{2} \times AB \times BC\right) \times 2.5 \quad (\because \text{ Width} = 2.5 \text{ m})$$

$$= \tfrac{1}{2} \times 6 \times 1.4676 \times 2.5 \quad (\because BC = 1.4676 \text{ m})$$

$$= \mathbf{11.007 \text{ m}^3}. \text{ Ans.}$$

3.8.2 Liquid Container Subjected to Constant Vertical Acceleration.

Fig. 3.49 shows a tank containing a liquid and the tank is moving vertically upward with a constant acceleration. The liquid in the tank will be subjected to the same vertical acceleration. To obtain the expression for the pressure at any point in the liquid mass subjected to vertical upward acceleration, consider a vertical elementary prism of liquid $CDFE$.

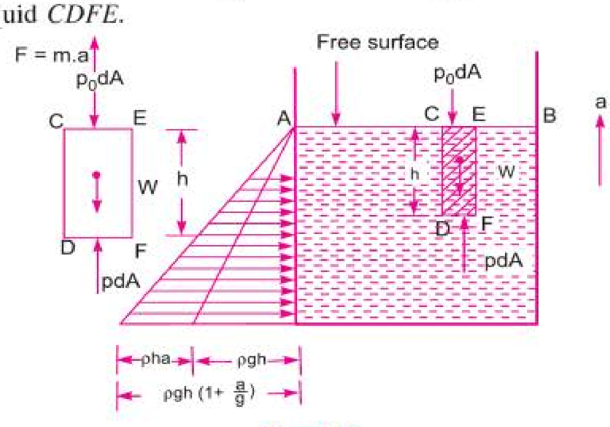


Fig. 3.49

Let dA = Cross-sectional area of prism

h = Height of prism

p_0 = Atmospheric pressure acting on the face CE

p = Pressure at a depth h acting on the face DF

The forces acting on the elementary prism are :

(*i*) Pressure force equal to $p_0 \times dA$ acting on the face *CE* vertically downward

(*ii*) Pressure force equal to $p \times dA$ acting on the face *DF* vertically upward

(*iii*) Weight of the prism equal to $\rho \times g \times dA \times h$ acting through C.G. of the element vertically downward.

According to Newton's second law of motion, the net force acting on the element must be equal to mass multiplied by acceleration in the same direction.

$\therefore$ Net force in vertically upward direction = Mass $\times$ acceleration

$$p \times dA - p_0 \times dA - \rho g dA \,.\, h = (\rho \times dA \times h) \times a \qquad (\because \text{Mass} = \rho \times dA \times h)$$

or
$$p - p_0 - \rho g h = \rho h \times a \qquad (\text{Cancelling } dA \text{ from both sides})$$

or
$$p - p_0 = \rho g h + \rho h a$$

$$= \rho g h \left[1 + \frac{a}{g}\right] \qquad ...(3.21)$$

But $(p - p_0)$ is the gauge pressure. Hence gauge pressure at any point in the liquid mass subjected to a constant vertical upward acceleration, is given by

$$p_g = \rho g h \left[1 + \frac{a}{g}\right] \qquad ...(3.22)$$

$$= \rho g h + \rho h a \qquad ...(3.22A)$$

where $p_g = p - p_0$ = gauge pressure

In equation (3.22) ρ, g and a are constant. Hence variation of guage pressure is linear. Also when $h = 0$, $p_g = 0$. This means $p - p_0 = 0$ or $p = p_0$. Hence when $h = 0$, the pressure is equal to atmospheric pressure. Hence free surface of liquid subjected to constant vertical acceleration will be horizontal.

From equation (3.22*A*) it is also clear that the pressure at any point in the liquid mass is greater than the hydrostatic pressure (hydrostatic pressure is = $\rho g h$) by an amount of $\rho \times h \times a$.

Fig. 3.49 shows the variation of pressure for the liquid mass subjected to a constant vertical upward acceleration.

If the tank containing liquid is moving vertically downward with a constant acceleration, then the gauge pressure at any point in the liquid at a depth of h from the free surface will be given by

$$(p - p_0) = \rho g h \left[1 - \frac{a}{g}\right] = \rho g h - \rho h a \qquad ...(3.23)$$

The above equation shows that the pressure at any point in the liquid mass is less than the hydrostatic pressure by an amount of $\rho h a$. Fig. 3.50 shows the variation of pressure for the liquid mass subjected to a constant vertical downward acceleration.

If the tank containing liquid is moving downward with a constant acceleration equal to g (*i.e.,* when $a = g$), then equation reduces to $p - p_0 = 0$ or $p = p_0$. This means the pressure at any point in the liquid is equal to surrounding atmospheric pressure. There will be no force on the walls or on the base of the tank.

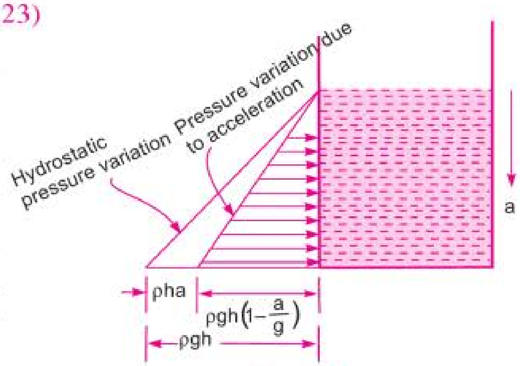


Fig. 3.50

Note. If a tank containing a liquid is subjected to a constant acceleration in the inclined direction, then the acceleration may be resolved along the horizontal direction and vertical direction. Then each of these cases may be separately analysed in accordance with the above procedure.

Problem 3.37 *A tank containing water upto a depth of 500 mm is moving vertically upward with a constant acceleration of 2.45 m/s². Find the force exerted by water on the side of the tank. Also calculate the force on the side of the tank when the width of tank is 2 m and*

(i) tank is moving vertically downward with a constant acceleration of 2.45 m/s², and

(ii) the tank is not moving at all.

Solution. Given :

Depth of water, $h = 500 \text{ mm} = 0.5 \text{ m}$

Vertical acceleration, $a = 2.45 \text{ m/s}^2$

Width of tank, $b = 2 \text{ m}$

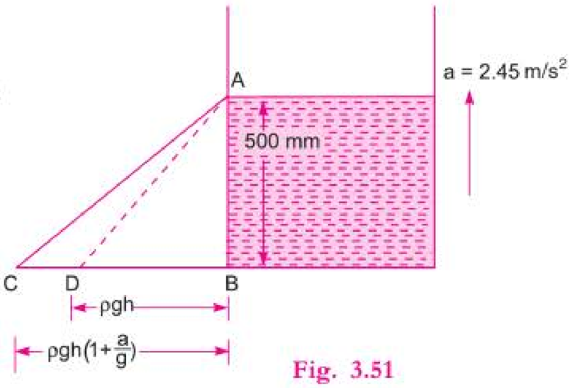


Fig. 3.51

To find the force exerted by water on the side of the tank when moving vertically upward, let us first find the pressure at the bottom of the tank.

The gauge pressure at the bottom (*i.e.*, at point *B*) for this case is given by equation as

$$p_B = \rho g h \left(1 + \frac{a}{g}\right)$$

$$= 1000 \times 9.81 \times 0.5 \left(1 + \frac{2.45}{9.81}\right) = 6131.25 \text{ N/m}^2$$

This pressure is represented by line *BC*.

Now the force on the side AB = Area of triangle $ABC \times$ Width of tank

$$= \left(\tfrac{1}{2} \times AB \times BC\right) \times b$$

$$= \left(\tfrac{1}{2} \times 0.5 \times 6131.25\right) \times 2 \qquad (\because BC = 6131.25 \text{ and } b = 2 \text{ m})$$

= 3065.6 N. Ans.

(*i*) **Force on the side of the tank, when tank is moving vertically downward.**

The pressure variation is shown in Fig. 3.52. For this case, the pressure at the bottom of the tank (*i.e.*, at point *B*) is given by equation (3.23) as

$$p_B = \rho g h \left(1 - \frac{a}{g}\right)$$

$$= 1000 \times 9.81 \times 0.5 \left(1 - \frac{2.45}{9.81}\right)$$

$$= 3678.75 \text{ N/m}^2$$

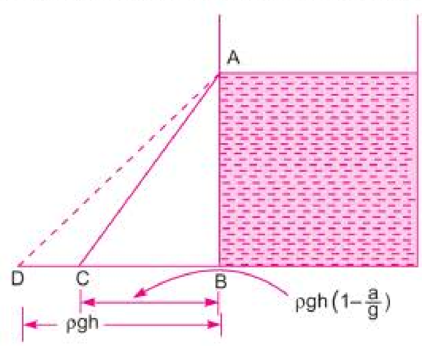


Fig. 3.52

This pressure is represented by line *BC*.

Now the force on the side AB = Area of triangle $ABC \times$ Width

$$= \left(\tfrac{1}{2} \times AB \times BC\right) \times b$$

$$= \left(\tfrac{1}{2} \times 0.5 \times 3678.75\right) \times 2 \qquad (\because BC = 3678.75,\ b = 2)$$

= 1839.37 N. Ans.

(*ii*) **Force on the side of the tank, when tank is stationary.**

The pressure at point B is given by,

$$p_B = \rho g h = 1000 \times 9.81 \times 0.5 = 4905 \text{ N/m}^2$$

This pressure is represented by line BD in Fig. 3.52

Force on the side AB = Area of triangle $ABD \times$ Width

$$= \left(\tfrac{1}{2} \times AB \times BD\right) \times b$$

$$= \left(\tfrac{1}{2} \times 0.5 \times 4905\right) \times 2 \qquad (\because BD = 4905)$$

$$= \mathbf{2452.5 \text{ N. Ans.}}$$

For this case, the force on AB can also be obtained as

$$F_{AB} = \rho g A.\bar{h}$$

where $A = AB \times \text{Width} = 0.5 \times 2 = 1 \text{ m}^2$

$$\bar{h} = \frac{AB}{2} = \frac{0.5}{2} = 0.25 \text{ m} = 1000 \times 9.81 \times 1 \times 0.25$$

$$= \mathbf{2452.5 \text{ N. Ans.}}$$

Problem 3.38 *A tank contains water upto a depth of 1.5 m. The length and width of the tank are 4 m and 2 m respectively. The tank is moving up an inclined plane with a constant acceleration of 4 m/s². The inclination of the plane with the horizontal is 30° as shown in Fig. 3.53. Find,*

(i) the angle made by the free surface of water with the horizontal.

(ii) the pressure at the bottom of the tank at the front and rear ends.

Solution. Given :

Depth of water, $h = 1.5$ m ; Length, $L = 4$ m and Width, $b = 2$ m

Constant acceleration along the inclined plane,

$$a = 4 \text{ m/s}^2$$

Inclination of plane, $\alpha = 30°$

Let θ = Angle made by the free surface of water after the acceleration is imparted to the tank

p_A = Pressure at the bottom of the tank at the front end and p_D = Pressure at the bottom of the tank at the rear end.

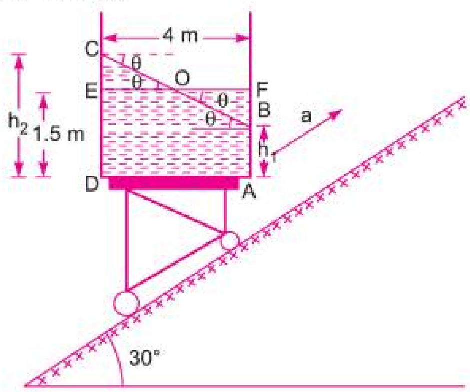


Fig. 3.53

This problem can be done by resolving the given acceleration along the horizontal direction and vertical direction. Then each of these cases may be separately analysed according to the set procedure.

Horizontal and vertical components of the acceleration are :

$$a_x = a \cos \alpha = 4 \cos 30° = 3.464 \text{ m/s}^2$$

$$a_y = a \sin \alpha = 4 \sin 30° = 2 \text{ m/s}^2$$

When the tank is stationary on the inclined plane, free surface of liquid will be along EF as shown in Fig. 3.53. But when the tank is moving upward along the inclined plane the free surface of liquid will be along BC. When the tank containing a liquid is moving up an inclined plane with a constant acceleration, the angle made by the free surface of the liquid with the horizontal is given by

$$\tan\theta = \frac{a_x}{a_y + g} = \frac{3.464}{2 + 9.81} = 0.2933$$

$\therefore$ $\theta = \tan^{-1} 0.2933 = \mathbf{16.346°}$ **or** $\mathbf{16° \; 20.8'}$. **Ans.**

Now let us first find the depth of liquid at the front and rear end of the tank.

Depth of liquid at front end $= h_1 = AB$

Depth of liquid at rear end $= h_2 = CD$

From Fig. 3.53, in triangle COE, $\tan\theta = \dfrac{CE}{EO}$

or $CE = EO \tan\theta = 2 \times 0.2933$ $\quad$ ($\because$ $EO = 2$ m, $\tan\theta = 0.2933$)

$= 0.5866$ m

$\therefore$ $CD = h_2 = ED + CE = 1.5 + 0.5866 = 2.0866$ m

Similarly $h_1 = AB = AF - BF$

$= 1.5 - 0.5866$ $\quad$ ($\because$ $AF = 1.5$, $BF = CE = 0.5866$)

$= 0.9134$ m

The pressure at the bottom of tank at the rear end is given by,

$$p_D = \rho g h_2 \left(1 + \frac{a_y}{g}\right)$$

$$= 1000 \times 9.81 \times 2.0866 \left(1 + \frac{2}{9.81}\right) = \mathbf{24642.7\ N/m^2}. \ \textbf{Ans.}$$

The pressure at the bottom of tank at the front end is given by

$$p_A = \rho g h_1 \left(1 + \frac{a_y}{g}\right)$$

$$= 1000 \times 9.81 \times 0.9134 \left(1 + \frac{2}{9.81}\right) = \mathbf{10787.2\ N/m^2}. \ \textbf{Ans.}$$

HIGHLIGHTS

1. When the fluid is at rest, the shear stress is zero.
2. The force exerted by a static fluid on a vertical, horizontal or an inclined plane immersed surface,

$$F = \rho g A \bar{h}$$

where ρ = Density of the liquid,

A = Area of the immersed surface, and

$\bar{h}$ = Depth of the centre of gravity of the immersed surface from free surface of the liquid.

3. Centre of pressure is defined as the point of application of the resultant pressure.
4. The depth of centre of pressure of an immersed surface from free surface of the liquid,

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h} \qquad \text{for vertically immersed surface.}$$

$$= \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h} \qquad \text{for inclined immersed surface.}$$

5. The centre of pressure for a plane vertical surface lies at a depth of two-third the height of the immersed surface.

6. The total force on a curved surface is given by $F = \sqrt{F_x^2 + F_y^2}$

where F_x = Horizontal force on curved surface and is equal to total pressure force on the projected area of the curved surface on the vertical plane,

$= \rho g A \bar{h}$

and F_y = Vertical force on sub-merged curved surface and is equal to the weight of liquid actually or imaginary supported by the curved surface.

7. The inclination of the resultant force on curved surface with horizontal, $\tan \theta = \dfrac{F_y}{F_x}$.

8. The resultant force on a sluice gate, $F = F_1 - F_2$

where F_1 = Pressure force on the upstream side of the sluice gate and

F_2 = Pressure force on the downstream side of the sluice gate.

9. For a lock gate, the reaction between the two gates is equal to the reaction at the hinge, $R = P$.

Also the reaction between the two gates, $P = \dfrac{F}{2 \sin \theta}$

where F = Resultant water pressure on the lock gate = $F_1 - F_2$

and θ = Inclination of the gate with the normal to the side of the lock.

EXERCISE

(A) THEORETICAL PROBLEMS

1. What do you understand by 'Total Pressure' and 'Centre of Pressure' ?
2. Derive an expression for the force exerted on a sub-merged vertical plane surface by the static liquid and locate the position of centre of pressure.
3. Prove that the centre of pressure of a completely sub-merged plane surface is always below the centre of gravity of the sub-merged surface or at most coincide with the centre of gravity when the plane surface is horizontal.
4. Prove that the total pressure exerted by a static liquid on an inclined plane sub-merged surface is the same as the force exerted on a vertical plane surface as long as the depth of the centre of gravity of the surface is unaltered.
5. Derive an expression for the depth of centre of pressure from free surface of liquid of an inclined plane surface sub-merged in the liquid.
6. (*a*) How would you determine the horizontal and vertical components of the resultant pressure on a sub-merged curved surface ?
   (*b*) Explain the procedure of finding hydrostatic forces on curved surfaces.
   (Delhi University, Dec. 2002)
7. Explain how you would find the resultant pressure on a curved surface immersed in a liquid.
8. Why the resultant pressure on a curved sub-merged surface is determined by first finding horizontal and vertical forces on the curved surface ? Why is the same method not adopted for a plane inclined surface sub-merged in a liquid ?

9. Describe briefly with sketches the various methods used for measuring pressure exerted by fluids.
10. Prove that the vertical component of the resultant pressure on a sub-merged curved surface is equal to the weight of the liquid supported by the curved surface.
11. What is the difference between sluice gate and lock gate ?
12. Prove that the reaction between the gates of a lock is equal to the reaction at the hinge.
13. Derive an expression for the reaction between the gates as $P = \dfrac{F}{2 \sin \theta}$

    where F = Resultant water pressure on lock gate, θ = inclination of the gate with normal to the side of the lock.
14. When will centre of pressure and centre of gravity of an immersed plane surface coincide ?
15. Find an expression for the force exerted and centre of pressure for a completely sub-merged inclined plane surface. Can the same method be applied for finding the resultant force on a curved surface immersed in the liquid ? If not, why ?
16. What do you understand by the hydrostatic equation ? With the help of this equation derive the expressions for the total thrust on a sub-merged plane area and the buoyant force acting on a sub-merged body.

(B) NUMERICAL PROBLEMS

1. Determine the total pressure and depth of centre of pressure on a plane rectangular surface of 1 m wide and 3 m deep when its upper edge is horizontal and (*a*) coincides with water surface (*b*) 2 m below the free water surface. [**Ans.** (*a*) 44145 N, 2.0 m, (*b*) 103005 N, 3.714 m]
2. Determine the total pressure on a circular plate of diameter 1.5 m which is placed vertically in water in such a way that centre of plate is 2 m below the free surface of water. Find the position of centre of pressure also. [**Ans.** 34668.54 N, 2.07 m]
3. A rectangular sluice gate is situated on the vertical wall of a lock. The vertical side of the sluice is 6 m in length and depth of centroid of area is 8 m below the water surface. Prove that the depth of centre of pressure is given by 8.475 m.
4. A circular opening, 3 m diameter, in a vertical side of a tank is closed by a disc of 3 m diameter which can rotate about a horizontal diameter. Calculate : (*i*) the force on the disc, and (*ii*) the torque required to maintain the disc in equilibrium in the vertical position when the head of water above the horizontal diameter is 6 m. [**Ans.** (*i*) 416.05 kN, (*ii*) 39005 Nm]
5. The pressure at the centre of a pipe of diameter 3 m is 29.43 N/cm^2. The pipe contains oil of sp. gr. 0.87 and is filled with a gate valve. Find the force exerted by the oil on the gate and position of centre of pressure. [**Ans.** 2.08 MN, .016 m below centre of pipe]
6. Determine the total pressure and centre of pressure on an isosceles triangular plate of base 5 m and altitude 5 m when the plate is immersed vertically in an oil of sp. gr. 0.8. The base of the plate is 1 m below the free surface of water. [**Ans.** 261927 N, 3.19 m]
7. The opening in a dam is 3 m wide and 2 m high. A vertical sluice gate is used to cover the opening. On the upstream of the gate, the liquid of sp. gr. 1.5, lies upto a height of 2.0 m above the top of the gate, whereas on the downstream side, the water is available upto a height of the top of the gate. Find the resultant force acting on the gate and position of centre of pressure. Assume that the gate is higher at the bottom. [**Ans.** 206010 N, 0.964 m above the hinge]

8. A caisson for closing the entrance to a dry dock is of trapezoidal form 16 m wide at the top and 12 m wide at the bottom and 8 m deep. Find the total pressure and centre of pressure on the caisson if the water on the outside is 1 m below the top level of the caisson and dock is empty. [**Ans.** 3.164 MN, 4.56 m below water surface]

9. A sliding gate 2 m wide and 1.5 m high lies in a vertical plane and has a co-efficient of friction of 0.2 between itself and guides. If the gate weighs one tonne, find the vertical force required to raise the gate if its upper edge is at a depth of 4 m from free surface of water. [**Ans.** 37768.5 N]

10. A tank contains water upto a height of 1 m above the base. An immiscible liquid of sp. gr. 0.8 is filled on the top of water upto 1.5 m height. Calculate : (*i*) total pressure on one side of the tank, (*ii*) the position of centre of pressure for one side of the tank, which is 3 m wide. [**Ans.** 76518 N, 1.686 m from top]

11. A rectangular tank 4 m long, 1.5 m wide contains water upto a height of 2 m. Calculate the force due to water pressure on the base of the tank. Find also the depth of centre of pressure from free surface. [**Ans.** 117720 N, 2 m from free surface]

12. A rectangular plane surface 1 m wide and 3 m deep lies in water in such a way that its plane makes an angle of 30° with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge of the plate is 2 m below the free water surface. [**Ans.** 80932.5 N, 2.318 m]

13. A circular plate 3.0 m diameter is immersed in water in such a way that the plane of the plate makes an angle of 60° with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge of the plate is 2 m below the free water surface. [**Ans.** 228.69 kN, 3.427 m from free surface]

14. A rectangular gate 6 m × 2 m is hinged at its base and inclined at 60° to the horizontal as shown in Fig. 3.54. To keep the gate in a stable position, a counter weight of 29430 N is attached at the upper end of the gate. Find the depth of water at which the gate begins to fall. Neglect the weight of the gate and also friction at the hinge and pulley. [**Ans.** 3.43 m]

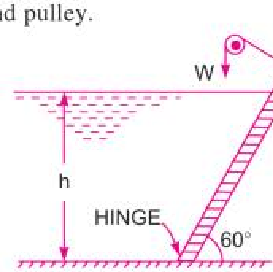


Fig. 3.54

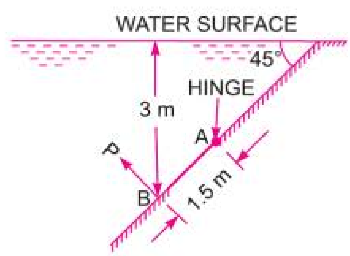


Fig. 3.55

15. An inclined rectangular gate of width 5 m and depth 1.5 m is installed to control the discharge of water as shown in Fig. 3.55. The end *A* is hinged. Determine the force normal to the gate applied at *B* to open it. [**Ans.** 97435.8 N]

16. A gate supporting water is shown in Fig. 3.56. Find the height '*h*' of the water so that the gate begins to tip about the hinge. Take the width of the gate as unity. [**Ans.** $3 \times \sqrt{3}$ m]

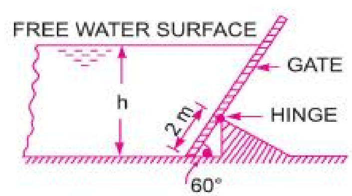


Fig. 3.56

17. Find the total pressure and depth of centre of pressure on a triangular plate of base 3 m and height 3 m which is immersed in water in such a way that plane of the plate makes an angle of 60° with the free surface. The base of the plate is parallel to water surface and at a depth of 2 m from water surface. [**Ans.** 126.52 kN, 2.996 m]

18. Find the horizontal and vertical components of the total force acting on a curved surface AB, which is in the form of a quadrant of a circle of radius 2 m as shown in Fig. 3.57. Take the width of the gate 2 m. [**Ans.** $F_x = 117.72$ kN, $F_y = 140.114$ kN]

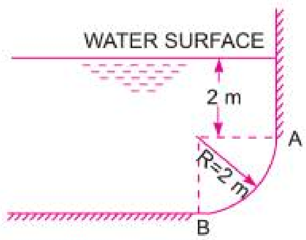


Fig. 3.57

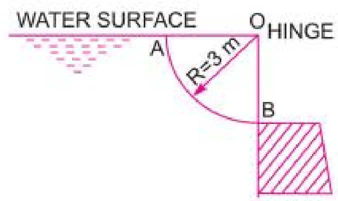


Fig. 3.58

19. Fig. 3.58 shows a gate having a quadrant shape of radius of 3 m. Find the resultant force due to water per metre length of the gate. Find also the angle at which the total force will act. [**Ans.** 82.201 kN, $\theta = 57° \, 31'$]

20. A roller gate is shown in Fig. 3.59. It is cylindrical form of 6.0 m diameter. It is placed on the dam. Find the magnitude and direction of the resultant force due to water acting on the gate when the water is just going to spill. The length of the gate is given 10 m. [**Ans.** 2.245 MN, $\theta = 38° \, 8'$]

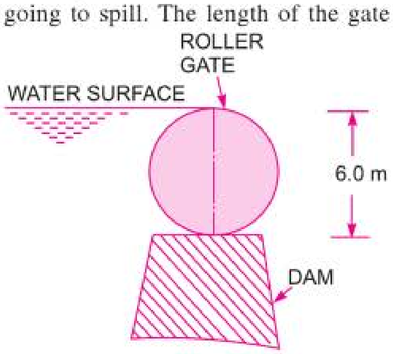


Fig. 3.59

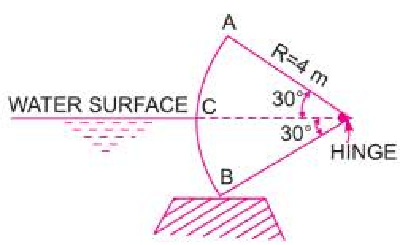


Fig. 3.60

21. Find the horizontal and vertical components of the water pressure exerted on a tainter gate of radius 4 m as shown in Fig. 3.60. Consider width of the gate unity. [**Ans.** $F_x = 19.62$ kN, $F_y = 7102.44$ N]

22. Find the magnitude and direction of the resultant water pressure acting on a curved face of a dam which is shaped according to the relation $y = \dfrac{x^2}{6}$ as shown in Fig. 3.61. The height of water retained by the dam is 12 m. Take the width of dam as unity. [**Ans.** 970.74 kN, $\theta = 43° \, 19'$]

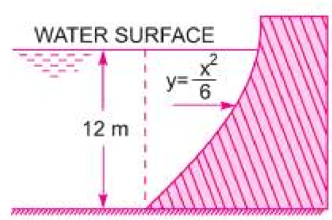


Fig. 3.61

23. Each gate of a lock is 5 m high and is supported by two hinges placed on the top and bottom of the gate. When the gates are closed, they make an angle of 120°. The width of the lock is 4 m. If the depths of water on the two sides of the gates are 4 m and 3 m respectively, determine : (*i*) the magnitude of resultant pressure on each gate, and (*ii*) magnitude of the hinge reactions. [**Ans.** (*i*) 79.279 kN, (*ii*) $R_T = 27.924$ kN, $R_B = 51.355$ kN]

24. The end gates *ABC* of a lock are 8 m high and when closed make an angle of 120°. The width of lock is 10 m. Each gate is supported by two hinges located at 1 m and 5 m above the bottom of the lock. The depth of water on the upstream and downstream sides of the lock are 6 m and 4 m respectively. Find :

(*i*) Resultant water force on each gate.

(*ii*) Reaction between the gates *AB* and *BC*, and
(*iii*) Force on each hinge, considering the reaction of the gate acting in the same horizontal plane as resultant water pressure. [**Ans.** 566.33 kN, (*ii*) 566.33 kN, and (*iii*) $R_T = 173.64$ kN, $R_B = 392.69$ kN]

25. A hollow circular plate of 2 m external and 1 m internal diameter is immersed vertically in water such that the centre of plate is 4 m deep from water surface. Find the total pressure and depth of centre of pressure. [**Ans.** 92.508 kN, 4.078 m]

26. A rectangular opening 2 m wide and 1 m deep in the vertical side of a tank is closed by a sluice gate of the same size. The gate can turn about the horizontal centroidal axis. Determine : (*i*) the total pressure on the sluice gate and (*ii*) the torque on the sluice gate. The head of water above the upper edge of the gate is 1.5 m. [**Ans.** (*i*) 39.24 kN, (*ii*) 1635 Nm]

27. Determine the total force and location of centre of pressure on one face of the plate shown in Fig. 3.62 immersed in a liquid of specific gravity 0.9. [**Ans.** 62.4 kN, 3.04 m]

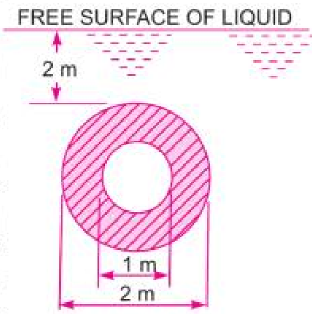


Fig. 3.62

28. A circular opening, 3 m diameter, in the vertical side of water tank is closed by a disc of 3 m diameter which can rotate about a horizontal diameter ? Calculate: (*i*) the force on the disc, and (*ii*) the torque required to maintain the disc in equilibrium in the vertical position when the head of water above the horizontal diameter is 4 m. [**Ans.** (*i*) 270 kN, and (*ii*) 38 kN m]

29. A penstock made up by a pipe of 2 m diameter contains a circular disc of same diameter to act as a valve which controls the discharge passing through it. It can rotate about a horizontal diameter. If the head of water above its centre is 20 m, find the total force acting on the disc and the torque required to maintain it in the vertical position.

30. A circular drum 1.8 m diameter and 1.2 m height is submerged with its axis vertical and its upper end at a depth of 1.8 m below water level. Determine :
(*i*) total pressure on top, bottom and curved surfaces of the drum,
(*ii*) resultant pressure on the whole surface, and
(*iii*) depth of centre of pressure on curved surface.

31. A circular plate of diameter 3 m is immersed in water in such a way that its least and greatest depth from the free surface of water are 1 m and 3 m respectively. For the front side of the plate, find (*i*) total force exerted by water and (*ii*) the position of centre of pressure. [**Ans.** (*i*) 138684 N ; (*ii*) 2.125 m]

32. A tank contains water upto a height of 10 m. One of the sides of the tank is inclined. The angle between free surface of water and inclined side is 60°. The width of the tank is 5 m. Find : (*i*) the force exerted by water on inclined side and (*ii*) position of centre of pressure. [**Ans.** (*i*) 283.1901 kN, (*ii*) 6.67 m]

33. A circular plate of 3 m diameter is under water with its plane making an angle of 30° with the water surface. If the top edge of the plate is 1 m below the water surface, find the force on one side of the plate and its location. *(J.N.T.U., Hyderabad S 2002)*

[**Hint.** $d = 3$ m, $\theta = 30°$, height of top edge = 1 m, $\bar{h} = 1 + 1.5 \times \sin 30°$ $= 1.75$

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times \left(\frac{\pi}{4} \times 3^2\right) \times 1.75 = \mathbf{121.35\ kN.}$$

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h} = \frac{\frac{\pi}{64}\left(3^4\right)^4 \times \frac{1}{4}}{\frac{\pi}{4}\left(3^2\right) \times 1.75} + 1.75 = 0.08 + 1.75 = \mathbf{1.83\ m.}]$$

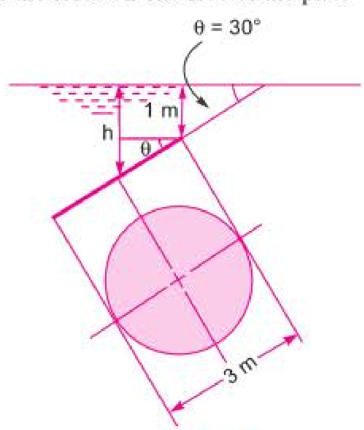


Fig. 3.63

4

CHAPTER

BUOYANCY AND FLOATATION

▶ 4.1 INTRODUCTION

In this chapter, the equilibrium of the floating and sub-merged bodies will be considered. Thus the chapter will include : 1. Buoyancy, 2. Centre of buoyancy, 3. Metacentre, 4. Metacentric height, 5. Analytical method for determining metacentric height, 6. Conditions of equilibrium of a floating and sub-merged body, and 7. Experimental method for metacentric height.

▶ 4.2 BUOYANCY

When a body is immersed in a fluid, an upward force is exerted by the fluid on the body. This upward force is equal to the weight of the fluid displaced by the body and is called the force of buoyancy or simply buoyancy.

▶ 4.3 CENTRE OF BUOYANCY

It is defined as the point, through which the force of buoyancy is supposed to act. As the force of buoyancy is a vertical force and is equal to the weight of the fluid displaced by the body, the centre of buoyancy will be the centre of gravity of the fluid displaced.

Problem 4.1 *Find the volume of the water displaced and position of centre of buoyancy for a wooden block of width 2.5 m and of depth 1.5 m, when it floats horizontally in water. The density of wooden block is 650 kg/m³ and its length 6.0 m.*

Solution. Given :

Width $= 2.5$ m

Depth $= 1.5$ m

Length $= 6.0$ m

Volume of the block $= 2.5 \times 1.5 \times 6.0 = 22.50 \text{ m}^3$

Density of wood, $\rho = 650 \text{ kg/m}^3$

$\therefore$ Weight of block $= \rho \times g \times \text{Volume}$

$= 650 \times 9.81 \times 22.50 \text{ N} = 143471 \text{ N}$

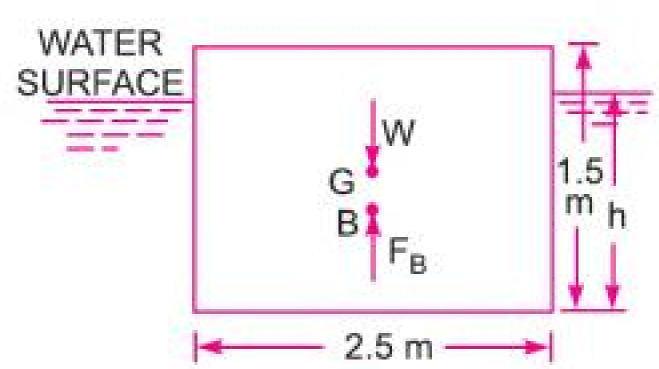


Fig. 4.1

For equilibrium the weight of water displaced = Weight of wooden block

$= 143471$ N

$\therefore$ Volume of water displaced

$$= \frac{\text{Weight of water displaced}}{\text{Weight density of water}} = \frac{143471}{1000 \times 9.81} = \mathbf{14.625\ m^3}.\ \textbf{Ans.}$$

($\because$ Weight density of water $= 1000 \times 9.81$ N/m^3)

Position of Centre of Buoyancy. Volume of wooden block in water

= Volume of water displaced

or $2.5 \times h \times 6.0 = 14.625$ m^3, where h is depth of wooden block in water

$$\therefore \quad h = \frac{14.625}{2.5 \times 6.0} = 0.975 \text{ m}$$

$$\therefore \quad \text{Centre of Buoyancy} = \frac{0.975}{2} = \mathbf{0.4875\ m\ from\ base.}\ \textbf{Ans.}$$

Problem 4.2 *A wooden log of 0.6 m diameter and 5 m length is floating in river water. Find the depth of the wooden log in water when the sp. gravity of the log is 0.7.*

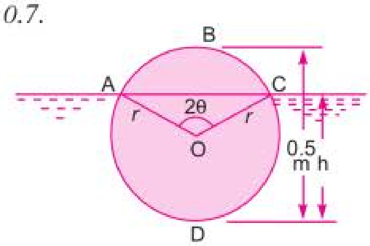


Fig. 4.2

Solution. Given :

Dia. of log $= 0.6$ m

Length, $L = 5$ m

Sp. gr., $S = 0.7$

$\therefore$ Density of log $= 0.7 \times 1000 = 700$ kg/m^3

$\therefore$ Weight density of log, $w = \rho \times g$

$= 700 \times 9.81$ N/m^3

Find depth of immersion or h

Weight of wooden log = Weight density $\times$ Volume of log

$$= 700 \times 9.81 \times \frac{\pi}{4} (D)^2 \times L$$

$$= 700 \times 9.81 \times \frac{\pi}{4} (.6)^2 \times 5 \text{ N} = 989.6 \times 9.81 \text{ N}$$

For equilibrium,

Weight of wooden log = Weight of water displaced

= Weight density of water $\times$ Volume of water displaced

$$\therefore \quad \text{Volume of water displaced} = \frac{989.6 \times 9.81}{1000 \times 9.81} = 0.9896 \text{ m}^3$$

($\because$ Weight density of water $= 1000 \times 9.81$ N/m^3)

Let h is the depth of immersion

$\therefore$ Volume of log inside water = Area of $ADCA \times$ Length

= Area of $ADCA \times 5.0$

But volume of log inside water = Volume of water displaced = 0.9896 m^3

$\therefore \quad 0.9896 = \text{Area of } ADCA \times 5.0$

$\therefore$ Area of $ADCA$ $= \dfrac{0.9896}{5.0} = 0.1979 \text{ m}^2$

But area of $ADCA$ = Area of curved surface $ADCOA$ + Area of ΔAOC

$$= \pi r^2 \left[\frac{360^\circ - 2\theta}{360^\circ}\right] + \frac{1}{2} r \cos\theta \times 2r \sin\theta$$

$$= \pi r^2 \left[1 - \frac{\theta}{180^\circ}\right] + r^2 \cos\theta \sin\theta$$

$$\therefore \quad 0.1979 = \pi (.3)^2 \left[1 - \frac{\theta}{180^\circ}\right] + (.3)^2 \cos\theta \sin\theta$$

$$0.1979 = .2827 - .00157\,\theta + 0.9 \cos\theta \sin\theta$$

or $\quad .00157\,\theta - .09 \cos\theta \sin\theta = .2827 - .1979 = 0.0848$

$$\theta - \frac{.09}{.00157} \cos\theta \sin\theta = \frac{.0848}{.00157}$$

or $\quad \theta - 57.32 \cos\theta \sin\theta = 54.01.$

or $\quad \theta - 57.32 \cos\theta \sin\theta - 54.01 = 0$

For $\theta = 60^\circ$, $\quad 60 - 57.32 \times 0.5 \times .866 - 54.01 = 60 - 24.81 - 54.01 = -18.82$

For $\theta = 70^\circ$, $\quad 70 - 57.32 \times .342 \times 0.9396 - 54.01 = 70 - 18.4 - 54.01 = -2.41$

For $\theta = 72^\circ$, $\quad 72 - 57.32 \times .309 \times .951 - 54.01 = 72 - 16.84 - 54.01 = +1.14$

For $\theta = 71^\circ$, $\quad 71 - 57.32 \times .325 \times .9455 - 54.01 = 71 - 17.61 - 54.01 = -0.376$

$\therefore \quad \theta = 71.5^\circ$, $\quad 71.5 - 57.32 \times .3173 \times .948 - 54.01 = 71.5 - 17.24 - 54.01 = +.248$

Then $\quad h = r + r \cos 71.5^\circ$

$= 0.3 + 0.3 \times 0.3173 =$ **0.395 m. Ans.**

Problem 4.3 *A stone weighs 392.4 N in air and 196.2 N in water. Compute the volume of stone and its specific gravity.*

Solution. Given :

Weight of stone in air = 392.4 N

Weight of stone in water = 196.2 N

For equilibrium,

Weight in air – Weight of stone in water = Weight of water displaced

or $\quad 392.4 - 196.2 = 196.2 = 1000 \times 9.81 \times$ Volume of water displaced

$\therefore$ Volume of water displaced

$= \dfrac{196.2}{1000 \times 9.81} = \dfrac{1}{50} \text{ m}^3 = \dfrac{1}{50} \times 10^6 \text{ cm}^3 = \mathbf{2 \times 10^4 \text{ cm}^3}$. **Ans.**

= Volume of stone

$\therefore$ Volume of stone $= \mathbf{2 \times 10^4 \text{ cm}^3}$. **Ans.**

Specific Gravity of Stone

Mass of stone $= \dfrac{\text{Weight in air}}{g} = \dfrac{392.4}{9.81} = 40 \text{ kg}$

Density of stone $= \dfrac{\text{Mass in air}}{\text{Volume}} = \dfrac{40.0 \text{ kg}}{\dfrac{1}{50}\text{m}^3} = 40 \times 50 = 2000 \dfrac{\text{kg}}{\text{m}^3}$

$\therefore$ Sp. gr. of stone $= \dfrac{\text{Density of stone}}{\text{Density of water}} = \dfrac{2000}{1000} = $ **2.0. Ans.**

Problem 4.4 *A body of dimensions 1.5 m × 1.0 m × 2 m, weighs 1962 N in water. Find its weight in air. What will be its specific gravity ?*

Solution. Given :

Volume of body $= 1.50 \times 1.0 \times 2.0 = 3.0 \text{ m}^3$

Weight of body in water $= 1962$ N

Volume of the water displaced = Volume of the body = 3.0 m^3

$\therefore$ Weight of water displaced $= 1000 \times 9.81 \times 3.0 = 29430$ N

For the equilibrium of the body

Weight of body in air – Weight of water displaced = Weight in water

$\therefore \qquad W_{air} - 29430 = 1962$

$W_{air} = 29430 + 1962 = 31392 \text{ N}$

Mass of body $= \dfrac{\text{Weight in air}}{g} = \dfrac{31392}{9.81} = 3200 \text{ kg}$

Density of the body $= \dfrac{\text{Mass}}{\text{Volume}} = \dfrac{3200}{3.0} = 1066.67$

$\therefore$ Sp. gravity of the body $= \dfrac{1066.67}{1000} = $ **1.067. Ans.**

Problem 4.5 *Find the density of a metallic body which floats at the interface of mercury of sp. gr. 13.6 and water such that 40% of its volume is sub-merged in mercury and 60% in water.*

Solution. Let the volume of the body $= V \text{ m}^3$

Then volume of body sub-merged in mercury

$$= \frac{40}{100} V = 0.4 \; V \text{ m}^3$$

Volume of body sub-merged in water

$$= \frac{60}{100} \times V = 0.6 \; V \text{ m}^3$$

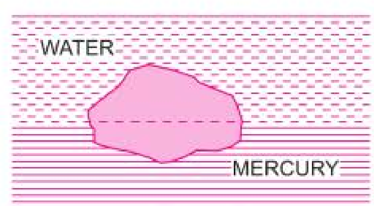


Fig. 4.3

For the equilibrium of the body

Total buoyant force (upward force) = Weight of the body

But total buoyant force = Force of buoyancy due to water + Force of buoyancy due to mercury

Force of buoyancy due to water = Weight of water displaced by body

= Density of water × g × Volume of water displaced

= 1000 × g × Volume of body in water

$= 1000 \times g \times 0.6 \times V$ N

and Force of buoyancy due to mercury $=$ Weight of mercury displaced by body

$= g \times$ Density of mercury $\times$ Volume of mercury displaced

$= g \times 13.6 \times 1000 \times$ Volume of body in mercury

$= g \times 13.6 \times 1000 \times 0.4\ V$ N

Weight of the body $=$ Density $\times g \times$ Volume of body $= \rho \times g \times V$

where ρ is the density of the body

$\therefore$ For equilibrium, we have

Total buoyant force $=$ Weight of the body

$1000 \times g \times 0.6 \times V + 13.6 \times 1000 \times g \times .4\ V = \rho \times g \times V$

or $\rho = 600 + 13600 \times .4 = 600 + 54400 = 6040.00$ kg/m^3

$\therefore$ Density of the body $=$ **6040.00 kg/m^3. Ans.**

Problem 4.6 *A float valve regulates the flow of oil of sp. gr. 0.8 into a cistern. The spherical float is 15 cm in diameter. AOB is a weightless link carrying the float at one end, and a valve at the other end which closes the pipe through which oil flows into the cistern. The link is mounted in a frictionless hinge at O and the angle AOB is 135°. The length of OA is 20 cm, and the distance between the centre of the float and the hinge is 50 cm. When the flow is stopped AO will be vertical. The valve is to be pressed on to the seat with a force of 9.81 N to completely stop the flow of oil into the cistern. It was observed that the flow of oil is stopped when the free surface of oil in the cistern is 35 cm below the hinge. Determine the weight of the float.*

Solution. Given :

Sp. gr. of oil $= 0.8$

$\therefore$ Density of oil, $\rho_0 = 0.8 \times 1000$

$= 800$ kg/m^3

Dia. of float, $D = 15$ cm

$\angle AOB = 135°$

$OA = 20$ cm

Force, $P = 9.81$ N

$OB = 50$ cm

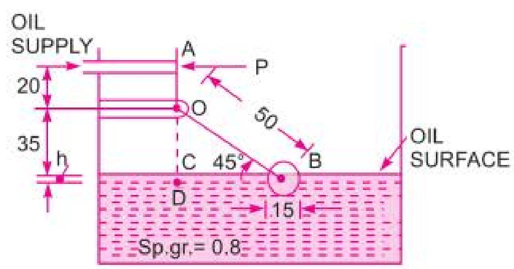


Fig. 4.4

Find the weight of the float. Let it is equal to W.

When the flow of oil is stopped, the centre of float is shown in Fig. 4.4

The level of oil is also shown. The centre of float is below the level of oil, by a depth 'h'.

From ΔBOD, $\sin 45° = \dfrac{OD}{OB} = \dfrac{OC + CD}{OB} = \dfrac{35 + h}{50}$

$\therefore$ $50 \times \sin 45° = 35 + h$

or $h = 50 \times \dfrac{1}{\sqrt{2}} - 35 = 35.355 - 35 = 0.355$ cm $= .00355$ m.

The weight of float is acting through B, but the upward buoyant force is acting through the centre of weight of oil displaced.

Volume of oil displaced $= \dfrac{2}{3}\pi r^3 + h \times \pi r^2$ $\left\{ r = \dfrac{D}{2} = \dfrac{15}{2} = 7.5 \text{ cm} \right\}$

$$= \frac{2}{3} \times \pi \times (.075)^3 + .00355 \times \pi \times (.075)^2 = 0.000945 \text{ m}^3$$

$\therefore$ Buoyant force $=$ Weight of oil displaced

$$= \rho_0 \times g \times \text{Volume of oil}$$
$$= 800 \times 9.81 \times .000945 = 7.416 \text{ N}$$

The buoyant force and weight of the float passes through the same vertical line, passing through B. Let the weight of float is W. Then net vertical force on float

$$= \text{Buoyant force} - \text{Weight of float} = (7.416 - W)$$

Taking moments about the hinge O, we get

$$P \times 20 = (7.416 - W) \times BD = (7.416 - W) \times 50 \times \cos 45°$$

or
$$9.81 \times 20 = (7.416 - W) \times 35.355$$

$\therefore$
$$W = 7.416 - \frac{20 \times 9.81}{35.355} = 7.416 - 5.55 = \mathbf{1.866\ N.}\ \textbf{Ans.}$$

▶ 4.4 META-CENTRE

It is defined as the point about which a body starts oscillating when the body is tilted by a small angle. The meta-centre may also be defined as the point at which the line of action of the force of buoyancy will meet the normal axis of the body when the body is given a small angular displacement.

Consider a body floating in a liquid as shown in Fig. 4.5 (*a*). Let the body is in equilibrium and G is the centre of gravity and B the centre of buoyancy. For equilibrium, both the points lie on the normal axis, which is vertical.

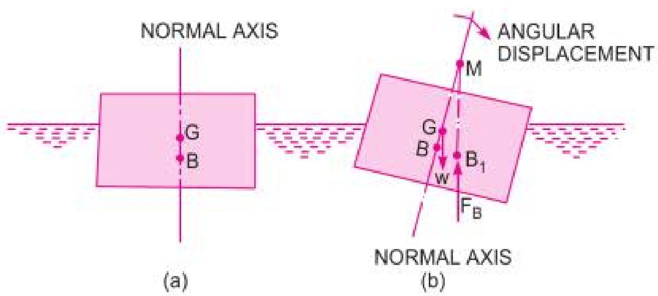


Fig. 4.5 *Meta-centre*

Let the body is given a small angular displacement in the clockwise direction as shown in Fig. 4.5 (*b*). The centre of buoyancy, which is the centre of gravity of the displaced liquid or centre of gravity of the portion of the body sub-merged in liquid, will now be shifted towards right from the normal axis. Let it is at B_1 as shown in Fig. 4.5 (*b*). The line of action of the force of buoyancy in this new position, will intersect the normal axis of the body at some point say M. This point M is called **Meta-centre.**

▶ 4.5 META-CENTRIC HEIGHT

The distance MG, *i.e.*, the distance between the meta-centre of a floating body and the centre of gravity of the body is called meta-centric height.

▶ 4.6 ANALYTICAL METHOD FOR META-CENTRE HEIGHT

Fig. 4.6 (*a*) shows the position of a floating body in equilibrium. The location of centre of gravity and centre of buoyancy in this position is at G and B. The floating body is given a small angular displacement in the clockwise direction. This is shown in Fig. 4.6 (*b*). The new centre of buoyancy is at B_1. The vertical line through B_1 cuts the normal axis at M. Hence M is the meta-centre and GM is meta-centric height.

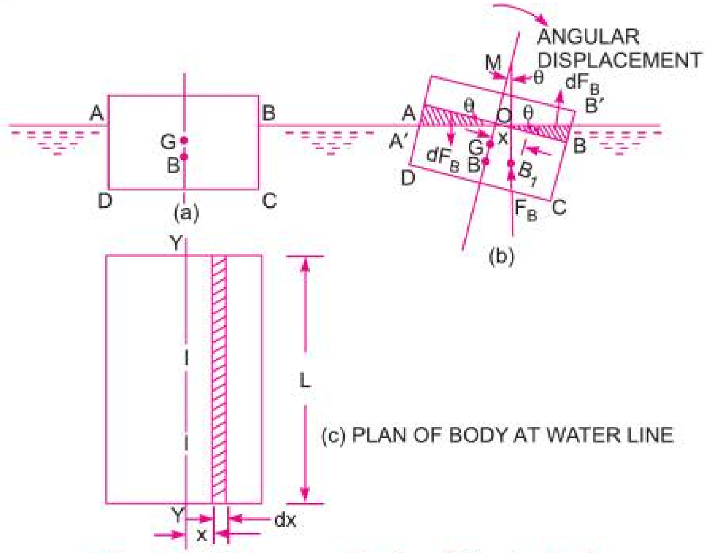


Fig. 4.6 *Meta-centre height of floating body.*

The angular displacement of the body in the clockwise direction causes the wedge-shaped prism BOB' on the right of the axis to go inside the water while the identical wedge-shaped prism represented by AOA' emerges out of the water on the left of the axis. These wedges represent a gain in buoyant force on the right side and a corresponding loss of buoyant force on the left side. The gain is represented by a vertical force dF_B acting through the $C.G.$ of the prism BOB' while the loss is represented by an equal and opposite force dF_B acting vertically downward through the centroid of AOA'. The couple due to these buoyant forces dF_B tends to rotate the ship in the counterclockwise direction. Also the moment caused by the displacement of the centre of buoyancy from B to B_1 is also in the counterclockwise direction. Thus these two couples must be equal.

Couple Due to Wedges. Consider towards the right of the axis a small strip of thickness dx at a distance x from O as shown in Fig. 4.5 (*b*). The height of strip $x \times \angle BOB' = x \times \theta$.

$$\{\because \angle BOB' = \angle AOA' = BMB_1' = \theta\}$$

$\therefore$ Area of strip $= \text{Height} \times \text{Thickness} = x \times \theta \times dx$

If L is the length of the floating body, then

Volume of strip $= \text{Area} \times L$

$= x \times \theta \times L \times dx$

$\therefore$ Weight of strip $= \rho g \times \text{Volume} = \rho g x\ \theta L\ dx$

Similarly, if a small strip of thickness dx at a distance x from O towards the left of the axis is considered, the weight of strip will be $\rho g x \theta\ L\ dx$. The two weights are acting in the opposite direction and hence constitute a couple.

Moment of this couple $= \text{Weight of each strip} \times \text{Distance between these two weights}$

$= \rho g x\, \theta L\, dx\, [x + x]$

$= \rho g x\, \theta L\, dx \times 2x = 2\rho g x^2\, \theta L\, dx$

$\therefore$ Moment of the couple for the whole wedge

$$= \int 2\rho g x^2\, \theta L\, dx \qquad ...(4.1)$$

Moment of couple due to shifting of centre of buoyancy from B to B_1

$= F_B \times BB_1$

$= F_B \times BM \times \theta \qquad \{\because BB_1 = BM \times \theta \text{ if } \theta \text{ is very small}\}$

$= W \times BM \times \theta \qquad \{\because F_B = W\} \quad ...(4.2)$

But these two couples are the same. Hence equating equations (4.1) and (4.2), we get

$$W \times BM \times \theta = \int 2\rho g x^2\, \theta\, L dx$$

$$W \times BM \times \theta = 2\rho g \theta \int x^2 L dx$$

$$W \times BM = 2\rho g \int x^2 L dx$$

Now Ldx = Elemental area on the water line shown in Fig. 4.6 (c) and $= dA$

$$\therefore \qquad W \times BM = 2\rho g \int x^2 dA.$$

But from Fig. 4.5 (c) it is clear that $2 \int x^2\, dA$ is the second moment of area of the plan of the body at water surface about the axis Y-Y. Therefore

$$W \times BM = \rho g I \qquad \{\text{where } I = 2 \int x^2\, dA\}$$

$$\therefore \qquad BM = \frac{\rho g I}{W}$$

But $\qquad W$ = Weight of the body

= Weight of the fluid displaced by the body

$= \rho g \times$ Volume of the fluid displaced by the body

$= \rho g \times$ Volume of the body sub-merged in water

$= \rho g \times \forall$

$$\therefore \qquad BM = \frac{\rho g \times I}{\rho g \times \forall} = \frac{I}{\forall} \qquad ...(4.3)$$

$$GM = BM - BG = \frac{I}{\forall} - BG$$

$$\therefore \text{ Meta-centric height } = GM = \frac{I}{\forall} - BG. \qquad ...(4.4)$$

Problem 4.7 *A rectangular pontoon is 5 m long, 3 m wide and 1.20 m high. The depth of immersion of the pontoon is 0.80 m in sea water. If the centre of gravity is 0.6 m above the bottom of the pontoon, determine the meta-centric height. The density for sea water = 1025 kg/m³.*

Solution. Given :

Dimension of pontoon $= 5 \text{ m} \times 3 \text{ m} \times 1.20 \text{ m}$

Depth of immersion $= 0.8$ m

Distance $AG = 0.6$ m

Distance $AB = \frac{1}{2} \times$ Depth of immersion

$= \frac{1}{2} \times .8 = 0.4$ m

Density for sea water $= 1025$ kg/m^3

Meta-centre height GM, given by equation (4.4) is

$$GM = \frac{I}{\forall} - BG$$

where I = M.O. Inertia of the plan of the pontoon about Y-Y axis

$$= \frac{1}{12} \times 5 \times 3^3 \text{ m}^4 = \frac{45}{4} \text{ m}^4$$

$\forall$ = Volume of the body sub-merged in water

$= 3 \times 0.8 \times 5.0 = 12.0$ m^3

$BG = AG - AB = 0.6 - 0.4 = 0.2$ m

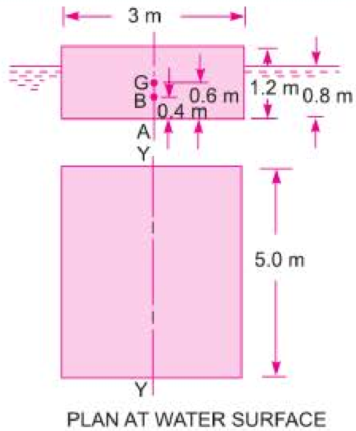


Fig. 4.7

$\therefore$ $$GM = \frac{45}{4} \times \frac{1}{12.0} - 0.2 = \frac{45}{48} - 0.2 = 0.9375 - 0.2 = \mathbf{0.7375 \text{ m.}} \text{ Ans.}$$

Problem 4.8 *A uniform body of size 3 m long × 2 m wide × 1 m deep floats in water. What is the weight of the body if depth of immersion is 0.8 m ? Determine the meta-centric height also.*

Solution. Given :

Dimension of body $= 3 \times 2 \times 1$

Depth of immersion $= 0.8$ m

Find (i) Weight of body, W

(ii) Meta-centric height, GM

(i) **Weight of Body, W**

= Weight of water displaced

$= \rho g \times$ Volume of water displaced

$= 1000 \times 9.81 \times$ Volume of body in water

$= 1000 \times 9.81 \times 3 \times 2 \times 0.8$ N

= **47088 N.** Ans.

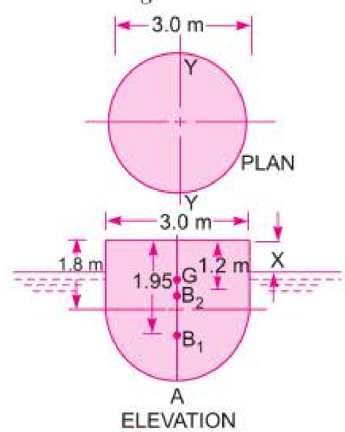


Fig. 4.8

(ii) **Meta-centric Height, GM**

Using equation (4.4), we get

$$GM = \frac{I}{\forall} - BG$$

where I = M.O.I about Y-Y axis of the plan of the body

$$= \frac{1}{12} \times 3 \times 2^3 = \frac{3 \times 2^3}{12} = 2.0 \text{ m}^4$$

$\forall$ = Volume of body in water

$= 3 \times 2 \times 0.8 = 4.8$ m^3

$$BG = AG - AB = \frac{1.0}{2} - \frac{0.8}{2} = 0.5 - 0.4 = 0.1$$

$\therefore$ $$GM = \frac{2.0}{4.8} - 0.1 = 0.4167 - 0.1 = \mathbf{0.3167 \text{ m.}} \text{ Ans.}$$

Problem 4.9 *A block of wood of specific gravity 0.7 floats in water. Determine the meta-centric height of the block if its size is 2 m × 1 m × 0.8 m.*

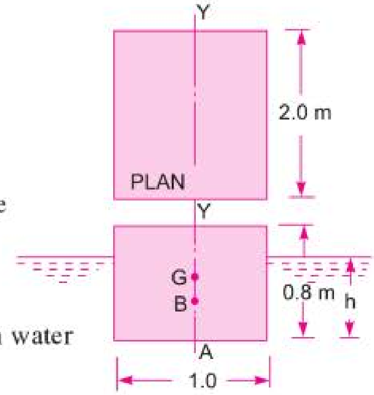


Fig. 4.9

Solution. Given :

Dimension of block $= 2 \times 1 \times 0.8$

Let depth of immersion $= h$ m

Sp. gr. of wood $= 0.7$

Weight of wooden piece = Weight density of wood* × Volume

$= 0.7 \times 1000 \times 9.81 \times 2 \times 1 \times 0.8$ N

Weight of water displaced = Weight density of water × Volume of the wood sub-merged in water

$= 1000 \times 9.81 \times 2 \times 1 \times h$ N

For equilibrium,

Weight of wooden piece = Weight of water displaced

$\therefore \quad 700 \times 9.81 \times 2 \times 1 \times 0.8 = 1000 \times 9.81 \times 2 \times 1 \times h$

$$\therefore \quad h = \frac{700 \times 9.81 \times 2 \times 1 \times 0.8}{1000 \times 9.81 \times 2 \times 1} = 0.7 \times 0.8 = 0.56 \text{ m}$$

$\therefore$ Distance of centre of Buoyancy from bottom, *i.e.*,

$$AB = \frac{h}{2} = \frac{0.56}{2} = 0.28 \text{ m}$$

and $\quad AG = 0.8/2.0 = 0.4$ m

$\therefore \quad BG = AG - AB = 0.4 - 0.28 = 0.12$ m

The meta-centric height is given by equation (4.4) or

$$GM = \frac{I}{\forall} - BG$$

where $\quad I = \frac{1}{12} \times 2 \times 1.0^3 = \frac{1}{6} \text{ m}^4$

$\forall$ = Volume of wood in water

$= 2 \times 1 \times h = 2 \times 1 \times .56 = 1.12 \text{ m}^3$

$$\therefore \quad GM = \frac{1}{6} \times \frac{1}{1.12} - 0.12 = 0.1488 - 0.12 = \mathbf{0.0288 \text{ m.}} \quad \textbf{Ans.}$$

Problem 4.10 *A solid cylinder of diameter 4.0 m has a height of 3 metres. Find the meta-centric height of the cylinder when it is floating in water with its axis vertical. The sp. gr. of the cylinder = 0.6.*

Solution. Given :

Dia. of cylinder, $\quad D = 4.0$ m

Height of cylinder, $\quad h = 3.0$ m

* Weight density of wood $= \rho \times g$, where ρ = density of wood
$= 0.7 \times 1000 = 700 \text{ kg/m}^3$. Hence w for wood $= 700 \times 9.81 \text{ N/m}^3$.

Sp. gr. of cylinder = 0.6

Depth of immersion of cylinder

$$= 0.6 \times 3.0 = 1.8 \text{ m}$$

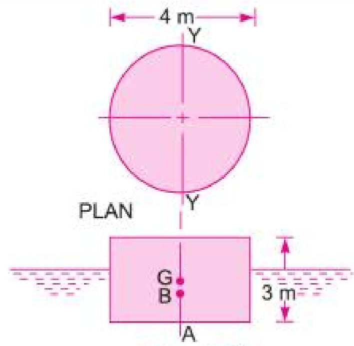


Fig. 4.10

$$\therefore \quad AB = \frac{1.8}{2} = 0.9 \text{ m}$$

and $$AG = \frac{3}{2} = 1.5 \text{ m}$$

$$\therefore \quad BG = AG - AB$$

$$= 1.5 - 0.9 = 0.6 \text{ m}$$

Now the meta-centric height GM is given by equation (4.4)

$$GM = \frac{I}{\forall} - BG$$

But I = M.O.I. about Y-Y axis of the plan of the body

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64} \times (4.0)^4$$

and $\forall$ = Volume of cylinder in water

$$= \frac{\pi}{4} D^2 \times \text{Depth of immersion}$$

$$= \frac{\pi}{4} (4)^2 \times 1.8 \text{ m}^3$$

$$\therefore \quad GM = \frac{\frac{\pi}{64} \times (4.0)^4}{\frac{\pi}{4} \times (4.0)^2 \times 1.8} - 0.6$$

$$= \frac{1}{16} \times \frac{4.0^2}{1.8} - 0.6 = \frac{1}{1.8} - 0.6 = 0.55 - 0.6 = \mathbf{-0.05 \text{ m.}} \text{ Ans.}$$

– ve sign means that meta-centre, (M) is below the centre of gravity (G).

Problem 4.11 *A body has the cylindrical upper portion of 3 m diameter and 1.8 m deep. The lower portion is a curved one, which displaces a volume of 0.6 m³ of water. The centre of buoyancy of the curved portion is at a distance of 1.95 m below the top of the cylinder. The centre of gravity of the whole body is 1.20 m below the top of the cylinder. The total displacement of water is 3.9 tonnes. Find the meta-centric height of the body.*

Solution. Given :

Dia. of body = 3.0 m

Depth of body = 1.8 m

Volume displaced by curved portion

$$= 0.6 \text{ m}^3 \text{ of water.}$$

Let B_1 is the centre of buoyancy of the curved surface and G is the centre of gravity of the whole body.

Then $CB_1 = 1.95$ m

$CG = 1.20$ m

Total weight of water displaced by body = 3.9 tonnes

$= 3.9 \times 1000 = 3900$ kgf

$= 3900 \times 9.81$ N $= 38259$ N

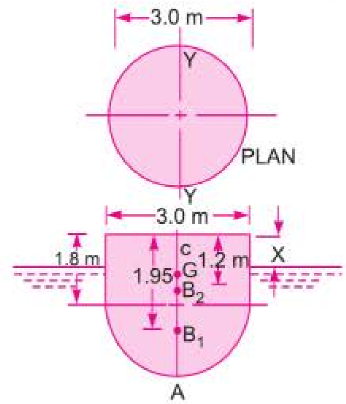


Fig. 4.11

Find **meta-centric** height of the body.

Let the height of the body above the water surface x m. Total weight of water displaced by body

= Weight density of water × [Volume of water displaced]

= 1000 × 9.81 × [Volume of the body in water]

= 9810 [Volume of cylindrical part in water + Volume of curved portion]

$$= 9810 \left[\frac{\pi}{4} \times D^2 \times \text{Depth of cylindrical part in water} + \text{Volume displaced by curved portion}\right]$$

or
$$38259 = 9810 \left[\frac{\pi}{4}(3)^2 \times (1.8 - x) + 0.6\right]$$

$$\therefore \quad \frac{\pi}{4}(3)^2 \times (1.8 - x) + 0.6 = \frac{38259}{9810} = 3.9$$

$$\therefore \quad \frac{\pi}{4} \times 3^2 \times (1.8 - x) = 3.9 - 0.6 = 3.3$$

or
$$1.8 - x = \frac{3.3 \times 4}{\pi \times 3 \times 3} = 0.4668$$

$$\therefore \quad x = 1.8 - .4668 = 1.33 \text{ m}$$

Let B_2 is the centre of buoyancy of cylindrical part and B is the centre of buoyancy of the whole body.

Then depth of cylindrical part in water = $1.8 - x = 0.467$ m

$$\therefore \quad CB_2 = x + \frac{.467}{2} = 1.33 + .2335 = 1.5635 \text{ m}.$$

The distance of the centre of buoyancy of the whole body from the top of the cylindrical part is given as

CB = (Volume of curved portion × CB_1 + Volume of cylindrical part in water × CB_2) ÷ (Total volume of water displaced)

$$= \frac{0.6 \times 1.95 + 3.3 \times 1.5635}{(0.6 + 3.3)} = \frac{1.17 + 5.159}{3.9} = 1.623 \text{ m}.$$

Then $BG = CB - CG = 1.623 - 1.20 = .423$ m.

Meta-centric height, GM, is given by

$$GM = \frac{I}{\forall} - BG$$

where I = M.O.I. of the plan of the body at water surface about Y-Y axis

$$= \frac{\pi}{64} \times D^4 = \frac{\pi}{64} \times 3^4 \text{ m}^4$$

$\forall$ = Volume of the body in water = 3.9 m^3

$$\therefore \quad GM = \frac{\pi}{64} \times \frac{3^4}{3.9} - .423 = 1.019 - .423 = \mathbf{0.596\ m.}\ \textbf{Ans.}$$

▶ 4.7 CONDITIONS OF EQUILIBRIUM OF A FLOATING AND SUB-MERGED BODIES

A sub-merged or a floating body is said to be stable if it comes back to its original position after a slight disturbance. The relative position of the centre of gravity (G) and centre of buoyancy (B_1) of a body determines the stability of a sub-merged body.

4.7.1 Stability of a Sub-merged Body. The position of centre of gravity and centre of buoyancy in case of a completely sub-merged body are fixed. Consider a balloon, which is completely sub-merged in air. Let the lower portion of the balloon contains heavier material, so that its centre of gravity is lower than its centre of buoyancy as shown in Fig. 4.12 (*a*). Let the weight of the balloon is W. The weight W is acting through G, vertically in the downward direction, while the buoyant force F_B is acting vertically up, through B. For the equilibrium of the balloon $W = F_B$. If the balloon is given an angular displacement in the clockwise direction as shown in Fig. 4.12 (*a*), then W and F_B constitute a couple acting in the anti-clockwise direction and brings the balloon in the original position. Thus the balloon in the position, shown by Fig. 4.12 (*a*) is in stable equilibrium.

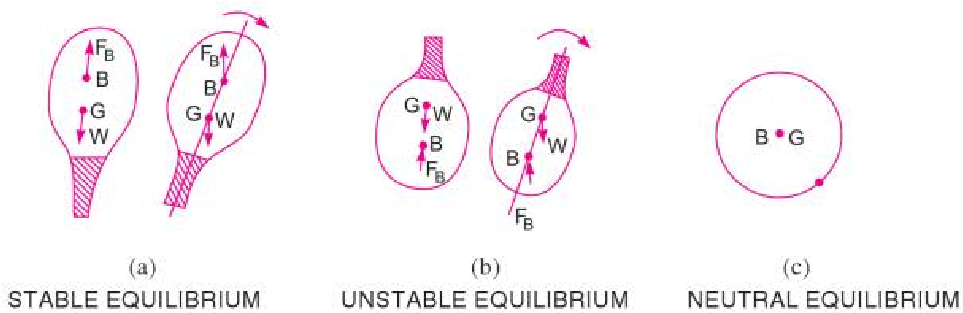


Fig. 4.12 *Stabilities of sub-merged bodies.*

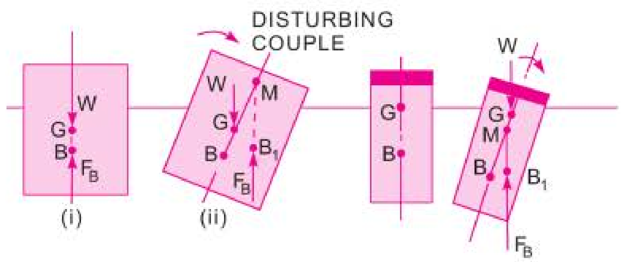
(*a*) **Stable Equilibrium.** When $W = F_B$ and point B is above G, the body is said to be in stable equilibrium.

(*b*) **Unstable Equilibrium.** If $W = F_B$, but the centre of buoyancy (B) is below centre of gravity (G), the body is in unstable equilibrium as shown in Fig. 4.12 (*b*). A slight displacement to the body, in the clockwise direction, gives the couple due to W and F_B also in the clockwise direction. Thus the body does not return to its original position and hence the body is in unstable equilibrium.

(*c*) **Neutral Equilibrium.** If $F_B = W$ and B and G are at the same point, as shown in Fig. 4.12 (*c*), the body is said to be in neutral equilibrium.

4.7.2 Stability of Floating Body. The stability of a floating body is determined from the position of Meta-centre (M). In case of floating body, the weight of the body is equal to the weight of liquid displaced.

(*a*) **Stable Equilibrium.** If the point M is above G, the floating body will be in stable equilibrium as shown in Fig. 4.13 (*a*). If a slight angular displacement is given to the floating body in the clockwise direction, the centre of buoyancy shifts from B to B_1 such that the vertical line through B_1 cuts at M. Then the buoyant force F_B through B_1 and weight W through G constitute a couple acting in the anti-clockwise direction and thus bringing the floating body in the original position.



(a) Stable equilibrium M is above G (b) Unstable equilibrium M is below G.

Fig. 4.13 *Stability of floating bodies.*

(*b*) **Unstable Equilibrium.** If the point M is below G, the floating body will be in unstable equilibrium as shown in Fig. 4.13 (*b*). The disturbing couple is acting in the clockwise direction. The couple due to buoyant force F_B and W is also acting in the clockwise direction and thus overturning the floating body.

(*c*) **Neutral Equilibrium.** If the point M is at the centre of gravity of the body, the floating body will be in neutral equilibrium.

Problem 4.12 *A solid cylinder of diameter 4.0 m has a height of 4.0 m. Find the meta-centric height of the cylinder if the specific gravity of the material of cylinder = 0.6 and it is floating in water with its axis vertical. State whether the equilibrium is stable or unstable.*

Solution. Given : $D = 4$ m

Height, $h = 4$ m

Sp. gr. $= 0.6$

Depth of cylinder in water $=$ Sp. gr. $\times h$

$= 0.6 \times 4.0 = 2.4$ m

$\therefore$ Distance of centre of buoyancy (B) from A

or $$AB = \frac{2.4}{2} = 1.2 \text{ m}$$

Distance of centre of gravity (G) from A

or $$AG = \frac{h}{2} = \frac{4.0}{2} = 2.0 \text{ m}$$

$\therefore$ $$BG = AG - AB = 2.0 - 1.2 = 0.8 \text{ m}$$

Now the meta-centric height GM is given by

$$GM = \frac{I}{\forall} - BG$$

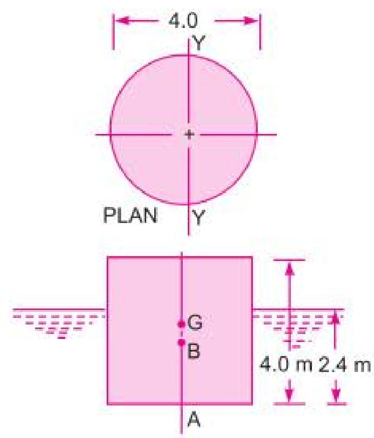


Fig. 4.14

where I = M.O.I. of the plan of the body about Y-Y axis

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64} \times (4.0)^4$$

$\forall$ = Volume of cylinder in water

$$= \frac{\pi}{4.0} \times D^2 \times \text{Depth of cylinder in water} = \frac{\pi}{4} \times 4^2 \times 2.4 \text{ m}^3$$

$$\therefore \qquad \frac{I}{\forall} = \frac{\frac{\pi}{64} \times 4^4}{\frac{\pi}{4} \times 4^2 \times 2.4} = \frac{1}{16} \times \frac{4^2}{2.4} = \frac{1}{2.4} = 0.4167 \text{ m}$$

$$\therefore \qquad GM = \frac{I}{\forall} - BG = 0.4167 - 0.8 = \mathbf{-0.3833 \text{ m.}} \text{ Ans.}$$

–ve sign means that the meta-centre (M) is below the centre of gravity (G). Thus the cylinder is in unstable equilibrium. **Ans.**

Problem 4.13 *A solid cylinder of 10 cm diameter and 40 cm long, consists of two parts made of different materials. The first part at the base is 1.0 cm long and of specific gravity = 6.0. The other part of the cylinder is made of the material having specific gravity 0.6. State, if it can float vertically in water.*

Solution. Given : $D = 10$ cm

Length, $L = 40$ cm

Length of 1st part, $l_1 = 1.0$ cm

Sp. gr., $S_1 = 6.0$

Density of 1st part, $\rho_1 = 6 \times 1000 = 6000$ kg/m^3

Length of 2nd part, $l_2 = 40 - 1.0 = 39.0$ cm

Sp. gr., $S_2 = 0.6$

Density of 2nd part, $\rho_2 = 0.6 \times 1000 = 600$ kg/m^3

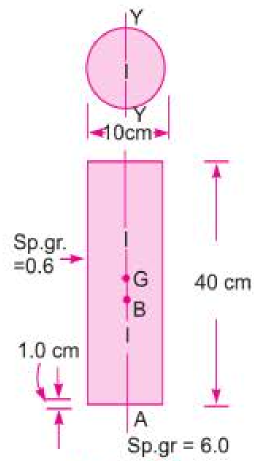


Fig. 4.15

The cylinder will float vertically in water if its meta-centric height GM is positive. To find meta-centric height, find the location of centre of gravity (G) and centre of buoyancy (B) of the combined solid cylinder. The distance of the centre of gravity of the solid cylinder from A is given as

AG = [(Weight of 1st part × Distance of C.G. of 1st part from A)
+ (Weight of 2nd part of cylinder
× Distance of C.G. of 2nd part from A)]
÷ [Weight of 1st part + weight of 2nd part]

$$= \frac{\left(\frac{\pi}{4} D^2 \times 1.0 \times 6.0 \times 0.5\right) + \left(\frac{\pi}{4} D^2 \times 39.0 \times 0.6 \times (1.0 \times 39/2)\right)}{\left(\frac{\pi}{4} D^2 \times 1.0 \times 6.0 + \frac{\pi}{4} D^2 \times 39 \times 0.6\right)}$$

$$= \frac{1.0 \times 6.0 \times 0.5 + 39.0 \times .6 \times (20.5)}{1.0 \times 6.0 + 39.0 \times 0.6}$$

Cancel $\frac{\pi}{4} D^2$ in the Numerator and Denominator $= \frac{3.0 + 479.7}{6.0 + 23.4} = \frac{482.7}{29.4} = 16.42.$

To find the centre of buoyancy of the combined two parts or of the cylinder, determine the depth of immersion of the cylinder. Let the depth of immersion of the cylinder is h. Then

Weight of the cylinder = Weight of water displaced

$$\frac{\pi}{4} \times (.1)^2 \times \frac{39.0}{100} \times 600 \times 9.81 + \frac{\pi}{4} (.1)^2 \times \frac{1.0}{100} \times 6000 \times 9.81 = \frac{\pi}{4} (.1)^2 \times \frac{h}{100} \times 1000 \times 9.81$$

[$\because$ h is in cm]

or cancelling $\frac{\pi}{4} (.1)^2 \times \frac{1000 \times 9.81}{100}$ throughout, we get

$$39.0 \times 0.6 + 1.0 \times 6.0 = h \quad \text{or} \quad h = 23.4 + 6.0 = 29.4$$

$\therefore$ The distance of the centre of the buoyancy B, of the cylinder from A is

$$AB = h/2 = \frac{29.4}{2} = 14.7$$

$$\therefore \qquad BG = AG - AB = 16.42 - 14.70 = 1.72 \text{ cm.}$$

Meta-centric height GM is given by

$$GM = \frac{I}{\forall} - BG$$

where I = M.O.I. of plan of the body about Y-Y axis

$$= \frac{\pi}{64} D^4 = \frac{\pi}{.64} (10)^4 \text{ cm}^4$$

$\forall$ = Volume of cylinder in water

$$= \frac{\pi}{4} D^2 \times h = \frac{\pi}{4} (10)^2 \times 29.4 \text{ m}^3$$

$$\therefore \qquad \frac{I}{\forall} = \frac{\pi}{64} (10)^4 \Big/ \frac{\pi}{4} (10)^2 \times 29.4 = \frac{1}{16} \times \frac{10^2}{29.4} = \frac{100}{19 \times 29.4} = 0.212$$

$$\therefore \qquad GM = 0.212 - 1.72 = -1.508 \text{ cm}$$

As GM is – ve, it means that the Meta-centre M is below the centre of gravity (G). Thus the cylinder is in unstable equilibrium and so it cannot float vertically in water. **Ans.**

Problem 4.14 *A rectangular pontoon 10.0 m long, 7 m broad and 2.5 m deep weighs 686.7 kN. It carries on its upper deck an empty boiler of 5.0 m diameter weighing 588.6 kN. The centre of gravity of the boiler and the pontoon are at their respective centres along a vertical line. Find the meta-centric height. Weight density of sea water is 10.104 kN/m³.*

Solution. Given : Dimension of pontoon = $10 \times 7 \times 2.5$

Weight of pontoon, $W_1 = 686.7$ kN

Dia. of boiler, $D = 5.0$ m

Weight of boiler, $W_2 = 588.6$ kN

w for sea water $= 10.104$ kN/m³

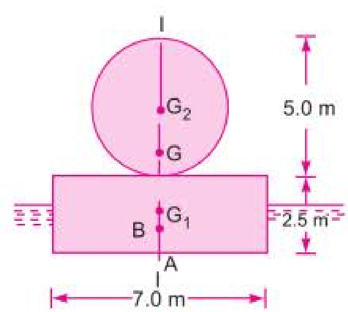


Fig. 4.16

To find the meta-centric height, first determine the common centre of gravity G and common centre of buoyancy B of the boiler and pontoon. Let G_1 and G_2 are the centre of gravities of pontoon and boiler respectively. Then

$$AG_1 = \frac{2.5}{2} = 1.25 \text{ m}$$

$$AG_2 = 2.5 + \frac{5.0}{2} = 2.5 + 2.5 = 5.0 \text{ m}$$

The distance of common centre of gravity G from A is given as

$$AG = \frac{W_1 \times AG_1 + W_2 \times AG_2}{W_1 + W_2}$$

$$= \frac{686.7 \times 1.25 + 588.6 \times 5.0}{(686.7 + 588.6)} = 2.98 \text{ m.}$$

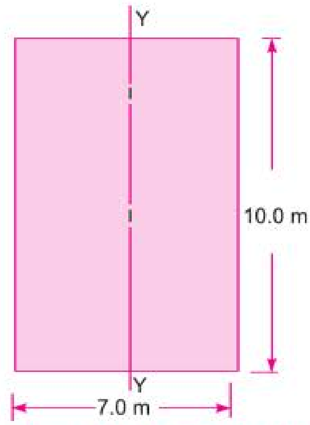


Fig. 4.17 *Plan of the body at water-line*

Let h is the depth of immersion. Then

Total weight of pontoon and boiler = Weight of sea water displaced

or $(686.7 + 588.6) = w \times$ Volume of the pontoon in water

$= 10.104 \times L \times b \times$ Depth of immersion

$\therefore \quad 1275.3 = 10.104 \times 10 \times 7 \times h$

$$h = \frac{1275.3}{10 \times 7 \times 10.104} = 1.803 \text{ m}$$

$\therefore$ The distance of the common centre of buoyancy B from A is

$$AB = \frac{h}{2} = \frac{1.803}{2} = .9015 \text{ m}$$

$\therefore \quad BG = AG - AB = 2.98 - .9015 = 2.0785 \text{ m} \approx 2.078 \text{ m}$

Meta-centric height is given by $GM = \dfrac{I}{\forall} - BG$

where I = M.O.I. of the plan of the body at the water level along Y-Y

$$= \frac{1}{12} \times 10.0 \times 7^3 = \frac{10 \times 49 \times 7}{12} \text{ m}^4$$

$\forall$ = Volume of the body in water

$= L \times b \times h = 10.0 \times 7 \times 1.857$

$$\therefore \quad \frac{I}{\forall} = \frac{10 \times 49 \times 7}{12 \times 10 \times 7 \times 1.857} = \frac{49}{12 \times 1.857} = 2.198 \text{ m}$$

$$\therefore \quad GM = \frac{I}{\forall} - BG = 2.198 - 2.078 = 0.12 \text{ m.}$$

$\therefore$ Meta-centric height of both the pontoon and boiler = **0.12 m. Ans.**

Problem 4.15 *A wooden cylinder of sp. gr. = 0.6 and circular in cross-section is required to float in oil (sp. gr. = 0.90). Find the L/D ratio for the cylinder to float with its longitudinal axis vertical in oil, where L is the height of cylinder and D is its diameter.*

Solution. Given :

Dia. of cylinder $= D$

Height of cylinder $= L$

Sp. gr. of cylinder, $S_1 = 0.6$

Sp. gr. of oil $S_2 = 0.9$

Let the depth of cylinder immersed in oil = h

For the principle of buoyancy

Weight of cylinder = wt. of oil displaced

$$\frac{\pi}{4} D^2 \times L \times 0.6 \times 1000 \times 9.81 = \frac{\pi}{4} D^2 \times h \times 0.9 \times 1000 \times 9.81$$

or $$L \times 0.6 = h \times 0.9$$

$\therefore$ $$h = \frac{0.6 \times L}{0.9} = \frac{2}{3} L.$$

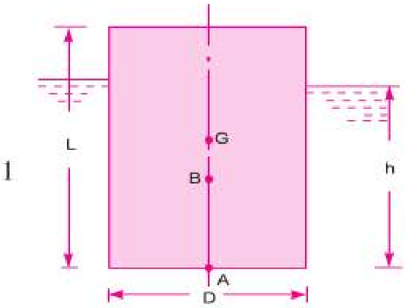


Fig. 4.18

The distance of centre of gravity G from A, $AG = \dfrac{L}{2}$

The distance of centre of buoyancy B from A,

$$AB = \frac{h}{2} = \frac{1}{2}\left[\frac{2}{3}L\right] = \frac{L}{3}$$

$\therefore$ $$BG = AG - AB = \frac{L}{2} - \frac{L}{3} = \frac{3L - 2L}{6} = \frac{L}{6}$$

The meta-centric height GM is given by

$$GM = \frac{I}{\forall} - BG$$

where $I = \dfrac{\pi}{64} D^4$ and $\forall$ = Volume of cylinder in oil = $\dfrac{\pi}{4} D^2 \times h$

$\therefore$ $$\frac{I}{\forall} = \left(\frac{\pi}{64} D^4 \bigg/ \frac{\pi}{4} D^2 h\right) = \frac{1}{16}\,\frac{D^2}{h} = \frac{D^2}{16 \times \frac{2}{3} L} = \frac{3D^2}{32L} \qquad \left\{\because\ h = \frac{2}{3}L\right\}$$

$\therefore$ $$GM = \frac{3D^2}{32L} - \frac{L}{6}.$$

For stable equilibrium, GM should be +ve or

$$GM > 0 \quad \text{or} \quad \frac{3D^2}{32L} - \frac{L}{6} > 0$$

or $$\frac{3D^2}{32L} > \frac{L}{6} \quad \text{or} \quad \frac{3 \times 6}{32} > \frac{L^2}{D^2}$$

or $$\frac{L^2}{D^2} < \frac{18}{32} \quad \text{or} \quad \frac{9}{16}$$

$\therefore$ $$\frac{L}{D} < \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$\therefore$ **L/D < 3/4. Ans.**

Problem 4.16 *Show that a cylindrical buoy of 1 m diameter and 2.0 m height weighing 7.848 kN will not float vertically in sea water of density 1030 kg/m³. Find the force necessary in a vertical chain attached at the centre of base of the buoy that will keep it vertical.*

Solution. Given : Dia. of buoy, $D = 1$ m

Height , $H = 2.0$ m

Weight, $W = 7.848$ kN

$= 7.848 \times 1000 = 7848$ N

Density, $\rho = 1030$ kg/m^3

(*i*) Show the cylinder will not float vertically.

(*ii*) Find the force in the chain.

Part I. The cylinder will not float if meta-centric height is –ve.

Let the depth of immersion be h

Then for equilibrium, Weight of cylinder

= Weight of water displaced

= Density $\times g \times$ Volume of cylinder in water

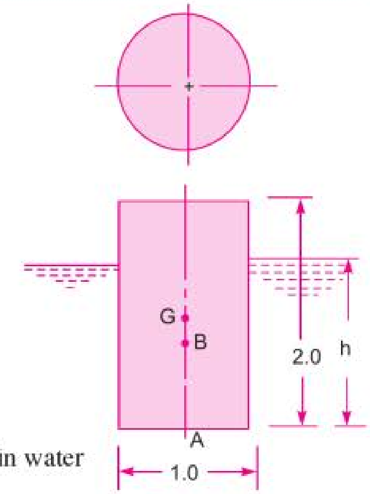


Fig. 4.19

$$\therefore \quad 7848 = 1030 \times 9.81 \times \frac{\pi}{4} D^2 \times h$$

$$= 10104.3 \times \frac{\pi}{4} (1)^2 \times h$$

$$\therefore \quad h = \frac{4 \times 7848}{10104.3 \times \pi} = 0.989 \text{ m.}$$

$\therefore$ The distance of centre of buoyancy B from A,

$$AB = \frac{h}{2} = \frac{0.989}{2} = 0.494 \text{ m.}$$

And the distance of centre of gravity G, from A is $AG = \dfrac{2.0}{2} = 1.0$ m

$$\therefore \quad BG = AG - AB = 1.0 - .494 = .506 \text{ m.}$$

Now meta-centric height GM is given by $GM = \dfrac{I}{\forall} - BG$

where $I = \dfrac{\pi}{64} D^4 = \dfrac{\pi}{64} \times (1)^4 \text{ m}^4$

and $\forall$ = Volume of cylinder in water $= \dfrac{\pi}{4} D^2 \times h = \dfrac{\pi}{4} 1^2 \times .989$

$$\therefore \quad \frac{I}{\forall} = \frac{\frac{\pi}{64} \times 1^4}{\frac{\pi}{4} D^2 \times h} = \frac{\frac{\pi}{64} \times 1^4}{\frac{\pi}{4} \times 1^2 \times .989}$$

$$= \frac{1}{16} \times 1^2 \times \frac{1}{.989} = \frac{1}{16 \times .989} = 0.063 \text{ m}$$

$$\therefore \quad GM = .063 - .506 = \mathbf{-0.443 \text{ m.}} \text{ Ans.}$$

As the meta-centric height is –ve, the point M lies below G and hence the cylinder will be in unstable equilibrium and hence cylinder will not float vertically.

Part II. Let the force applied in a vertical chain attached at the centre of the base of the buoy is T to keep the buoy vertical.

Now find the combined position of centre of gravity (G') and centre of buoyancy (B'). For the combined centre of buoyancy, let h' = depth of immersion when the force T is applied. Then

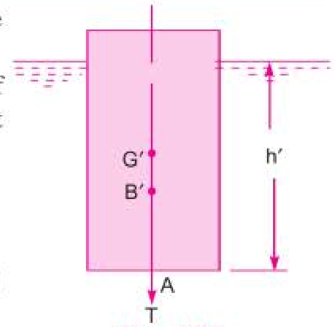


Fig. 4.20

Total downward force = Weight of water displaced

or $(7848 + T)$ = Density of water $\times\, g \times$ Volume of cylinder in water

$$= 1030 \times 9.81 \times \frac{\pi}{4} D' \times h' \text{ [where } h' = \text{depth of immersion]}$$

$$\therefore \qquad h' = \frac{7848 + T}{10104.3 \times \frac{\pi}{4} \times D^2} = \frac{7848 + T}{10104.3 \times \frac{\pi}{4} \times 1^2} = \frac{10104.3 + T}{7935.9} \text{ m}$$

$$\therefore \qquad AB' = \frac{h'}{2} = \frac{1}{2}\left[\frac{7848 + T}{7935.9}\right] = \frac{7848 + T}{15871.8} \text{ m.}$$

The combined centre of gravity (G') due to weight of cylinder and due to tension T in the chain from A is

$$AG' = [\text{Wt. of cylinder} \times \text{Distance of C.G. of cylinder from } A$$

$$+\ T \times \text{Distance of C.G. of } T \text{ from } A] + [\text{Weight of cylinder} + T]$$

$$= \left(7848 \times \frac{2}{2} + T \times 0\right) + [7848 + T] = \frac{7848}{7848 + T} \text{ m}$$

$$\therefore \qquad B'G' = AG' - AB' = \frac{7848}{(7848 + T)} - \frac{(7848 + T)}{15871.8}$$

The meta-centric height GM is given by $GM = \frac{I}{\forall} - B'G'$

where $\quad = \frac{\pi}{64} \times D^4 = \frac{\pi}{64} \times 1^4 = \frac{\pi}{64} \text{ m}^4$

and $\quad \forall = \frac{\pi}{4} D^2 \times h' = \frac{\pi}{4} \times 1^2 \times \frac{(7848 + T)}{7935.9} = \frac{\pi}{4} \times \frac{7848 + T}{7935.9}$

$$\therefore \qquad \frac{I}{\forall} = \frac{\frac{\pi}{64}}{\frac{\pi}{4} \frac{(7848 + T)}{7935.9}} = \frac{1}{16} \times \frac{7935.9}{(7848 + T)}$$

$$\therefore \qquad GM = \frac{7935.9}{16\,(7848 + T)} - \left[\frac{7848}{(7848 + T)} - \frac{(7848 + T)}{15871.8}\right]$$

For stable equilibrium GM should be positive

or $$GM > 0$$

or $$\frac{7935.9}{16\,(7848 + T)} - \left[\frac{7848}{(7848 + T)} - \frac{(7848 + T)}{15871.8}\right] > 0$$

or $$\frac{7935.9}{16(7848+T)} - \frac{7848}{(7848+T)} + \frac{7848+T}{15871.8} > 0$$

or $$\frac{7935.9 - 16 \times 7848}{16(7848+T)} + \frac{(7848+T)}{15871.8} > 0$$

or $$\frac{-117632}{16(7848+T)} + \frac{(7848+T)}{15871.8} > 0$$

or $$\frac{(7848+T)}{15871.8} > \frac{117632}{16(7848+T)}$$

or $$(7848 + T)^2 > \frac{117632}{16.0} \times 15871.8$$

$$> 116689473.5$$

$$> (10802.3)^2$$

$\therefore$ $$7848 + T > 10802.3$$

$\therefore$ $$T > 10802.3 - 7848$$

$$> \mathbf{2954.3\ N.}$$ **Ans.**

$\therefore$ The force in the chain must be at least 2954.3 N so that the cylindrical buoy can be kept in vertical position. **Ans.**

Problem 4.17 *A solid cone floats in water with its apex downwards. Determine the least apex angle of cone for stable equilibrium. The specific gravity of the material of the cone is given 0.8.*

Solution. Given :

Sp. gr. of cone $= 0.8$

Density of cone, $\rho = 0.8 \times 1000 = 800\ kg/m^3$

Let $D =$ Dia. of the cone

$d =$ Dia. of cone at water level

$2\theta =$ Apex angle of cone

$H =$ Height of cone

$h =$ Depth of cone in water

$G =$ Centre of gravity of the cone

$B =$ Centre of buoyancy of the cone

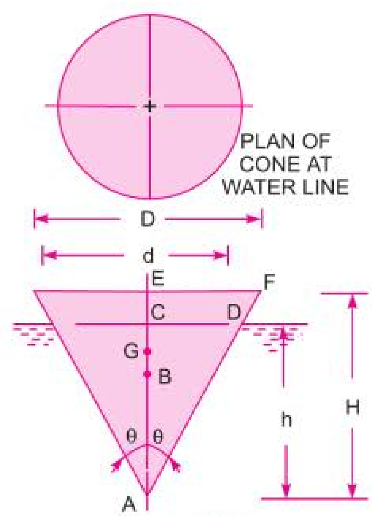


Fig. 4.21

For the cone, the distance of centre of gravity from the apex A is

$$AC = \tfrac{3}{4} \text{ height of cone} = \tfrac{3}{4} H$$

also $$AB = \tfrac{3}{4} \text{ depth of cone in water} = \tfrac{3}{4} h$$

Volume of water displaced $= \frac{1}{3}\pi r^2 \times h$

Volume of cone $= \frac{1}{3} \times \pi R^2 \times h$

$\therefore$ Weight of cone $= 800 \times g \times \frac{1}{3} \times \pi R^2 \times H$

Now from ΔAEF, $$\tan\theta = \frac{EF}{EA} = \frac{R}{H}$$

$\therefore$ $$R = H \tan\theta$$

Similarly, $$r = h \tan\theta$$

$\therefore$ Weight of cone $= 800 \times g \times \frac{1}{3} \times \pi \times (H \tan\theta)^2 \times H = \dfrac{800 \times g \times \pi \times H^3 \tan^2\theta}{3}$

$\therefore$ Weight of water displaced $= 1000 \times g \times \frac{1}{3} \times \pi r^2 \times h$

$$= 1000 \times g \times \tfrac{1}{3} \times \pi (h \tan\theta)^2 \times h = \frac{1000 \times g \times \pi \times h^3 \tan^2\theta}{3.0}$$

For equilibrium

Weight of cone = Weight of water displaced

or $$\frac{800 \times g \times \pi \times H^3 \tan^2\theta}{3.0} = \frac{1000 \times 9.81 \times \pi \times h^3 \times \tan^2\theta}{3.0}$$

or $$800 \times H^3 = 1000 \times h^3$$

$\therefore$ $$H^3 = \frac{1000}{800} \times h^3 \text{ or } \frac{H}{h} = \left(\frac{1000}{800}\right)^{1/3}$$

For stable equilibrium, Meta-centric height GM should be positive. But GM is given by

$$GM = \frac{I}{\forall} - BG$$

where I = M.O.I. of cone at water-line $= \dfrac{\pi}{64} d^4$

$\forall$ = Volume of cone in water $= \dfrac{1}{3}\dfrac{\pi}{4} d^2 \times h$

$\therefore$ $$\frac{I}{\forall} = \frac{\pi}{64} d^4 \Big/ \frac{1}{3} \times \frac{\pi}{4} d^2 \times h$$

$$= \frac{1 \times 3}{16} \times \frac{d^2}{h} = \frac{3d^2}{16h} = \frac{3}{16h} \times (2r)^2 = \frac{3}{4}\frac{r^2}{h}$$

$$= \frac{3}{4}\frac{(h \tan\theta)^2}{h} \qquad \{\because \quad r = h \tan\theta\}$$

$$= \tfrac{3}{4} h \tan^2\theta$$

and $$BG = AG - AB = \tfrac{3}{4} H - \tfrac{3}{4} h = \tfrac{3}{4}(H - h)$$

$\therefore$ $$GM = \tfrac{3}{4} h \tan^2\theta - \tfrac{3}{4}(H - h)$$

For stable equilibrium GM should be positive or

$$\tfrac{3}{4} h \tan^2\theta - \tfrac{3}{4}(H - h) > 0 \quad \text{or} \quad h \tan^2\theta - (H - h) > 0$$

or $$h \tan^2\theta > (H - h) \quad \text{or} \quad h \tan^2\theta + h > \text{H}$$

or $$h[\tan^2\theta + 1] > H \quad \text{or} \quad 1 + \tan^2\theta > H/h \quad \text{or} \quad \sec^2\theta > \frac{H}{h}$$

But $$\frac{H}{h} = \left(\frac{1000}{800}\right)^{1/3} = 1.077$$

$\therefore$ $$\sec^2\theta > 1.077 \text{ or } \cos^2\theta > \frac{1}{1.077} = 0.9285$$

$\therefore$ $$\cos\theta > 0.9635$$

$\therefore$ $$\theta > 15° \, 30' \quad \text{or} \quad 2\theta > 31°$$

$\therefore$ Apex angle (2θ) should be at least **31°**. **Ans.**

Problem 4.18 *A cone of specific gravity S, is floating in water with its apex downwards. It has a diameter D and vertical height H. Show that for stable equilibrium of the cone* $H < \frac{1}{2}\left[\frac{D^2 . S^{1/3}}{2 - S^{1/3}}\right]^{1/2}$.

Solution. Given :

Dia. of cone $= D$

Height of cone $= H$

Sp. gr. of cone $= S$

Let G = Centre of gravity of cone

B = Centre of buoyancy

2θ = Apex angle

A = Apex of the cone

h = Depth of immersion

d = Dia. of cone at water surface

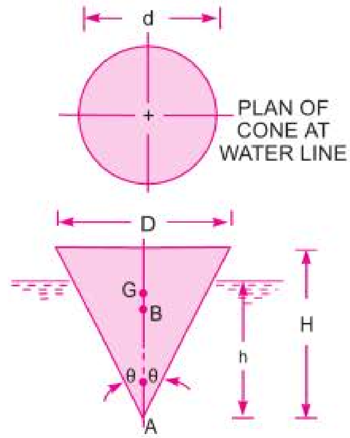


Fig. 4.22

Then $$AG = \frac{3}{4} H$$

$$AB = \frac{3}{4} h$$

Also weight of cone =Weight of water displaced.

$$1000\, S \times g \times \tfrac{1}{3}\, \pi R^2 \times \text{H} = 1000 \times g \times \frac{1}{3} \pi r^2 \times h \quad \text{or} \quad SR^2H = r^2 h$$

$\therefore$ $$h = \frac{SR^2 H}{r^2}$$

But $$\tan\theta = \frac{R}{H} = \frac{r}{h}$$

$\therefore$ $$R = H \tan\theta,\ r = h \tan\theta$$

$\therefore$ $$h = \frac{S \times (H \tan\theta)^2 \times H}{(h \tan\theta)^2}$$

$$h = \frac{S \times H^2 \times \tan^2\theta \times H}{h^2 \tan^2\theta} = \frac{SH^3}{h^2} \quad \text{or} \quad h^3 = SH^3$$

or $$h = (SH^3)^{1/3} = S^{1/3} H \qquad ...(1)$$

Distance, $$BG = AG - AB$$

$$= \frac{3}{4} H - \frac{3}{4} h = \frac{3}{4} (H - h) = \frac{3}{4} (H - S^{1/3} H) \qquad \{\because h = S^{1/3} H\}$$

$$= \frac{3}{4} H[1 - S^{1/3}] \qquad ...(2)$$

Also I = M.O. Inertia of the plan of body at water surface

$$= \frac{\pi}{64} d^4$$

$$\forall = \text{Volume of cone in water} = \frac{1}{3} \times \frac{\pi}{4} \times d^2 \times h = \frac{1}{3} \frac{\pi}{4} d^2\ [H.S^{1/3}]$$

$$\therefore \qquad \frac{I}{\forall} = \frac{\frac{\pi}{64} d^4}{\frac{1}{3} \times \frac{\pi}{4} d^2 \; H.S^{1/3}} = \frac{3d^2}{16.H.S^{1/3}}$$

Now Meta-centric height GM is given as

$$GM = \frac{I}{\forall} - BG = \frac{3d^2}{16.H.S^{1/3}} - \frac{3H}{4} [1 - S^{1/3}]$$

GM should be +ve for stable equilibrium or $GM > 0$

$$\text{or} \qquad \frac{3d^2}{16.H.S^{1/3}} - \frac{3H}{4} (1 - S^{1/3}) > 0$$

$$\text{or} \qquad \frac{3d^2}{16.H.S^{1/3}} > \frac{3H}{4} (1 - S^{1/3}) \qquad \text{...(3)}$$

Also we know $R = H \tan \theta$ and $r = h \tan \theta$

$$\therefore \qquad \frac{R}{r} = \frac{H}{h} = \frac{D}{d}$$

$$\therefore \qquad d = \frac{Dh}{H} = \frac{D}{H} \times HS^{1/3} = DS^{1/3}$$

Substituting the value of d in equation (3), we get

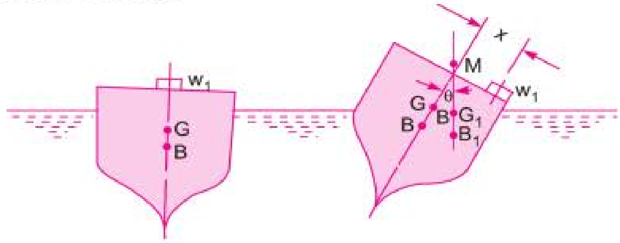
$$\frac{3\left(DS^{1/3}\right)^2}{16.H.S^{1/3}} > \frac{3H}{4} (1 - S^{1/3}) \quad \text{or} \quad \frac{D^2 . S^{1/3}}{4.H} > H (1 - S^{1/3})$$

$$\text{or} \qquad \frac{D^2 . S^{1/3}}{4\left(1 - S^{1/2}\right)} > H^2 \quad \text{or} \quad H^2 < \frac{D^2 . S^{1/3}}{4\left(1 - S^{1/3}\right)}$$

$$\text{or} \qquad H < \frac{1}{2}\left[\frac{D^2 . S^{1/3}}{1 - S^{1/3}}\right]^{1/2} . \textbf{ Ans.}$$

▶ 4.8 EXPERIMENTAL METHOD OF DETERMINATION OF META-CENTRIC HEIGHT

The meta-centric height of a floating vessel can be determined, provided we know the centre of gravity of the floating vessel. Let w_1 is a known weight placed over the centre of the vessel as shown in Fig. 4.23 (*a*) and the vessel is floating.



(a) Floating body (b) Tilted body

Fig. 4.23 *Meta-centric height.*

Let W = Weight of vessel including w_1

G = Centre of gravity of the vessel

B = Centre of buoyancy of the vessel

The weight w_1 is moved across the vessel towards right through a distance x as shown in Fig. 4.23 (*b*). The vessel will be tilted. The angle of heel θ is measured by means of a plumbline and a protractor attached on the vessel. The new centre of gravity of the vessel will shift to G_1 as the weight w_1 has been moved towards the right. Also the centre of buoyancy will change to B_1 as the vessel has tilted. Under equilibrium, the moment caused by the movement of the load w_1 through a distance x must be equal to the moment caused by the shift of the centre of gravity from G to G_1. Thus

The moment due to change of $G = GG_1 \times W = W \times GM \tan\theta$

The moment due to movement of $w_1 = w_1 \times x$

$\therefore \quad w_1 x = WGM \tan\theta$

Hence
$$GM = \frac{w_1 x}{W \tan\theta} \qquad \text{...(4.5)}$$

Problem 4.19 *A ship 70 m long and 10 m broad has a displacement of 19620 kN. A weight of 343.35 kN is moved across the deck through a distance of 6 m. The ship is tilted through 6°. The moment of inertia of the ship at water-line about its fore and aft axis is 75% of M.O.I. of the circumscribing rectangle. The centre of buoyancy is 2.25 m below water-line. Find the meta-centric height and position of centre of gravity of ship. Specific weight of sea water is 10104 N/m³.*

Solution. Given :

Length of ship, $L = 70$ m

Breadth of ship, $b = 10$ m

Displacement, $W = 19620$ kN

Angle of heel, $\theta = 6°$

M.O.I. of ship at water-line = 75% of M.O.I. of circumscribing rectangle

w for sea-water $= 10104 \text{ N/m}^3 = 10.104 \text{ kN/m}^3$

Movable weight, $w_1 = 343.35$ kN

Distance moved by w_1, $x = 6$ m

Centre of buoyancy = 2.25 m below water surface

Find (*i*) Meta-centric height, GM

(*ii*) Position of centre of gravity, G.

(*i*) **Meta-centric height, GM** is given by equation (4.5)

$$\therefore \quad GM = \frac{w_1 x}{W \tan\theta} = \frac{343.35 \text{ kN} \times 6.0}{19620 \text{ kN} \times \tan 6°}$$

$$= \frac{343.35 \text{ kN} \times 6.0}{19620 \text{ kN} \times .1051} = \mathbf{0.999 \text{ m.}} \textbf{ Ans.}$$

(*ii*) **Position of Centre of Gravity, G**

$$GM = \frac{I}{\forall} - BG$$

where I = M.O.I. of the ship at water-line about Y-Y

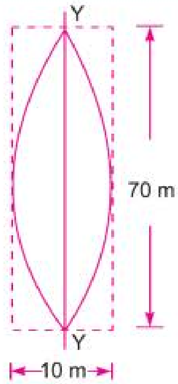


Fig. 4.24

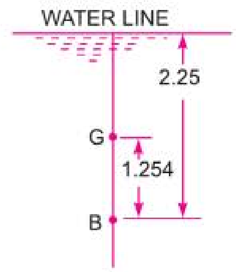


Fig. 4.25

$$= 75\% \text{ of } \frac{1}{12} \times 70 \times 10^3 = .75 \times \frac{1}{12} \times 70 \times 10^3 = 4375 \text{ m}^4$$

and $\forall$ = Volume of ship in water $= \dfrac{\text{Weight of ship}}{\text{Weight density of water}} = \dfrac{19620}{10.104} = 1941.74 \text{ m}^3$

$$\therefore \quad \frac{I}{\forall} = \frac{4375}{1941.74} = 2.253 \text{ m}$$

$$\therefore \quad GM = 2.253 - BG \text{ or } .999 = 2.253 - BG$$

$$\therefore \quad BG = 2.253 - .999 = 1.254 \text{ m.}$$

From Fig. 4.25, it is clear that the distance of G from free surface of the water = distance of B from water surface – BG

$$= 2.25 - 1.254 = \mathbf{0.996 \text{ m.}} \text{ Ans.}$$

Problem 4.20 *A pontoon of 15696 kN displacement is floating in water. A weight of 245.25 kN is moved through a distance of 8 m across the deck of pontoon, which tilts the pontoon through an angle 4°. Find meta-centric height of the pontoon.*

Solution. Given :

Weight of pontoon = Displacement

or $W = 15696$ kN

Movable weight, $w_1 = 245.25$ kN

Distance moved by weight w_1, $x = 8$ m

Angle of heel, $\theta = 4°$

The meta-centric height, GM is given by equation (4.5)

or
$$GM = \frac{w_1 x}{W \tan \theta} = \frac{245.25 \text{ kN} \times 8}{15696 \text{ kN} \times \tan 4°}$$

$$= \frac{1962}{15696 \times 0.0699} = \mathbf{1.788 \text{ m.}} \text{ Ans.}$$

▶ 4.9 OSCILLATION (ROLLING) OF A FLOATING BODY

Consider a floating body, which is tilted through an angle by an overturning couple as shown in Fig. 4.26. Let the overturning couple is suddenly removed. The body will start oscillating. Thus, the body will be in a state of oscillation as if suspended at the meta-centre M. This is similar to the case of a pendulum. The only force acting on the body is due to the restoring couple due to the weight W of the body force of buoyancy F_B.

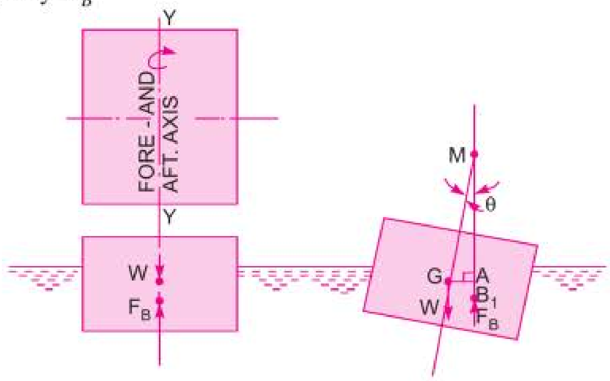


Fig. 4.26

$\therefore$ Restoring couple $= W \times \text{Distance } GA$

$$= W \times GM \sin\theta \quad \ldots(i)$$

This couple tries to decrease the angle

Angular acceleration of the body, $\alpha = -\dfrac{d^2\theta}{dt^2}$.

–ve sign has been introduced as the restoring couple tries to decrease the angle θ.

Torque due to inertia = Moment of Inertia about Y-Y $\times$ Angular acceleration

$$= I_{Y\text{-}Y} \times \left(-\frac{d^2\theta}{dt^2}\right)$$

But
$$I_{Y\text{-}Y} = \frac{W}{g} K^2$$

where W = Weight of body, K = Radius of gyration about Y-Y

$\therefore$ Inertia torque
$$= \frac{W}{g} K^2 \left(-\frac{d^2\theta}{dt^2}\right) = -\frac{W}{g} K^2 \frac{d^2\theta}{dt^2} \quad \ldots(ii)$$

Equating (i) and (ii), we get

$$W \times GM \sin\theta = -\frac{W}{g} K^2 \frac{d^2\theta}{dt^2} \quad \text{or} \quad GM \sin\theta = -\frac{K^2}{g} \frac{d^2\theta}{dt^2}$$

For small angle θ, $\sin\theta \approx \theta$

$\therefore$
$$GM \times \theta = -\frac{K^2}{g} \frac{d^2\theta}{dt^2} \quad \text{or} \quad \frac{K^2}{g} \frac{d^2\theta}{dt^2} + GM \times \theta = 0$$

Dividing by $\dfrac{K^2}{g}$, we get $\dfrac{d^2\theta}{dt^2} + \dfrac{GM \times g \times \theta}{K^2} = 0$

The above equation is a differential equation of second degree. The solution is

$$\theta = C_1 \sin\sqrt{\frac{GM.g}{K^2}} \times t + C_2 \cos\sqrt{\frac{GM.g \times t}{K^2}} \quad \ldots(iii)$$

where C_1 and C_2 are constants of integration.

The values of C_1 and C_2 are obtained from boundary conditions which are

(*i*) at $t = 0$, $\theta = 0$

(*ii*) at $t = \frac{T}{2}$, $\theta = 0$

where T is the time period of one complete oscillation.

Substituting the 1st boundary condition in (*iii*), we get

$$0 = C_1 \times 0 + C_2 \times 1 \qquad \{\because \sin\theta = 0, \cos\theta = 1\}$$

$$\therefore \qquad C_2 = 0$$

Substituting 2nd boundary conditions in (*iii*), we get

$$0 = C_1 \sin \sqrt{\frac{GM.g}{K^2}} \times \frac{T}{2}$$

But C_1 cannot be equal to zero and so the other alternative is

$$\sin \sqrt{\frac{GM.g}{K^2}} \times \frac{T}{2} = 0 = \sin \pi \qquad \{\because \sin \pi = 0\}$$

$$\therefore \qquad \sqrt{\frac{GM.g}{K^2}} \times \frac{T}{2} = \pi \quad \text{or} \quad T = 2\pi \sqrt{\frac{K^2}{GM.g}} \qquad \text{...(4.6)}$$

$\therefore$ Time period of oscillation is given by equation (4.6).

Problem 4.21 *The least radius of gyration of a ship is 8 m and meta-centric height 70 cm. Calculate the time period of oscillation of the ship.*

Solution. Given :

Least radius of gyration, $K = 8$ m

Meta-centric height, $GM = 70$ cm $= 0.70$ m

The time period of oscillation is given by equation (4.6).

$$T = 2\pi \sqrt{\frac{K^2}{GM.g}} = 2\pi \sqrt{\frac{8 \times 8}{0.7 \times 9.81}} = \textbf{19.18 sec. Ans.}$$

Problem 4.22 *The time period of rolling of a ship of weight 29430 kN in sea water is 10 seconds. The centre of buoyancy of the ship is 1.5 m below the centre of gravity. Find the radius of gyration of the ship if the moment of inertia of the ship at the water line about fore and aft axis is 1000 m^4. Take specific weight of sea water as = 10100 N/m^3.*

Solution. Given :

Time period, $T = 10$ sec

Distance between centre of buoyancy and centre of gravity, $BG = 1.5$ m

Moment of Inertia, $I = 10000 \text{ m}^4$

Weight, $W = 29430 \text{ kN} = 29430 \times 1000$ N

Let the radius of gyration $= K$

First calculate the meta-centric height GM, which is given as

$$GM = BM - BG = \frac{I}{\forall} - BG$$

where I = M.O. Inertia

and $\forall$ = Volume of water displaced

$$= \frac{\text{Weight of ship}}{\text{Sp. weight of sea water}} = \frac{29430 \times 1000}{10104} = 2912.6 \text{ m}^3$$

$$\therefore \quad GM = \frac{10000}{2912.6} - 1.5 = 3.433 - 1.5 = 1.933 \text{ m.}$$

Using equation (4.6), we get $T = 2\pi \sqrt{\dfrac{K^2}{GM \times g}}$

or $$10 = 2\pi \sqrt{\frac{K^2}{1.933 \times 9.81}} = \frac{2\pi K}{\sqrt{1.933 \times 9.81}}$$

or $$K = \frac{10 \times \sqrt{1.933 \times 9.81}}{2\pi} = \mathbf{6.93 \text{ m.}}$$ **Ans.**

HIGHLIGHTS

1. The upward force exerted by a liquid on a body when the body is immersed in the liquid is known as buoyancy or force of buoyancy.
2. The point through which force of buoyancy is supposed to act is called centre of buoyancy.
3. The point about which a body starts oscillating when the body is tilted is known as meta-centre.
4. The distance between the meta-centre and centre of gravity is known as meta-centric height.
5. The meta-centric height (GM) is given by $GM = \dfrac{I}{\forall} - BG$

   where I = Moment of Inertia of the floating body (in plan) at water surface about the axis Y-Y

   $\forall$ = Volume of the body sub-merged in water

   BG = Distance between centre of gravity and centre of buoyancy.
6. Conditions of equilibrium of a floating and sub-merged body are :

Equilibrium	*Floating Body*	*Sub-merged Body*
(*i*) Stable Equilibrium	M is above G	B is above G
(*ii*) Unstable Equilibrium	M is below G	B is below G
(*iii*) Neutral Equilibrium	M and G coincide	B and G coincide

7. The value of meta-centric height GM, experimentally is given as $GM = \dfrac{w_1 x}{W \tan \theta}$

   where w_1 = Movable weight

   x = Distance through which w_1 is moved

   W = Weight of the ship or floating body including w_1

   θ = Angle through the ship or floating body is tilted due to the movement of w_1.
8. The time period of oscillation or rolling of a floating body is given by $T = 2\pi \sqrt{\dfrac{K^2}{GM \times g}}$

   where K = Radius of gyration, GM = Meta-centric height

   T = Time of one complete oscillation.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define the terms 'buoyancy' and 'centre of buoyancy'.
2. Explain the terms 'meta-centre' and 'meta-centric height'.
3. Derive an expression for the meta-centric height of a floating body.
4. Show that the distance between the meta-centre and centre of buoyancy is given by $BM = \frac{I}{\forall}$

   where I = Moment of inertia of the plan of the floating body at water surface about longitudinal axis.

   $\forall$ = Volume of the body sub-merged in liquid.
5. What are the conditions of equilibrium of a floating body and a sub-merged body ?
6. How will you determine the meta-centric height of a floating body experimentally ? Explain with neat sketch.
7. Select the correct statement :

   (*a*) The buoyant force for a floating body passes through the

   (*i*) centre of gravity of the body (*ii*) centroid of volume of the body

   (*iii*) meta-centre of the body (*iv*) centre of gravity of the sub-merged part of the body

   (*v*) centroid of the displaced volume.

   (*b*) A body sub-merged in liquid is in equilibrium when :

   (*i*) its meta-centre is above the centre of gravity

   (*ii*) its meta-centre is above the centre of buoyancy

   (*iii*) its centre of gravity is above the centre of buoyancy

   (*iv*) its centre of buoyancy is above the centre of gravity

   (*v*) none of these. [**Ans.** 7 (*a*) (*v*), (*b*) (*iv*)]
8. Derive an expression for the time period of the oscillation of a floating body in terms of radius of gyration and meta-centric height of the floating body.
9. Define the terms : meta-centre, centre of buoyancy, meta-centric height, gauge pressure and absolute pressure.
10. What do you understand by the hydrostatic equation ? With the help of this equation, derive the expression for the buoyant force acting on a sub-merged body.
11. With neat sketches, explain the conditions of equilibrium for floating and sub-merged bodies.
12. Differentiate between :

    (*i*) Dynamic viscosity and kinematic viscosity, (*ii*) Absolute and gauge pressure (*iii*) Simple and differential manometers (*iv*) Centre of gravity and centre of buoyancy.

    (*Delhi University, Dec. 2002*)

(B) NUMERICAL PROBLEMS

1. A wooden block of width 2 m, depth 1.5 m and length 4 m floats horizontally in water. Find the volume of water displaced and position of centre of buoyancy. The specific gravity of the wooden block is 0.7.

   [**Ans.** 8.4 m^3, 0.525 m from the base]

2. A wooden log of 0.8 m diameter and 6 m length is floating in river water. Find the depth of wooden log in water when the sp. gr. of the wooden log is 0.7. **[Ans.** 0.54 m]
3. A stone weighs 490.5 N in air and 196.2 N in water. Determine the volume of stone and its specific gravity. **[Ans.** 0.03 m^3 or 3×10^4 cm^3, 1.67]
4. A body of dimensions 2.0 m × 1.0 m × 3.0 m weighs 3924 N in water. Find its weight in air. What will be its specific gravity ? **[Ans.** 62784 N, 1.0667]
5. A metallic body floats at the interface of mercury of sp. gr. 13.6 and water in such a way that 30% of its volume is sub-merged in mercury and 70% in water. Find the density of the metallic body. **[Ans.** 4780 kg/m^3]
6. A body of dimensions 0.5 m × 0.5 m × 1.0 m and of sp. gr. 3.0 is immersed in water. Determine the least force required to lift the body. **[Ans.** 4905 N]
7. A rectangular pontoon is 4 m long, 3 m wide and 1.40 m high. The depth of immersion of the pontoon is 1.0 m in sea-water. If the centre of gravity is 0.70 m above the bottom of the pontoon, determine the meta-centric height. Take the density of sea-water as 1030 kg/m^3. **[Ans.** 0.45 m]
8. A uniform body of size 4 m long × 2 m wide × 1 m deep floats in water. What is the weight of the body if depth of immersion is 0.6 m ? Determine the meta-centric height also. **[Ans**. 47088 N, 0.355 m]
9. A block of wood of specific gravity 0.8 floats in water. Determine the meta-centric height of the block if its size is 3 m × 2 m × 1 m. **[Ans.** 0.316 m]
10. A solid cylinder of diameter 3.0 m has a height of 2 m. Find the meta-centric height of the cylinder when it is floating in water with its axis vertical. The sp. gr. of the cylinder is 0.7. **[Ans.** 0.1017 m]
11. A body has the cylindrical upper portion of 4 m diameter and 2 m deep. The lower portion is a curved one, which displaces a volume of 0.9 m^3 of water. The centre of buoyancy of the curved portion is at a distance of 2.10 m below the top of the cylinder. The centre of gravity of the whole body is 1.50 m below the top of the cylinder. The total displacement of water is 4.5 tonnes. Find the meta-centric height of the body. **[Ans.** 2.387 m]
12. A solid cylinder of diameter 5.0 m has a height of 5.0 m. Find the meta-centric height of the cylinder if the specific gravity of the material of cylinder is 0.7 and it is floating in water with its axis vertical. State whether the equilibrium is stable or unstable. **[Ans.** – 0.304 m, Unstable Equilibrium]
13. A solid cylinder of 15 cm diameter and 60 cm long, consists of two parts made of different materials. The first part at the base is 1.20 cm long and of specific gravity = 5.0. The other parts of the cylinder is made of the material having specific gravity 0.6. State, if it can float vertically in water. **[Ans.** GM = – 5.26, Unstable, Equilibrium]
14. A rectangular pontoon 8.0 m long, 7 m broad and 3.0 m deep weighs 588.6 kN. It carries on its upper deck an empty boiler of 4.0 m diameter weighing 392.4 kN. The centre of gravity of the boiler and the pontoon are at their respective centres along a vertical line. Find the meta-centric height. Weight density of sea-water is 10104 N/m^3. **[Ans.** 0.325 m]
15. A wooden cylinder of sp. gr. 0.6 and circular in cross-section is required to float in oil (sp. gr. 0.8). Find the L/D ratio for the cylinder to float with its longitudinal axis vertical in oil where L is the height of cylinder and D is its diameter. **[Ans.** $(L/D) < 0.8164$]
16. Show that a cylindrical buoy of 1.5 m diameter and 3 m long weighing 2.5 tonnes will not float vertically in sea-water of density 1030 kg/m^3. Find the force necessary in a vertical chain attached at the centre of the base of the buoy that will keep it vertical. **[Ans.** 10609.5 N]
17. A solid cone floats in water its apex downwards. Determine the least apex angle of cone for stable equilibrium. The specific gravity of the material of the cone is given 0.7. **[Ans.** 39° 7′]
18. A ship 60 m long and 12 m broad has a displacement of 19620 kN. A weight of 294.3 kN is moved across the deck through a distance of 6.5 m. The ship is tilted through 5°. The moment of inertia of the ship at

water line about its force and aft axis is 75% of moment of inertia the circumscribing rectangle. The centre of buoyancy is 2.75 m below water line. Find the meta-centric height and position of centre of gravity of ship. Take specific weight of sea water = 10104 N/m^3. [**Ans.** 1.1145 m, 0.53 m below water surface]

19. A pontoon of 1500 tonnes displacement is floating in water. A weight of 20 tonnes is moved through a distance of 6 m across the deck of pontoon, which tilts the pontoon through an angle of 5°. Find meta-centric height of the pontoon. [**Ans.** 0.9145 m]

20. Find the time period of rolling of a solid circular cylinder of radius 2.5 m and 5.0 m long. The specific gravity of the cylinder is 0.9 and is floating in water with its axis vertical. [**Ans.** 0.35 sec]

5

CHAPTER

KINEMATICS OF FLOW AND IDEAL FLOW

A. KINEMATICS OF FLOW

▶ 5.1 INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

▶ 5.2 METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods. They are —(*i*) Lagrangian Method, and (*ii*) Eulerian Method. In the Lagrangian method, a **single fluid particle** is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density etc., are described **at a point** in flow field. The Eulerian method is commonly used in fluid mechanics.

▶ 5.3 TYPES OF FLUID FLOW

The fluid flow is classified as :

(*i*) Steady and unsteady flows ;

(*ii*) Uniform and non-uniform flows ;

(*iii*) Laminar and turbulent flows ;

(*iv*) Compressible and incompressible flows ;

(*v*) Rotational and irrotational flows ; and

(*vi*) One, two and three-dimensional flows.

5.3.1 Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

5.3.2 Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity

∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$

5.3.3 Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number,

where D = Diameter of pipe

V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

5.3.4 Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant}.$$

5.3.5 Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

5.3.6 One-, Two- and Three-Dimensional Flows. **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only, say x. For a steady one-dimensional flow, the velocity is a function of one-space-co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence mathematically, for one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$

where u, v and w are velocity components in x, y and z directions respectively.

Two-dimensional flow is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say x and y. For a steady two-dimensional flow the velocity is a function of two space co-ordinates only. The variation of velocity in the third direction is negligible. Thus, mathematically for two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$

Three-dimensional flow is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are functions of three space co-ordinates (x, y and z) only. Thus, mathematically, for three-dimensional flow

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

▶ 5.4 RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(*i*) For liquids the units of Q are m^3/s or litres/s

(*ii*) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section

Then discharge $\quad Q = A \times V. \quad$ ...(5.1)

▶ 5.5 CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig. 5.1.

Let V_1 = Average velocity at cross-section 1-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section 1-1

and V_2, ρ_2, A_2 are corresponding values at section, 2-2.

Then rate of flow at section 1-1 = $\rho_1 A_1 V_1$

Rate of flow at section 2-2 = $\rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

or $$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \text{...(5.2)}$$

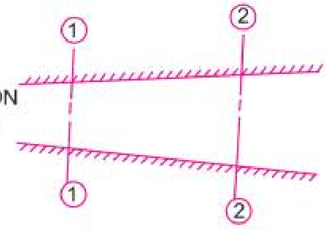


Fig. 5.1 *Fluid flowing through a pipe.*

Equation (5.2) is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (5.2) reduces to

$$A_1 V_1 = A_2 V_2 \quad \text{...(5.3)}$$

Problem 5.1 *The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.*

Solution. Given :

At section 1, $D_1 = 10 \text{ cm} = 0.1 \text{ m}$

$$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2 = 0.007854 \text{ m}^2$$

$$V_1 = 5 \text{ m/s.}$$

At section 2, $D_2 = 15 \text{ cm} = 0.15 \text{ m}$

$$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$$

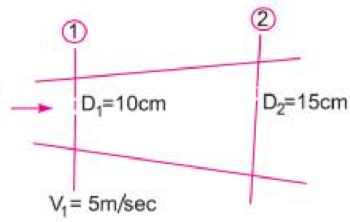


Fig. 5.2

(*i*) Discharge through pipe is given by equation (5.1)

or $$Q = A_1 \times V_1$$

$$= 0.007854 \times 5 = \mathbf{0.03927\ m^3/s.}\ \textbf{Ans.}$$

Using equation (5.3), we have $A_1 V_1 = A_2 V_2$

(*ii*) $\therefore$ $$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = \mathbf{2.22\ m/s.}\ \textbf{Ans.}$$

Problem 5.2 *A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.*

Solution. Given :

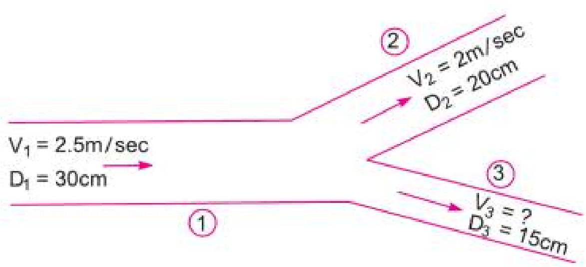


Fig. 5.3

$$D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore \quad A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

$$V_1 = 2.5 \text{ m/s}$$

$$D_2 = 20 \text{ cm} = 0.20 \text{ m}$$

$$\therefore \quad A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2,$$

$$V_2 = 2 \text{ m/s}$$

$$D_3 = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \quad A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$$

Find (*i*) Discharge in pipe 1 or Q_1

(*ii*) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \qquad \text{...(1)}$$

(*i*) **The discharge $\mathbf{Q_1}$** in pipe 1 is given by

$$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3\text{/s} = \mathbf{0.1767 \text{ m}^3\text{/s}}. \text{ Ans.}$$

(*ii*) **Value of $\mathbf{V_3}$**

$$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3\text{/s}$$

Substituting the values of Q_1 and Q_2 in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$$\therefore \quad Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3\text{/s}$$

But $\quad Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$

$$\therefore \quad V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44 \text{ m/s.}} \text{ Ans.}$$

Problem 5.3 *Water flows through a pipe AB 1.2 m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter. At C, the pipe branches. Branch CD is 0.8 m in diameter and carries one-third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.*

Solution. Given :

Diameter of pipe *AB*, $D_{AB} = 1.2$ m

Velocity of flow through *AB*, $V_{AB} = 3.0$ m/s

Dia. of pipe *BC*, $D_{BC} = 1.5$ m

Dia. of branched pipe *CD*, $D_{CD} = 0.8$ m

Velocity of flow in pipe *CE*, $V_{CE} = 2.5$ m/s

Let the flow rate in pipe $AB = Q$ m^3/s

Velocity of flow in pipe $BC = V_{BC}$ m/s

Velocity of flow in pipe $CD = V_{CD}$ m/s

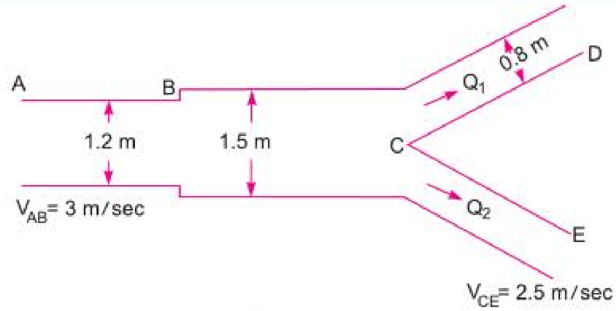


Fig. 5.4

Diameter of pipe $CE = D_{CE}$

Then flow rate through $CD = Q/3$

and flow rate through $CE = Q - Q/3 = \dfrac{2Q}{3}$

(*i*) Now volume flow rate through $AB = Q = V_{AB} \times$ Area of AB

$$= 3.0 \times \frac{\pi}{4}(D_{AB})^2 = 3.0 \times \frac{\pi}{4}(1.2)^2 = \mathbf{3.393\ m^3/s.}\ \textbf{Ans.}$$

(*ii*) Applying continuity equation to pipe AB and pipe BC,

$$V_{AB} \times \text{Area of pipe } AB = V_{BC} \times \text{Area of pipe } BC$$

or $$3.0 \times \frac{\pi}{4}(D_{AB})^2 = V_{BC} \times \frac{\pi}{4}(D_{BC})^2$$

or $$3.0 \times (1.2)^2 = V_{BC} \times (1.5)^2 \qquad \left[\text{Divide by } \frac{\pi}{4}\right]$$

or $$V_{BC} = \frac{3 \times 1.2^2}{1.5^2} = \mathbf{1.92\ m/s.}\ \textbf{Ans.}$$

(*iii*) The flow rate through pipe

$$CD = Q_1 = \frac{Q}{3} = \frac{3.393}{3} = 1.131\ \text{m}^3/\text{s}$$

$\therefore$ $$Q_1 = V_{CD} \times \text{Area of pipe } CD \times \frac{\pi}{4}(D_{CD})^2$$

or $$1.131 = V_{CD} \times \frac{\pi}{4} \times 0.8^2 = 0.5026\ V_{CD}$$

$\therefore$ $$V_{CD} = \frac{1.131}{0.5026} = \mathbf{2.25\ m/s.}\ \textbf{Ans.}$$

(*iv*) Flow rate through CE,

$$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.262\ \text{m}^3/\text{s}$$

$\therefore$ $$Q_2 = V_{CE} \times \text{Area of pipe } CE = V_{CE}\frac{\pi}{4}(D_{CE})^2$$

or $$2.263 = 2.5 \times \frac{\pi}{4} \times (D_{CE})^2$$

or $$D_{CE} = \sqrt{\frac{2.263 \times 4}{2.5 \times \pi}} = \sqrt{1.152} = 1.0735\ \text{m}$$

$\therefore$ Diameter of pipe CE = **1.0735 m. Ans.**

Problem 5.4 *A 25 cm diameter pipe carries oil of sp. gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.*

Solution. Given :

at section 1, $D_1 = 25 \text{ cm} = 0.25 \text{ m}$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049 \text{ m}^3$$

$V_1 = 3$ m/s

at section 2, $D_2 = 20 \text{ cm} = 0.2 \text{ m}$

$$A_2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$V_2 = ?$

Mass rate of flow of oil = ?

Applying continuity equation at sections 1 and 2,

$$A_1V_1 = A_2V_2$$

or $$0.049 \times 3.0 = 0.0314 \times V_2$$

$\therefore$ $$V_2 = \frac{0.049 \times 3.0}{0.0314} = \mathbf{4.68 \text{ m/s. Ans.}}$$

Mass rate of flow of oil $= \text{Mass density} \times Q = \rho \times A_1 \times V_1$

Sp. gr. of oil $= \dfrac{\text{Density of oil}}{\text{Density of water}}$

$\therefore$ Density of oil = Sp. gr. of oil × Density of water

$$= 0.9 \times 1000 \text{ kg/m}^3 = \frac{900 \text{ kg}}{\text{m}^3}$$

$\therefore$ Mass rate of flow $= 900 \times 0.049 \times 3.0$ kg/s $=$ **132.23 kg/s. Ans.**

Problem 5.5 *A jet of water from a 25 mm diameter nozzle is directed vertically upwards. Assuming that the jet remains circular and neglecting any loss of energy, that will be the diameter at a point 4.5 m above the nozzle, if the velocity with which the jet leaves the nozzle is 12 m/s.*

Solution. Given :

Dia. of nozzle, $D_1 = 25 \text{ mm} = 0.025 \text{ m}$

Velocity of jet at nozzle, $V_1 = 12$ m/s

Height of point A, $h = 4.5$ m

Let the velocity of the jet at a height 4.5 m = V_2

Consider the vertical motion of the jet from the outlet of the nozzle to the point A (neglecting any loss of energy).

Initial velocity, $u = V_1 = 12$ m/s

Final velocity, $V = V_2$

Value of $g = -9.81 \text{ m/s}^2$ and $h = 4.5$ m

Using, $V^2 - u^2 = 2gh$, we get

$$V_2^2 - 12^2 = 2 \times (-9.81) \times 4.5$$

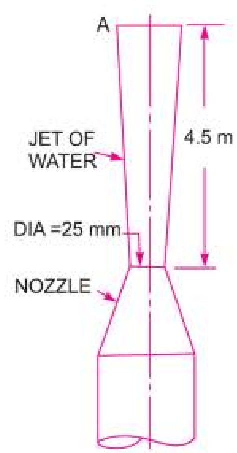


Fig. 5.5

$$\therefore \qquad V_2 = \sqrt{12^2 - 2 \times 9.81 \times 4.5} = \sqrt{144 - 88.29} = 7.46 \text{ m/s}$$

Now applying continuity equation to the outlet of nozzle and at point A, we get

$$A_1 V_1 = A_2 V_2$$

or
$$A_2 = \frac{A_1 V_1}{V_2} = \frac{\frac{\pi}{4} D_1^2 \times V_1}{V_2} = \frac{\pi \times (0.025)^2 \times 12}{4 \times 7.46} = 0.0007896$$

Let D_2 = Diameter of jet at point A.

Then
$$A_2 = \frac{\pi}{2} D_2^2 \text{ or } 0.0007896 = \frac{\pi}{4} \times D_2^2$$

$$\therefore \qquad D_2 = \sqrt{\frac{0.0007896 \times 4}{\pi}} = 0.0317 \text{ m} = \mathbf{31.7 \text{ mm.}} \text{ Ans.}$$

▶ 5.6 CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$
$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u\, dydz + \frac{\partial}{\partial x}(\rho u\, dydz)\, dx$

$\therefore$ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$
$$= \rho u\, dydz - \rho u\, dydz - \frac{\partial}{\partial x}(\rho u\, dydz) dx$$
$$= -\frac{\partial}{\partial x}(\rho u\, dydz)\, dx$$
$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction
$$= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$$

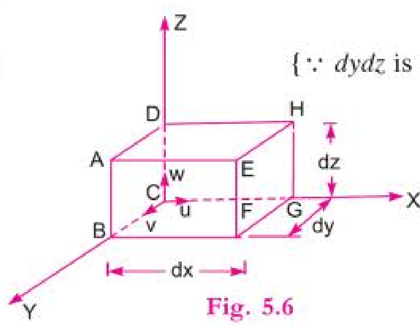


Fig. 5.6

$$\therefore \qquad \text{Net gain of masses} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is ρ. *dx*. *dy*. *dz* and its rate of increase with time is $\frac{\partial}{\partial t}$ (ρ *dx*. *dy*. *dz*) or

$\frac{\partial \rho}{\partial t}$. *dx dy dz*.

Equating the two expressions,

$$\text{or} \qquad -\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\ dxdydz$$

$$\text{or} \qquad \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad ...(5.3A)$$

Equation (5.3*A*) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t}$ = 0 and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \qquad ...(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \qquad ...(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \qquad ...(5.5)$$

5.6.1 Continuity Equation in Cylindrical Polar Co-ordinates. The continuity equation in cylindrical polar co-ordinates (*i.e.*, *r*, θ, *z* co-ordinates) is derived by the procedure given below.

Consider a two-dimensional incompressible flow field. The two-dimensional polar co-ordinates are *r* and θ. Consider a fluid element *ABCD* between the radii *r* and *r* + *dr* as shown in Fig. 5.7. The angle subtended by the element at the centre is *d*θ. The components of the velocity *V* are u_r in the radial direction and u_θ in the tangential direction. The sides of the element are having the lengths as

Side $AB = rd\theta$, $BC = dr$, $DC = (r + dr)\, d\theta$, $AD = dr$.

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

Consider the flow in radial direction

Mass of fluid entering the face AB per unit time

= ρ × Velocity in *r*-direction × Area

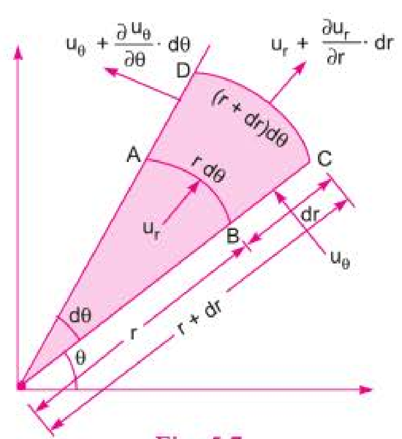


Fig. 5.7

$= \rho \times u_r \times (AB \times 1)$ ($\because$ Area $= AB \times$ Thickness $= rd\theta \times 1$)

$= \rho \times u_r \times (rd\theta \times 1) = \rho \, . \, u_r . \, rd\theta$

Mass of fluid leaving the face CD per unit time

$= \rho \times$ Velocity $\times$ Area

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}.dr\right) \times (CD \times 1) \qquad (\because \text{Area} = CD \times 1)$$

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r} dr\right) \times (r + dr)d\theta \qquad [\because CD = (r + dr)\, d\theta]$$

$$= \rho \times \left[u_r \times r + u_r dr + r\frac{\partial u_r}{\partial r} dr + \frac{\partial u_r}{\partial r}(dr)^2\right] d\theta$$

$$= \rho\left[u_r \times r + u_r \times dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

[The term containing $(dr)^2$ is very small and has been neglected]

$\therefore$ Gain of mass in r-direction per unit time

= (Mass through AB – Mass through CD) per unit time

$$= \rho.\, u_r.\, rd\theta - \rho\left[u_r.r + u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= \rho.\, u_r.\, rd\theta - \rho.\, u_r.\, r.\, d\theta - \rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= -\rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right].\, d\theta$$

$$= -\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r.\, dr.\, d\theta$$

[This is written in this form because ($r.\ d\theta.\ dr.$ 1) is equal to volume of element]

Now consider the flow in θ*-direction*

Gain in mass in θ-direction per unit time

= (Mass through BC – Mass through AD) per unit time

= [$\rho \times$ Velocity through $BC \times$ Area – $\rho \times$ Velocity through $AD \times$ Area]

$$= \left[\rho.u_\theta.dr \times 1 - \rho\left(u_\theta + \frac{\partial u_\theta}{\partial \theta}.d\theta\right) \times dr \times 1\right]$$

$$= -\rho\left(\frac{\partial u_\theta}{\partial \theta}.d\theta\right) dr \times 1 \qquad (\because \text{Area} = dr \times 1)$$

$$= -\rho\frac{\partial u_\theta}{\partial \theta}.\frac{r.d\theta.dr}{r} \qquad [\text{Multiplying and dividing by } r]$$

$\therefore$ Total gain in fluid mass per unit time

$$= -\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right].\, r.\, dr.\, d\theta - \rho\frac{\partial u_\theta}{\partial \theta}.\frac{rd\theta.\, dr}{r} \qquad ...(5.5A)$$

But mass of fluid element $= \rho \times$ Volume of fluid element

$= \rho \times [rd\theta \times dr \times 1]$

$= \rho \times rd\theta \,.\, dr$

Rate of increase of fluid mass in the element with time

$$= \frac{\partial}{\partial t} [\rho \,.\, rd\theta \,.\, dr] = \frac{\partial \rho}{\partial t} \,.\, rd\theta \, dr \qquad ...(5.5B)$$

($\because$ $rd\theta \,.\, dr \,.\, 1$ is the volume of element and is a constant quantity)

Since the mass is neither created nor destroyed in the fluid element, hence net gain of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Hence equating the two expressions given by equations (5.5 *A*) and (5.5 *B*), we get

$$-\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r \,.\, dr. \, d\theta - \rho \, \frac{\partial u_\theta}{\partial \theta} \, \frac{rd\theta \,.\, dr}{r} = \frac{\partial \rho}{\partial t} \, rd\theta \, dr$$

or
$$-\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] - \rho \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = \frac{\partial \rho}{\partial t} \qquad \text{[Cancelling } rdr \,.\, d\theta \text{ from both sides]}$$

or
$$\frac{\partial \rho}{\partial t} + \rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] + \rho \, \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = 0 \qquad ...(5.5C)$$

Equation (5.5 *C*) is the continuity equation in polar co-ordinates for two-dimensional flow.

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.5 *C*) reduces to

$$\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] + \rho \, \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = 0$$

or
$$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} \,.\, \frac{1}{r} = 0$$

or
$$u_r + r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

or
$$\frac{\partial}{\partial r} \, (ru_r) + \frac{\partial}{\partial \theta} \, (u_\theta) = 0 \quad \left[\because \frac{\partial}{\partial r}(r \,.\, u_r) = r \,.\, \frac{\partial u_r}{\partial r} + u_r\right] \qquad ...(5.5D)$$

Equation (5.5 *D*) represents the continuity equation in polar co-ordinates for two-dimensional steady incompressible flow.

Problem 5.5A *Examine whether the following velocity components represent a physically possible flow ?*

$$u_r = r \sin \theta, \; u_\theta = 2r \cos \theta.$$

Solution. Given : $u_r = r \sin \theta$ and $u_\theta = 2r \cos \theta$

For physically possible flow, the continuity equation,

$$\frac{\partial}{\partial r} \, (ru_r) + \frac{\partial}{\partial \theta} \, (u_\theta) = 0 \text{ should be satisfied.}$$

Now $u_r = r \sin \theta$

Multiplying the above equation by r, we get

$$ru_r = r^2 \sin \theta$$

Differentiating the preceding equation w.r.t. r, we get

$$\frac{\partial}{\partial r}(ru_r) = \frac{\partial}{\partial r}(r^2 \sin\theta)$$

$$= 2r \sin\theta \qquad (\because \ \sin\theta \text{ is constant w.r.t. } r)$$

Now $\qquad u_\theta = 2r\cos\theta$

Differentiating the above equation w.r.t. θ, we get

$$\frac{\partial}{\partial\theta}(u_\theta) = \frac{\partial}{\partial\theta}(2r\cos\theta)$$

$$= 2r(-\sin\theta) \qquad (\because \ 2r \text{ is constant w.r.t. } \theta)$$

$$= -2r\sin\theta$$

$$\therefore \qquad \frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial\theta}(u_\theta) = 2r\sin\theta - 2r\sin\theta = 0$$

Hence the continuity equation is satisfied. Hence the given velocity components represent a physically possible flow.

▶ 5.7 VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $\qquad V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\qquad \frac{dx}{dt} = u, \frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$

$$\left.\begin{aligned} \therefore \qquad a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ \text{Similarly,} \qquad a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned}\right] \qquad ...(5.6)$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity or $$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and} \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right] \quad \text{...(5.7)}$$

Acceleration vector $$\left.\begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2}\,. \end{aligned}\right] \quad \text{...(5.8)}$$

5.7.1 Local Acceleration and Convective Acceleration. **Local acceleration** is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the equation given by (5.6), the expression $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as local acceleration.

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expressions other than $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ in equation (5.6) are known as convective acceleration.

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 yj + 2tk.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time $t = 1$.

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$
$$v = -10(2)^2(1) = -40 \text{ units}$$
$$w = 2 \times 1 = 2 \text{ units}$$

$\therefore$ Velocity vector V at (2, 1, 3) $= 32i - 40j + 2k$

or Resultant velocity $= \sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \mathbf{51.26 \text{ units.}} \text{ Ans.}$$

Acceleration is given by equation (5.6)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -\ 20xy,\ \frac{\partial v}{\partial y} = -\ 10x^2,\ \frac{\partial v}{\partial z} = 0\ \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0,\ \frac{\partial w}{\partial y} = 0,\ \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3\,(12x^2) + (-\ 10x^2y)\,(0) + 2t \times (0) + 0$$
$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$
$$a_y = 4x^3\,(-\ 20xy) + (-\ 10x^2y)\,(-\ 10x^2) + 2t\,(0) + 0$$
$$= -\ 80x^4y + 100x^4y$$
$$= -\ 80\,(2)^4\,(1) + 100\,(2)^4 \times 1 = -\ 1280 + 1600 = 320 \text{ units.}$$
$$a_z = 4x^3\,(0) + (-\ 10x^2y)\,(0) + (2t)\,(0) + 2.1 = 2.0 \text{ units}$$

$\therefore$ Acceleration is $\quad A = a_xi + a_yj + a_zk =$ **1536i + 320j + 2k. Ans.**

or Resultant $\quad A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$$= \sqrt{2359296 + 102400 + 4} = \mathbf{1568.9 \text{ units.}}$$ **Ans.**

Problem 5.7 *The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :*

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2$, $w = 2xyz$.

Solution. The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Case I. $\quad u = x^2 + y^2 + z^2 \quad \therefore \quad \frac{\partial u}{\partial x} = 2x$

$$v = xy^2 - yz^2 + xy \quad \therefore \quad \frac{\partial v}{\partial y} = 2xy - z^2 + x$$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation.

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or $\quad \frac{\partial w}{\partial z} = -\ 3x - 2xy + z^2$ or $\partial w = (-\ 3x - 2xy + z^2)\ \partial z$

Integration of both sides gives $\int dw = \int\left(-3x - 2xy + z^2\right) dz$

or $$w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + \text{Constant of integration},$$

where constant of integration cannot be a function of z. But it can be a function of x and y that is $f(x, y)$.

$$\therefore \quad w = \left(\mathbf{-3xz - 2xyz + \frac{z^3}{3}}\right) + \mathbf{f(x, y)}. \text{ **Ans.**}$$

Case II. $$v = 2y^2 \quad \therefore \quad \frac{\partial v}{\partial y} = 4y$$

$$w = 2xyz \quad \therefore \quad \frac{\partial w}{\partial z} = 2xy$$

Substituting the values of $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or $$\frac{\partial u}{\partial x} = -4y - 2xy \text{ or } du = (-4y - 2xy)\, dx$$

Integrating, we get $$u = -4xy - 2y\frac{x^2}{2} + f(y, z) = \mathbf{-4xy - x^2y + f(y, z)}. \text{ **Ans.**}$$

Problem 5.8 *A fluid flow field is given by*

$$V = x^2yi + y^2zj - (2xyz + yz^2)k$$

Prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2, 1, 3).

Solution. For the given fluid flow field $u = x^2y \quad \therefore \quad \frac{\partial u}{\partial x} = 2xy$

$$v = y^2z \quad \therefore \quad \frac{\partial v}{\partial y} = 2yz$$

$$w = -2xyz - yz^2 \quad \therefore \quad \frac{\partial w}{\partial z} = -2xy - 2yz.$$

For a case of possible steady incompressible fluid flow, the continuity equation (5.4) should be satisfied.

i.e., $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

Hence the velocity field $V = x^2yi + y^2zj - (2xyz + yz^2)\,k$ is a possible case of fluid flow. **Ans.**

Velocity at (2, 1, 3)

Substituting the values $x = 2$, $y = 1$ and $z = 3$ in velocity field, we get

$$V = x^2yi + y^2zj - (2xyz + yz^2)\,k$$
$$= 2^2 \times 1i + 1^2 \times 3j - (2 \times 2 \times 1 \times 3 + 1 \times 3^2)\,k$$
$$= \mathbf{4i + 3j - 21k}. \textbf{ Ans.}$$

and Resultant velocity $= \sqrt{4^2 + 3^2 + (-21)^2} = \sqrt{16 + 9 + 441} = \sqrt{466}$ = **21.587 units. Ans.**

Acceleration at (2, 1, 3)

The acceleration components a_x, a_y and a_z for steady flow are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$u = x^2y, \frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 \text{ and } \frac{\partial u}{\partial z} = 0$$

$$v = y^2z, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2yz, \frac{\partial v}{\partial z} = y^2$$

$$w = -2xyz - yz^2, \frac{\partial w}{\partial x} = -2yz, \frac{\partial w}{\partial y} = -2xz - z^2, \frac{\partial w}{\partial z} = -2xy - 2yz.$$

Substituting these values in acceleration components, we get acceleration at (2, 1, 3)

$$a_x = x^2y\,(2xy) + y^2z\,(x)^2 - (2xyz + yz^2)\,(0)$$
$$= 2x^3y^2 + x^2y^2z$$
$$= 2\,(2)^3 1^2 + 2^2 \times 1^2 \times 3 = 2 \times 8 + 12$$
$$= 16 + 12 = 28 \text{ units}$$
$$a_y = x^2y\,(0) + y^2z\,(2yz) - (2xyz + yz^2)\,(y^2)$$
$$= 2y^3z^2 - 2xy^3z - y^3z^2$$
$$= 2 \times 1^3 \times 3^2 - 2 \times 2 \times 1^3 \times 3 - 1^3 \times 3^2 = 18 - 12 - 9 = -3 \text{ units}$$
$$a_z = x^2y\,(-2yz) + y^2z\,(-2xz - z^2) - (2xyz + yz^2)\,(-2xy - 2yz)$$
$$= -2x^2y^2z - 2xy^2z^2 - y^2z^3 + [4x^2y^2z + 2xy^2z^2 + 4xy^2z^2 + 2y^2z^3]$$
$$= -2 \times 2^2 \times 1^2 \times 3 - 2 \times 2 \times 1^2 \times 3^2 - 1^2 \times 3^3$$
$$+ [4 \times 2^2 \times 1^2 \times 3 + 2 \times 2 \times 1^2 \times 3^2 + 4 \times 2 \times 1^2 \times 3^2 + 2 \times 1^2 \times 3^3]$$
$$= -24 - 36 - 27 + [48 + 36 + 72 + 54]$$
$$= -24 - 36 - 27 + 48 + 36 + 72 + 54 = 123$$

∴ Acceleration $= a_xi + a_yj + a_zk$ = **28i – 3j + 123k. Ans.**

or Resultant acceleration $= \sqrt{28^2 + (-3)^2 + 123^2} = \sqrt{784 + 9 + 15129}$

$= \sqrt{15922}$ = **126.18 units. Ans.**

Problem 5.9 *Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m diameter to 0.2 m diameter over 2 m length. The rate of flow is 20 lit/s. If the rate of flow changes uniformly from 20 l/s to 40 l/s in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.*

Solution. Given :

Diameter at section 1, $D_1 = 0.4$ m ; $D_2 = 0.2$ m, $L = 2$ m, $Q = 20$ l/s $= 0.02$ m^3/s as one litre $= 0.001\text{m}^3 = 1000$ cm^3

Find (*i*) Convective acceleration at middle *i.e.*, at *A* when $Q = 20$ l/s.

(*ii*) Total acceleration at *A* when Q changes from 20 l/s to 40 l/s in 30 seconds.

Case I. In this case, the rate of flow is constant and equal to 0.02 m^3/s. The velocity of flow is in x-direction only. Hence this is one-dimensional flow and velocity components in y and z directions are zero or $v = 0$, $z = 0$.

$$\therefore \quad \text{Convective acceleration} = u\frac{\partial u}{\partial y} \text{ only} \qquad \ldots(i)$$

Let us find the value of u and $\dfrac{\partial u}{\partial x}$ at a distance x from inlet

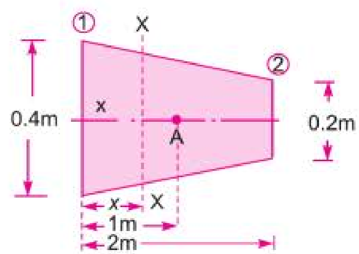


Fig. 5.8

The diameter (D_x) at a distance x from inlet or at section X-X is given by,

$$D_x = 0.4 - \frac{0.4 - 0.2}{2} \times x$$
$$= (0.4 - 0.1\,x) \text{ m}$$

The area of cross-section (A_x) at section X-X is given by,

$$A_x = \frac{\pi}{4} D_x^2 = \frac{\pi}{4}(0.4 - 0.1\,x)^2$$

Velocity (u) at the section X-X in terms of Q (*i.e.*, in terms of rate of flow)

$$u = \frac{Q}{\text{Area}} = \frac{Q}{A_x} = \frac{Q}{\frac{\pi}{4} D_x^2} = \frac{4Q}{\pi\,(0.4 - 0.1x)^2}$$

$$= \frac{1.273Q}{(0.4 - 0.1x)^2} = 1.273\,Q\,(0.4 - 0.1\,x)^{-2} \text{ m/s} \qquad \ldots(ii)$$

To find $\dfrac{\partial u}{\partial x}$, we must differentiate equation (*ii*) with respect to x.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[1.273\,Q\,(0.4 - 0.1\,x)^{-2}]$$
$$= 1.273\,Q\,(-2)\,(0.4 - 0.1\,x)^{-1} \times (-0.1) \qquad \text{[Here Q is constant]}$$
$$= 0.2546\,Q\,(0.4 - 0.1\,x)^{-1} \qquad \ldots(iii)$$

Substituting the value of u and $\dfrac{\partial u}{\partial x}$ in equation (*i*), we get

$$\text{Convective acceleration} = [1.273\,Q\,(0.4 - 0.1\,x)^{-2}] \times [0.2546\,Q\,(0.4 - 0.1\,x)^{-1}]$$
$$= 1.273 \times 0.2546 \times Q^2 \times (0.4 - 0.1\,x)^{-3}$$

$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1\,x)^{-3} \quad [\because Q = 0.02 \text{ m}^3/\text{s}]$$

$\therefore$ Convective acceleration at the middle (where $x = 1$ m)

$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1 \times 1)^{-3} \text{ m/s}^2$$
$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.3)^{-3} \text{ m/s}^2$$
$$= \mathbf{0.0048 \text{ m/s}^2}.$$ **Ans.**

Case II. When Q changes from 0.02 m^3/s to 0.04 m^3/s in 30 seconds, find the total acceleration at $x = 1$ m and $t = 15$ seconds.

Total acceleration = Convective acceleration + Local acceleration at $t = 15$ seconds.

The rate of flow at $t = 15$ seconds is given by

$$Q = Q_1 + \frac{Q_2 - Q_1}{30} \times 15 \text{ where } Q_2 = 0.04 \text{ m}^3/\text{s and } Q_1 = 0.02 \text{ m}^3/\text{s}$$

$$= 0.02 + \frac{(0.04 - 0.02)}{30} \times 15 = 0.03 \text{ m}^3/\text{s}$$

The velocity (u) and gradient $\left(\frac{\partial u}{\partial x}\right)$ in terms of Q are given by equations (*ii*) and (*iii*) respectively

$$\therefore \text{ Convective acceleration} = u.\frac{\partial u}{\partial x}$$
$$= [\,1.273\, Q\,(0.4 - 0.1\,x)^{-2}] \times [0.2546\, Q\,(0.4 - 0.1\,x)^{-1}]$$
$$= 1.273 \times 0.2546\, Q^2 \times (0.4 - 0.1 \times 1)^{-3}$$

$\therefore$ Convective acceleration (when $Q = 0.03$ m^3/s and $x = 1$ m)

$$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.4 - 0.1 \times 1)^{-3}$$
$$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.3)^{-3} \text{ m/s}^2$$
$$= 0.0108 \text{ m/s}^2 \quad \text{...(iv)}$$

$$\text{Local acceleration} = \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\,[1.273\, Q\,(0.4 - 0.1\,x)^{-2}]$$

$[\because u$ from equation (*ii*) is $u = 1.273\, Q\,(0.4 - 0.1\,x)^{-2}]$

$$= 1.273 \times (0.4 - 0.1\,x)^{-2} \times \frac{\partial Q}{\partial t}$$

[$\because$ Local acceleration is at a point where x is constant but Q is changing]

Local acceleration (at $x = 1$ m)

$$= 1.273 \times (0.4 - 0.1 \times 1)^{-2} \times \frac{\partial Q}{\partial t}$$

$$= 1.273 \times (0.3)^{-2} \times \frac{0.02}{30} \quad \left[\because \frac{\partial Q}{\partial t} = \frac{Q_2 - Q_1}{t} = \frac{0.04 - 0.02}{30} = \frac{0.02}{30}\right]$$

$$= 0.00943 \text{ m/s}^2 \quad \text{...(v)}$$

Hence adding equations (*iv*) and (*v*), we get total acceleration.

$\therefore$ Total acceleration = Convective acceleration + Local acceleration

$$= 0.0108 + 0.00943 = \mathbf{0.02023 \text{ m/s}^2}.$$ **Ans.**

▶ 5.8 VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

5.8.1 Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \quad \text{...(5.9)}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by

$$\left.\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{aligned}\right\} \quad \text{...(5.9}A\text{)}$$

where u_r = velocity component in radial direction (*i.e.*, in r direction)
and u_θ = velocity component in tangential direction (*i.e.*, in θ direction)

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad \text{...(5.10)}$$

Equation (5.10) is a Laplace equation.

For two-dimension case, equation (5.10) reduces to $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. ...(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

* Please, refer to equation (5.17) on page 192.

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Substituting the values, of u, v and w from equation (5.9) in the above rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

and
$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$; $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$; etc.

$\therefore$
$$\omega_z = \omega_y = \omega_x = 0.$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential (ϕ) exists, the flow should be irrotational.
2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

5.8.2 Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \text{and} \quad \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\} \quad \text{...(5.12)}$$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \quad \text{...(5.12}A\text{)}$$

where u_r= radial velocity and u_θ = tangential velocity

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y}+\frac{\partial^2 \psi}{\partial x \partial y}=0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$.

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]=\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial \psi}{\partial y^2}=0$

which is Laplace equation for ψ.

The **properties** of stream function (ψ) are :

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational.
2. If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

5.8.3 Equipotential Line. A line along which the velocity potential ϕ is constant, is called equipotential line.

For equipotential line $\quad \phi = \text{Constant}$

$\therefore \quad d\psi = 0$

But $\quad \phi = f(x, y)$ for steady flow

$$\therefore \quad d\phi=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy$$

$$= -udx - vdy \qquad \left\{\because \frac{\partial \phi}{\partial x}=-u, \frac{\partial \phi}{\partial x}=-v\right\}$$

$$= -(udx + vdy).$$

For equipotential line, $\quad d\phi = 0$

or $\quad -(udx + vdy) = 0$ or $udx + vdy = 0$

$$\therefore \quad \frac{dy}{dx}=-\frac{u}{v} \qquad ...(5.13)$$

But $\quad \frac{dy}{dx}$ = Slope of equipotential line.

5.8.4 Line of Constant Stream Function

$\psi = \text{Constant}$

$\therefore \quad d\psi = 0$

But $$d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=+vdx-udy \qquad \left\{\because \frac{\partial \psi}{\partial x}=v;\frac{\partial \psi}{\partial y}=-u\right\}$$

For a line of constant stream function

$$= d\psi = 0 \text{ or } vdx - udy = 0$$

or

$$\frac{dy}{dx} = \frac{v}{u} \qquad ...(5.14)$$

But $\frac{dy}{dx}$ is slope of stream line.

From equations (5.13) and (5.14) it is clear that the product of the slope of the equipotential line and the slope of the stream line at the point of intersection is equal to – 1. Thus the equipotential lines are othogonal to the stream lines at all points of intersection.

5.8.5 Flow Net. A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analysing two-dimensional irrotational flow problems.

5.8.6 Relation between Stream Function and Velocity Potential Function

From equation (5.9),

we have

$$u = -\frac{\partial\phi}{\partial x} \text{ and } v = -\frac{\partial\phi}{\partial y}$$

From equation (5.12), we have $u = -\frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$

Thus, we have

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y} \text{ and } v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

Hence

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \text{and} \quad \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x} \end{aligned}\right\} \qquad ...(5.15)$$

Problem 5.10 *The velocity potential function (ϕ) is given by an expression*

$$\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$$

(*i*) *Find the velocity components in x and y direction.*
(*ii*) *Show that ϕ represents a possible case of flow.*

Solution. Given :

$$\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$$

The partial derivatives of ϕ w.r.t. x and y are

$$\frac{\partial\phi}{\partial x} = -\frac{y^3}{3} - 2x + \frac{3x^2y}{3} \qquad ...(1)$$

and

$$\frac{\partial\phi}{\partial y} = -\frac{3xy^2}{3} + \frac{x^3}{3} + 2y \qquad ...(2)$$

(*i*) The velocity components u and v are given by equation (5.9)

$$u = -\frac{\partial \phi}{\partial x} = -\left[-\frac{y^3}{3} - 2x + \frac{3x^2 y}{3}\right] = \frac{y^3}{3} + 2x - x^2 y$$

$$\therefore \quad u = \mathbf{\frac{y^2}{3} + 2x - x^2y.}\ \textbf{Ans.}$$

$$\therefore \quad v = -\frac{\partial \phi}{\partial y} = -\left[-\frac{3xy^2}{3} + \frac{x^3}{3} + 2y\right] = \frac{3xy^2}{3} - \frac{x^3}{3} - 2y = \mathbf{xy^2 - \frac{x^3}{3} - 2y.}$$

Ans.

(*ii*) The given value of ϕ, will represent a possible case of flow if it satisfies the Laplace equation, *i.e.*,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

From equations (1) and (2), we have

Now
$$\frac{\partial \phi}{\partial x} = -y^3/3 - 2x + x^2 y$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} = -2 + 2xy$$

and
$$\frac{\partial \phi}{\partial y} = -xy^2 + \frac{x^3}{3} + 2y$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial y^2} = -2xy + 2$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (-2 + 2xy) + (-2xy + 2) = 0$$

$\therefore$ Laplace equation is satisfied and hence ϕ represent a possible case of flow. **Ans.**

Problem 5.11 *The velocity potential function is given by $\phi = 5(x^2 - y^2)$. Calculate the velocity components at the point (4, 5).*

Solution.
$$\phi = 5(x^2 - y^2)$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 10x$$

$$\frac{\partial \phi}{\partial y} = -10y.$$

But velocity components u and v are given by equation (5.9) as

$$u = -\frac{\partial \phi}{\partial x} = -10x$$

$$v = -\frac{\partial \phi}{\partial y} = -(-10y) = 10y$$

The velocity components at the point (4, 5), *i.e.*, at $x = 4$, $y = 5$

$u = -10 \times 4 =$ **– 40 units. Ans.**

$v = 10 \times 5 =$ **50 units. Ans.**

Problem 5.12 *A stream function is given by $\psi = 5x - 6y$.*
Calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

Solution. $\psi = 5x - 6y$

$$\therefore \quad \frac{\partial \psi}{\partial x} = 5 \text{ and } \frac{\partial \psi}{\partial y} = -6.$$

But the velocity components u and v in terms of stream function are given by equation (5.12) as

$$u = -\frac{\partial \psi}{\partial y} = -(-6) = \mathbf{6 \text{ units/sec.}} \text{ Ans.}$$

$$v = \frac{\partial \psi}{\partial x} = \mathbf{5 \text{ units/sec.}} \text{ Ans.}$$

Resultant velocity $= \sqrt{u^2 + v^2} = \sqrt{6^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61} = 7.81$ unit/sec

Direction is given by, $\tan \theta = \dfrac{v}{u} = \dfrac{5}{6} = 0.833$

$$\therefore \quad \theta = \tan^{-1} .833 = \mathbf{39° \ 48'}. \text{ Ans.}$$

Problem 5.13 *If for a two-dimensional potential flow, the velocity potential is given by*

$$\phi = x(2y - 1)$$

determine the velocity at the point P (4, 5). Determine also the value of stream function ψ at the point P.

Solution. Given : $\phi = x(2y - 1)$

(*i*) The velocity components in the direction of x and y are

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}[x(2y-1)] = -[2y - 1] = 1 - 2y$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}[x(2y-1)] = -[2x] = -2x$$

At the point P (4, 5), *i.e.*, at $x = 4$, $y = 5$

$$u = 1 - 2 \times 5 = -9 \text{ units/sec}$$

$$v = -2 \times 4 = -8 \text{ units/sec}$$

$\therefore$ Velocity at P $= -9i - 8j$

or Resultant velocity at P $= \sqrt{9^2 + 8^2} = \sqrt{81 + 64} = 12.04$ units/sec $=$ **12.04 units/sec. Ans.**

(*ii*) **Value of Stream Function at P**

We know that
$$\frac{\partial \psi}{\partial y} = -u = -(1 - 2y) = 2y - 1 \qquad \ldots(i)$$

and
$$\frac{\partial \psi}{\partial x} = v = -2x \qquad \ldots(ii)$$

Integrating equation (*i*) w.r.t. 'y', we get

$$\int d\psi = \int (2y - 1)\, dy \text{ or } \psi = \frac{2y^2}{2} - y + \text{Constant of integration.}$$

The constant of integration is not a function of y but it can be a function of x. Let the value of constant of integration is k. Then

$$\psi = y^2 - y + k. \qquad ...(iii)$$

Differentiating the above equation w.r.t. 'x', we get

$$\frac{\partial \psi}{\partial x} = \frac{\partial k}{\partial x}.$$

But from equation (*ii*), $\dfrac{\partial \psi}{\partial x} = -2x$

Equating the value of $\dfrac{\partial \psi}{\partial x}$, we get $\dfrac{\partial k}{\partial x} = -2x$.

Integrating this equation, we get $k = \int -2x dx = -\dfrac{2x^2}{2} = -x^2$.

Substituting this value of k in equation (*iii*), we get $\psi = \mathbf{y^2 - y - x^2}$. **Ans.**

∴ Stream function ψ at P (4, 5) $= 5^2 - 5 - 4^2 = 25 - 5 - 16 =$ **4 units. Ans.**

Problem 5.14 *The stream function for a two-dimensional flow is given by $\psi = 2xy$, calculate the velocity at the point P (2, 3). Find the velocity potential function ϕ.*

Solution. Given : $\psi = 2xy$

The velocity components u and v in terms of ψ are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(2xy) = -2x$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y.$$

At the point P (2, 3), we get $u = -2 \times 2 = -4$ units/sec

$v = 2 \times 3 = 6$ units/sec

∴ Resultant velocity at $P = \sqrt{u^2 + v^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} = 7.21$ units/sec.

Velocity Potential Function ϕ

We know
$$\frac{\partial \phi}{\partial x} = -u = -(-2x) = 2x \qquad ...(i)$$

$$\frac{\partial \phi}{\partial y} = -v = -2y \qquad ...(ii)$$

Integrating equation (*i*), we get

$$\int d\phi = \int 2x dx$$

or
$$\phi = \frac{2x^2}{2} + C = x^2 + C \qquad ...(iii)$$

where C is a constant which is independent of x but can be a function of y.

Differentiating equation (*iii*) w.r.t. 'y', we get $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial C}{\partial y}$

But from (*ii*), $$\frac{\partial \phi}{\partial y} = -2y$$

$$\therefore \qquad \frac{\partial C}{\partial y} = -2y$$

Integrating this equation, we get $C = \int -2y \, dy = -\frac{2y^2}{2} = -y^2$

Substituting this value of C in equation (*iii*), we get $\phi = \mathbf{x^2 - y^2}$. **Ans.**

Problem 5.15 *Sketch the stream lines represented by $\psi = x^2 + y^2$. Also find out the velocity and its direction at point (1, 2).*

Solution. Given : $\psi = x^2 + y^2$

The velocity components u and v are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(x^2 + y^2) = -2y$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

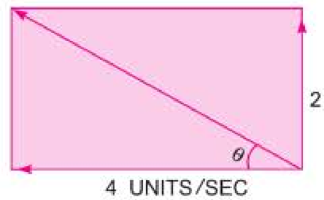


Fig. 5.9

At the point (1, 2), the velocity components are

$$u = -2 \times 2 = -4 \text{ units/sec}$$
$$v = 2 \times 1 = 2 \text{ units/sec}$$

Resultant velocity $$= \sqrt{u^2 + v^2} = \sqrt{(-4)^2 + 2^2}$$
$$= \sqrt{20} = 4.47 \text{ units/sec}$$

and $$\tan \theta = \frac{v}{u} = \frac{2}{4} = \frac{1}{2}$$

$$\therefore \qquad \theta = \tan^{-1} .5 = 26° \, 34'$$

$\therefore$ Resultant velocity makes an angle of 26° 34′ with x-axis.

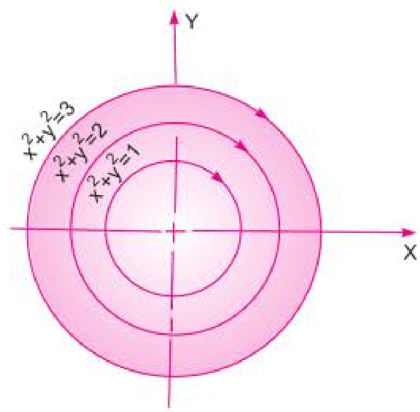


Fig. 5.10

Sketch of Stream Lines

$$\psi = x^2 + y^2$$

Let $\psi = 1, 2, 3$ and so on.

Then we have
$$1 = x^2 + y^2$$
$$2 = x^2 + y^2$$
$$3 = x^2 + y^2$$

and so on.

Each equation is a equation of a circle. Thus we shall get concentric circles of different diameters as shown in Fig. 5.10.

Problem 5.16 *The velocity components in a two-dimensional flow field for an incompressible fluid are as follows :*

$$u = \frac{y^3}{3} + 2x - x^2 y \text{ and } v = xy^2 - 2y - x^3/3$$

obtain an expression for the stream function ψ.

Solution. Given : $u = y^3/3 + 2x - x^2y$

$v = xy^2 - 2y - x^3/3.$

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = xy^2 - 2y - x^3/3 \quad \ldots(i)$$

$$\frac{\partial \psi}{\partial y} = -u = -y^3/3 - 2x + x^2y \quad \ldots(ii)$$

Integrating (i) w.r.t. x, we get $\psi = \int (xy^2 - 2y - x^3/3)\, dx$

or
$$\psi = \frac{x^2y^2}{2} - 2xy - \frac{x^4}{4 \times 3} + k, \quad \ldots(iii)$$

where k is a constant of integration which is independent of x but can be a function of y.

Differentiating equation (iii) w.r.t. y, we get

$$\frac{\partial \psi}{\partial y} = \frac{2x^2y}{2} - 2x + \frac{\partial k}{\partial y} = x^2y - 2x + \frac{\partial k}{\partial y}$$

But from (ii),
$$\frac{\partial \psi}{\partial y} = -y^3/3 - 2x + x^2y$$

Comparing the value of $\frac{\partial \psi}{\partial y}$, we get $x^2y - 2x + \frac{\partial k}{\partial y} = -y^3/3 - 2x + x^2y$

$\therefore$
$$\frac{\partial k}{\partial y} = -y^3/3$$

Integrating, we get
$$k = \int (-y^3/3)\, dy = \frac{-y^4}{4 \times 3} = \frac{-y^4}{12}$$

Substituting this value in (iii), we get

$$\psi = \frac{\mathbf{x^2y^2}}{\mathbf{2}} - \mathbf{2xy} - \frac{\mathbf{x^4}}{\mathbf{12}} - \frac{\mathbf{y^4}}{\mathbf{12}}.\ \textbf{Ans.}$$

Problem 5.17 *In a two-dimensional incompressible flow, the fluid velocity components are given by*

$u = x - 4y$ *and* $v = -y - 4x.$

Show that velocity potential exists and determine its form. Find also the stream function.

Solution. Given : $u = x - 4y$ and $v = -y - 4x$

$\therefore$
$$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1$$

$\therefore$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

Hence flow is continuous and velocity potential exists.

Let ϕ = Velocity potential.

Let velocity components in terms of velocity potential is given by

$$\frac{\partial \phi}{\partial x} = -u = -(x - 4y) = -x + 4y \qquad ...(i)$$

and

$$\frac{\partial \phi}{\partial y} = -v = -(-y - 4x) = y + 4x \qquad ...(ii)$$

Integrating equation (*i*), we get $\phi = -\frac{x^2}{2} + 4xy + C \qquad ...(iii)$

where C is a constant of integration, which is independent of x.

This constant can be a function of y.

Differentiating the above equation, *i.e.*, equation (*iii*) with respect to 'y', we get

$$\frac{\partial \phi}{\partial y} = 0 + 4x + \frac{\partial C}{\partial y}$$

But from equation (*iii*), we have $\frac{\partial \phi}{\partial y} = y + 4x$

Equating the two values of $\frac{\partial \phi}{\partial y}$, we get

$$4x + \frac{\partial C}{\partial y} = y + 4x \quad \text{or} \quad \frac{\partial C}{\partial y} = y$$

Integrating the above equation, we get

$$C = \frac{y^2}{2} + C_1$$

where C_1 is a constant of integration, which is independent of x and y.

Taking it equal to zero, we get $C = \frac{y^2}{2}$.

Substituting the value of C in equation (*iii*), we get

$$\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2}.$$ **Ans.**

Value of Stream functions

Let ψ = Stream function

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = -y - 4x \qquad ...(iv)$$

and

$$\frac{\partial \psi}{\partial y} = -u = -(x - 4y) = -x + 4y \qquad ...(v)$$

Integrating equation (*iv*) w.r.t. x, we get

$$\psi = -yx - \frac{4x^2}{2} + k \qquad ...(vi)$$

where k is a constant of integration which is independent of x but can be a function of y.

Differentiating equation (*vi*) w.r.t. *y*, we get $\dfrac{\partial \psi}{\partial y} = -x - 0 + \dfrac{\partial k}{\partial y}$

But from equation (*v*), we have $\dfrac{\partial \psi}{\partial y} = -x + 4y$

Equating the two values of $\dfrac{\partial \psi}{\partial y}$, we get $-x + \dfrac{\partial k}{\partial y} = -x + 4y$ or $\dfrac{\partial k}{\partial y} = 4y$

Integrating the above equation, we get $k = \dfrac{4y^2}{2} = 2y^2$

Substituting the value of *k* in equation (*vi*), we get

$$\psi = -yx - 2x^2 + 2y^2. \text{ Ans.}$$

5.9 TYPES OF MOTION

A fluid particle while moving may undergo anyone or combination of following four types of displacements :

(*i*) Linear Translation or Pure Translation,
(*ii*) Linear Deformation,
(*iii*) Angular Deformation, and
(*iv*) Rotation.

5.9.1 Linear Translation. It is defined as the movement of a fluid element in such a way that it moves bodily from one position to another position and the two axes *ab* and *cd* represented in new positions by *a′b′* and *c′d′* are parallel as shown in Fig. 5.11 (*a*).

5.9.2 Linear Deformation. It is defined as the deformation of a fluid element in linear direction when the element moves. The axes of the element in the deformed position and un-deformed position are parallel, but their lengths change as shown in Fig. 5.11 (*b*).

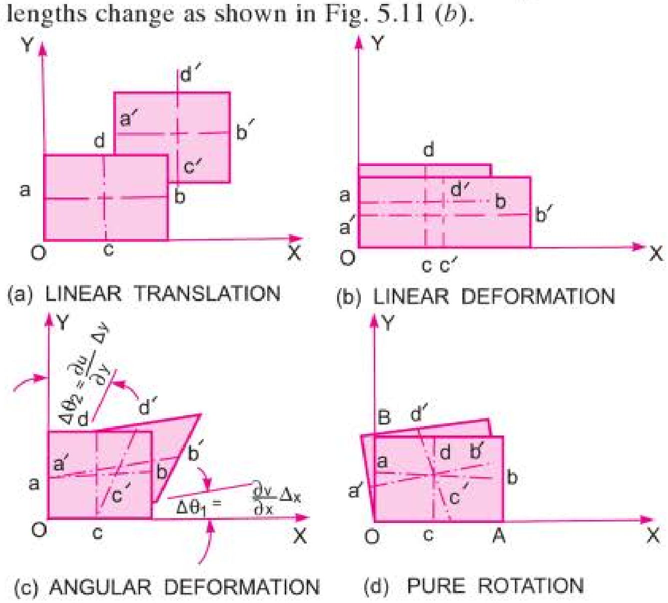


Fig. 5.11. *Displacement of a fluid element.*

5.9.3 Angular Deformation or Shear Deformation. It is defined as the average change in the angle contained by two adjacent sides. Let $\Delta\theta_1$ and $\Delta\theta_2$ is the change in angle between two adjacent sides of a fluid element as shown in Fig. 5.11 (*c*), then angular deformation or shear strain rate

$$= \frac{1}{2}\,[\Delta\theta_1 + \Delta\theta_2]$$

Now $$\Delta\theta_1 = \frac{\partial v}{\partial x} \times \frac{\Delta x}{\Delta x} = \frac{\partial v}{\partial x} \text{ and } \Delta\theta_2 = \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta y} = \frac{\partial u}{\partial y}.$$

$\therefore$ **Angular deformation** $= \frac{1}{2}\,[\Delta\theta_1 + \Delta\theta_2]$

or **Shear strain rate** $= \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$ ...(5.16)

5.9.4 Rotation. It is defined as the movement of a fluid element in such a way that both of its axes (horizontal as well as vertical) rotate in the same direction as shown in Fig. 5.11 (*d*). It is equal to $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ for a two-dimensional element in *x*-*y* plane. The rotational components are

$$\left.\begin{aligned} \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \end{aligned}\right\} \qquad \text{...(5.17)}$$

5.9.5 Vorticity. It is defined as the value twice of the rotation and hence it is given as 2ω.

Problem 5.18 *A fluid flow is given by $V = 8x^3 i - 10x^2 yj$.*
Find the shear strain rate and state whether the flow is rotational or irrotational.

Solution. Given : $V = 8x^3 i - 10x^2 yj$

$\therefore$ $$u = 8x^3,\ \frac{\partial u}{\partial x} = 24x^2,\ \frac{\partial u}{\partial y} = 0$$

and $$v = -\,10x^2 y,\ \frac{\partial v}{\partial x} = -\,20xy,\ \frac{\partial v}{\partial y} = -\,10x^2$$

(*i*) Shear strain rate is given by equation (5.16) as

$$= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\,(-\,20xy + 0) = \mathbf{-\,10xy}.\ \textbf{Ans.}$$

(*ii*) Rotation in x - y plane is given by equation (5.17) or

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\,(-\,20xy - 0) = -\,10xy$$

As rotation $\omega_z \neq 0$. Hence flow is rotational. **Ans.**

Problem 5.19 *The velocity components in a two-dimensional flow are*

$$u = y^3/3 + 2x - x^2y \text{ and } v = xy^2 - 2y - x^3/3.$$

Show that these components represent a possible case of an irrotational flow.

Solution. Given : $u = y^3/3 + 2x - x^2y$

$\therefore$ $$\frac{\partial u}{\partial x} = 2 - 2xy$$

$$\frac{\partial u}{\partial y} = \frac{3y^2}{3} - x^2 = y^2 - x^2$$

Also $v = xy^2 - 2y - x^3/3$

$\therefore$ $$\frac{\partial v}{\partial y} = 2xy - 2$$

$$\frac{\partial v}{\partial x} = y^2 - \frac{3x^2}{3} = y^2 - x^2.$$

(*i*) For a two-dimensional flow, continuity equation is $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Substituting the value of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

$\therefore$ It is a possible case of fluid flow.

(*ii*) Rotation, ω_z is given by $\omega_z = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) = \dfrac{1}{2}\,[(y^2 - x^2) - (y^2 - x^2)] = 0$

$\therefore$ Rotation is zero, which means it is case of irrotational flow. **Ans.**

▶ 5.10 VORTEX FLOW

Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known a 'Vortex Flow'. The vortex flow is of two types namely :

1. Forced vortex flow, and
2. Free vortex flow.

5.10.1 Forced Vortex Flow. Forced vortex flow is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow, rotates at constant angular velocity, ω. The tangential velocity of any fluid particle is given by

$$v = \omega \times r \qquad ...(5.18)$$

where r = Radius of fluid particle from the axis of rotation.

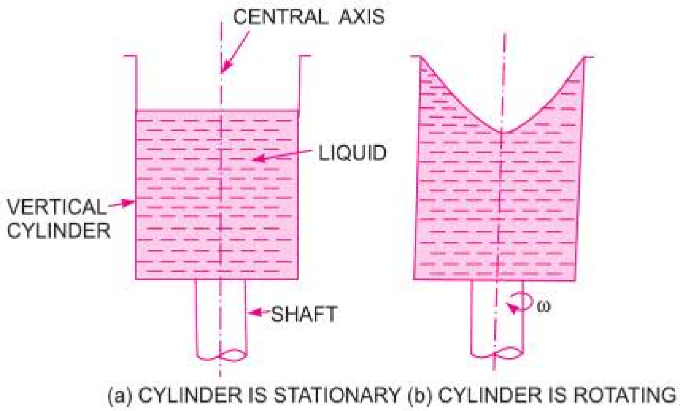


Fig. 5.12 *Forced vortex flow.*

Hence angular velocity ω is given by

$$\omega = \frac{v}{r} = \text{Constant.} \qquad ...(5.19)$$

Examples of forced vortex are :

1. A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity ω, as shown in Fig. 5.12.
2. Flow of liquid inside the impeller of a centrifugal pump.
3. Flow of water through the runner of a turbine.

5.10.2 Free Vortex Flow. When no external torque is required to rotate the fluid mass, that type of flow is called free vortex flow. Thus the liquid in case of free vortex is rotating due to the rotation which is imparted to the fluid previously.

Examples of the free vortex flow are :

1. Flow of liquid through a hole provided at the bottom of a container.
2. Flow of liquid around a circular bend in a pipe.
3. A whirlpool in a river.
4. Flow of fluid in a centrifugal pump casing.

The relation between velocity and radius, in free vortex is obtained by putting the value of external torque equal to zero, or, the time rate of change of angular momentum, *i.e.*, moment of momentum must be zero. Consider a fluid particle of mass 'm' at a radial distance r from the axis of rotation, having a tangential velocity v. Then

Angular momentum = Mass × Velocity = $m \times v$

Moment of momentum = Momentum × r = $m \times v \times r$

∴ Time rate of change of angular momentum = $\frac{\partial}{\partial t}(mvr)$

∴ For free vortex $\frac{\partial}{\partial t}(mvr) = 0$

Integrating, we get $mvr = \text{Constant}$ or $vr = \frac{\text{Constant}}{m} = \text{Constant}$ ...(5.20)

5.10.3 Equation of Motion for Vortex Flow. Consider a fluid element $ABCD$ (shown shaded) in Fig. 5.13 rotating at a uniform velocity in a horizontal plane about an axis perpendicular to the plane of paper and passing through O.

Let r = Radius of the element from O.

$\Delta\theta$ = Angle subtended by the element at O.

Δr = Radial thickness of the element.

ΔA = Area of cross-section of element.

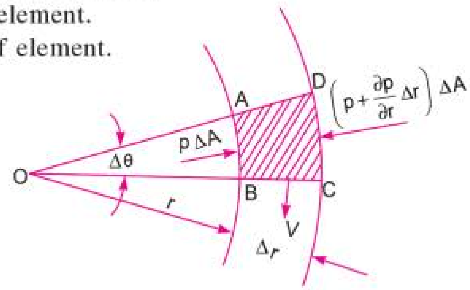


Fig. 5.13

The forces acting on the element are :

(*i*) Pressure force, $p\Delta A$, on the face AB.

(*ii*) Pressure force, $\left(p + \dfrac{\partial p}{\partial r}\Delta r\right)\Delta A$ on the face CD.

(*iii*) Centrifugal force, $\dfrac{mv^2}{r}$ acting in the direction away from the centre, O.

Now, the mass of the element = Mass density × Volume

$$= \rho \times \Delta A \times \Delta r$$

$\therefore$ Centrifugal force $= \rho\Delta A\Delta r \dfrac{v^2}{r}$.

Equating the forces in the radial direction, we get

$$\left(p + \frac{\partial p}{\partial r}\Delta r\right)\Delta A - p\Delta A = \rho\Delta A\Delta r\,\frac{v^2}{r}$$

or

$$\frac{\partial p}{\partial r}\Delta r\,\Delta A = \rho\Delta A\Delta r\,\frac{v^2}{r}.$$

Cancelling $\Delta r \times \Delta A$ from both sides, we get $\dfrac{\partial p}{\partial r} = \rho\,\dfrac{v^2}{r}$ ...(5.21)

Equation (5.21) gives the pressure variation along the radial direction for a forced or free vortex flow in a horizontal plane. The expression $\dfrac{\partial p}{\partial r}$ is called pressure gradient in the radial direction. As $\dfrac{\partial p}{\partial r}$ is positive, hence pressure increases with the increase of radius 'r'.

The pressure variation in the vertical plane is given by the hydrostatic law, *i.e.*,

$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{...(5.22)}$$

In equation (5.22), z is measured vertically in the upward direction.

The pressure, p varies with respect to r and z or p is a function of r and z and hence total derivative of p is

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz.$$

Substituting the values of $\dfrac{\partial p}{\partial r}$ from equation (5.21) and $\dfrac{\partial p}{\partial z}$ from equation (5.22), we get

$$dp = \rho \frac{v^2}{r} dr - \rho g dz \qquad ...(5.23)$$

Equation (5.23) gives the variation of pressure of a rotating fluid in any plane.

5.10.4 Equation of Forced Vortex Flow. For the forced vortex flow, from equation (5.18), we have

$$v = \omega \times r$$

where ω = Angular velocity = Constant.

Substituting the value of v in equation (5.23), we get

$$dp = \rho \times \frac{\omega^2 r^2}{r} dr - \rho g\, dz.$$

Consider two points 1 and 2 in the fluid having forced vortex flow as shown in Fig. 5.14. Integrating the above equation for points 1 and 2, we get

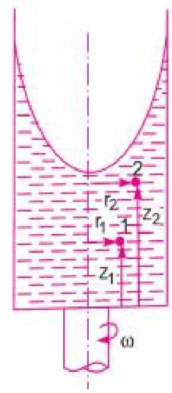


Fig. 5.14

$$\int_1^2 dp = \int_1^2 \rho\omega^2 r dr - \int_1^2 \rho g dz$$

or $$(p_2 - p_1) = \left[\rho\omega^2 \frac{r^2}{2}\right]_1^2 - \rho g\, [z]_1^2$$

or $$(p_2 - p_1) = \frac{\rho\omega^2}{2} [r_2^2 - r_1^2] - \rho g\, [z_2 - z_1]$$

$$= \frac{\rho}{2} [\omega^2 r_2^2 - \omega^2 r_1^2] - \rho g\, [z_2 - z_1]$$

$$= \frac{\rho}{2} [v_2^2 - v_1^2] - \rho g\, [z_2 - z_1] \quad \left\{\because \begin{array}{l} v_2 = \omega r_2 \\ v_1 = \omega r_1 \end{array}\right\}$$

If the points 1 and 2 lie on the free surface of the liquid, then $p_1 = p_2$ and hence above equation becomes

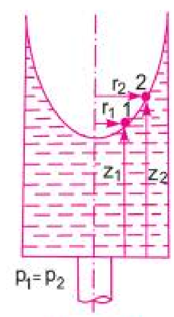


Fig. 5.15

$$0 = \frac{\rho}{2} [v_2^2 - v_1^2] - \rho g\, [z_2 - z_1]$$

or $$\rho g\, [z_2 - z_1] = \frac{\rho}{2} [v_2^2 - v_1^2]$$

or $$[z_2 - z_1] = \frac{1}{2g} [v_2^2 - v_1^2].$$

If the point 1 lies on the axis of rotation, then $v_1 = \omega \times r_1 = \omega \times 0 = 0$. The above equation becomes as

$$z_2 - z_1 = \frac{1}{2g} v_2^2 = \frac{v_2^2}{2g}$$

Let $z_2 - z_1 = Z$, then we have $Z = \dfrac{v_2^2}{2g} = \dfrac{\omega^2 \times r_2^2}{2g}$ ...(5.24)

Thus Z varies with the square of r. Hence equation (5.24) is an equation of parabola. This means the free surface of the liquid is a paraboloid.

Problem 5.20 *Prove that in case of forced vortex, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.*

Solution. Let R = radius of the cylinder.

$O\text{-}O$ = Initial level of liquid in cylinder when the cylinder is not rotating.

$\therefore$ Initial height of liquid $= (h + x)$

$\therefore$ Volume of liquid in cylinder $= \pi R^2 \times$ Height of liquid

$$= \pi R^2 \times (h + x) \qquad \ldots(i)$$

Let the cylinder is rotated at constant angular velocity ω. The liquid will rise at the ends and will fall at the centre.

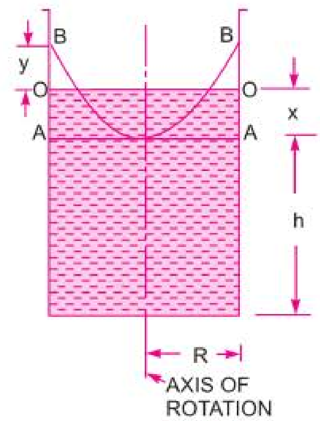


Fig. 5.16

Let y = Rise of liquid at the ends from $O\text{-}O$

x = Fall of liquid at the centre from $O\text{-}O$.

Then volume of liquid

= [Volume of cylinder upto level $B\text{-}B$] – [Volume of paraboloid]

= [$\pi R^2 \times$ Height of liquid upto level $B\text{-}B$]

$$- \left[\frac{\pi R^2}{2} \times \text{Height of paraboloid}\right]$$

$$= \pi R^2 \times (h + x + y) - \frac{\pi R^2}{2} \times (x + y)$$

$$= \pi R^2 \times h + \pi R^2 (x + y) - \frac{\pi R^2}{2} \times (x + y)$$

$$= \pi R^2 \times h + \frac{\pi R^2}{2} (x + y) \qquad \ldots(ii)$$

Equating (i) and (ii), we get

$$\pi R^2 (h + x) = \pi R^2 \times h + \frac{\pi R^2}{2} (x + y)$$

or
$$\pi R^2 h + \pi R^2 x = \pi R^2 \times h + \frac{\pi R^2}{2} x + \frac{\pi R^2}{2} y$$

or
$$\pi R^2 x - \frac{\pi R^2}{2} x = \frac{\pi R^2}{2} y \quad \text{or} \quad \frac{\pi R^2}{2} x = \frac{\pi R^2}{2} y \quad \text{or} \quad x = y$$

or Fall of liquid at centre = Rise of liquid at the ends.

Problem 5.21 *An open circular tank of 20 cm diameter and 100 cm long contains water upto a height of 60 cm. The tank is rotated about its vertical axis at 300 r.p.m., find the depth of parabola formed at the free surface of water.*

Solution. Given :

Diameter of cylinder $= 20$ cm

$\therefore$ Radius, $R = \dfrac{20}{2} = 10$ cm

Height of liquid, $H = 60$ cm

Speed, $N = 300$ r.p.m.

Angular velocity, $\omega = \dfrac{2\pi N}{60} = \dfrac{2 \times \pi \times 300}{60} = 31.41$ rad/sec.

Let the depth of parabola $= Z$

Using equation (5.24), $Z = \dfrac{\omega^2 r_2^2}{2g}$, where $r_2 = R$

$$= \frac{\omega^2 R^2}{2g} = \frac{(31.41)^2 \times (10)^2}{2 \times 981} = \textbf{50.28 cm. Ans.}$$

Problem 5.22 *An open circular cylinder of 15 cm diameter and 100 cm long contains water upto a height of 80 cm. Find the maximum speed at which the cylinder is to be rotated about its vertical axis so that no water spills.*

Solution. Given :

Diameter of cylinder $= 15$ cm

$\therefore$ Radius, $R = \dfrac{15}{2} = 7.5$ cm

Length of cylinder, $L = 100$ cm

Initial height of water $= 80$ cm.

Let the cylinder is rotated at an angular speed of ω rad/sec, when the water is about to spill. Then using,

Rise of liquid at ends = Fall of liquid at centre

But rise of liquid at ends = Length – Initial height

$= 100 - 80 = 20$ cm

$\therefore$ Fall of liquid at centre $= 20$ cm

$\therefore$ Height of parabola $= 20 + 20 = 40$ cm

$\therefore$ $Z = 40$ cm

Using the relation, $Z = \dfrac{\omega^2 R^2}{2g}$, we get $40 = \dfrac{\omega^2 (7.5)^2}{2 \times 981}$

$$\therefore \quad \omega^2 = \frac{40 \times 2 \times 981}{7.5 \times 7.5} = 1395.2$$

$$\therefore \quad \omega = \sqrt{1395.2} = 37.35 \text{ rad/s}$$

$\therefore$ Speed, N is given by $\omega = \dfrac{2\pi N}{60}$

or $$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 37.35}{2 \times \pi} = \textbf{356.66 r.p.m. Ans.}$$

Problem 5.23 *A cylindrical vessel 12 cm in diameter and 30 cm deep is filled with water upto the top. The vessel is open at the top. Find the quantity of liquid left in the vessel, when it is rotated about its vertical axis with a speed of (a) 3000 r.p.m., and (b) 600 r.p.m.*

Solution. Given :

Diameter of cylinder $= 12$ cm

$\therefore$ Radius, $R = 6$ cm

Initial height of water $= 30$ cm

Initial volume of water = Area × Initial height of water

$$= \frac{\pi}{4} \times 12^2 \times 30 \text{ cm}^3 = 3392.9 \text{ cm}^3$$

(*a*) Speed, $N = 300$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.41 \text{ rad/s}$$

Height of parabola is given by $Z = \dfrac{\omega^2 R^2}{2g} = \dfrac{(31.41)^2 \times 6^2}{2 \times 981} = 18.10$ cm.

As vessel is initially full of water, water will be spilled if it is rotated. Volume of water spilled is equal to the volume of paraboloid.

But volume of paraboloid = [Area of cross-section × Height of parabola] ÷ 2

$$= \frac{\pi}{4} D^2 \times \frac{Z}{2} = \frac{\pi}{4} \times 12^2 \times \frac{18.10}{2} = 1023.53 \text{ cm}^3$$

Volume of water left = Initial volume – Volume of water spilled

= 3392.9 – 1023.53 = **2369.37 cm³. Ans.**

(*b*) Speed, $N = 600$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 600}{60} = 62.82 \text{ rad/s}$$

Height of parabola, $Z = \dfrac{\omega^2 R^2}{2g} = \dfrac{(62.82)^2 \times 6^2}{2 \times 981} = 72.40$ cm.

As the height of parabola is more than the height of cylinder the shape of imaginary parabola will be as shown in Fig. 5.17.

Let r = Radius of the parabola at the bottom of the vessel.

Height of imaginary parabola

= 72.40 – 30 = 42.40 cm.

Volume of water left in the vessel

= Volume of water in portions *ABC* and *DEF*

= Initial volume of water

– Volume of paraboloid *AOF*

+ Volume of paraboloid *COD*.

Now volume of paraboloid

$$AOF = \frac{\pi}{4} \times D^2 \times \text{Height of parabola}$$

$$= \frac{\pi}{4} \times 12^2 \times \frac{72.4^2}{2} = 4094.12 \text{ cm}^3$$

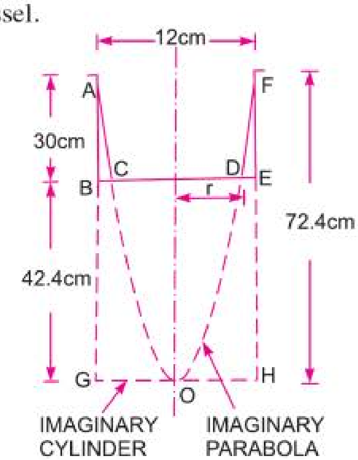


Fig. 5.17

For the imaginary parabola (*COD*), $\omega = 62.82$ rad/sec

$Z = 42.4$ cm

r = Radius at the bottom of vessel

Using the relation $$Z = \frac{\omega^2 r^2}{2g}, \text{ we get } 42.4 = \frac{62.82^2 \times r^2}{2 \times 981}$$

$$\therefore \quad r^2 = \frac{2 \times 981 \times 42.40}{62.82 \times 62.82} = 21.079$$

$$\therefore \quad r = \sqrt{21.079} = 4.59 \text{ cm}$$

$\therefore$ Volume of paraboloid COD

$$= \frac{1}{2} \times \text{Area at the top of the imaginary parabola} \times \text{Height of parabola}$$

$$= \frac{1}{2} \times \pi r^2 \times 42.4 = \frac{1}{2} \times \pi \times 4.59^2 \times 42.4 = 1403.89 \text{ cm}^3$$

$\therefore$ Volume of water left = 3392.9 – 4094.12 + 1403.89 = **702.67 cm³. Ans.**

Problem 5.24 *An open circular cylinder of 15 cm diameter and 100 cm long contains water upto a height of 70 cm. Find the speed at which the cylinder is to be rotated about its vertical axis, so that the axial depth becomes zero.*

Solution. Given :

Diameter of cylinder = 15 cm

$$\therefore \text{ Radius, } R = \frac{15}{2} = 7.5 \text{ cm}$$

Length of cylinder = 100 cm

Initial height of water = 70 cm.

When axial depth is zero, the depth of paraboloid = 100 cm.

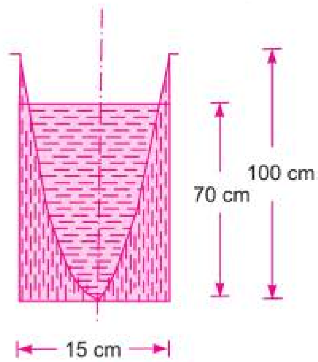


Fig. 5.18

Using the relation, $$Z = \frac{\omega^2 R^2}{2g}, \text{ we get}$$

$$100 = \frac{\omega^2 \times 7.5^2}{2 \times 9.81}$$

$$\therefore \quad \omega^2 = \frac{100 \times 2 \times 9.81}{7.5 \times 7.5}$$

$$\therefore \quad \omega = \sqrt{\frac{100 \times 2 \times 9.81}{7.5 \times 7.5}} = \frac{442.92}{7.5} = 59.05 \text{ rad/s}$$

$\therefore$ Speed, N is given by $$\omega = \frac{2\pi H}{60}$$

or $$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 59.05}{2\pi} = \mathbf{563.88 \text{ r.p.m.}} \textbf{ Ans.}$$

Problem 5.25 *For the problem (5.24), find the difference in total pressure force (i) at the bottom of cylinder, and (ii) at the sides of the cylinder due to rotation.*

Solution. (*i*) The data is given in Problem 5.24. The difference in total pressure force at the bottom of cylinder is obtained by finding total hydrostatic force at the bottom before rotation and after rotation.

Before rotation, force = $\rho g A\bar{h}$

where $\rho = 1000$ kg/m^3, A = Area of bottom $= \frac{\pi}{3} D^2 = \frac{\pi}{4} \times (0.15)^2$ m^2, $\bar{h} = 70$ cm $= 0.70$ m

$$\therefore \quad \text{Force} = 1000 \times 9.81 \times \frac{\pi}{4} \times (0.15)^2 \times 0.7 \text{ N} = 121.35 \text{ N}$$

After rotation, the depth of water at the bottom is not constant and hence pressure force due to the height of water, will not be constant. Consider a circular ring of radius r and width dr as shown in Fig. 5.19. Let the height of water from the bottom of the tank upto free surface of water at a radius

$$r = Z = \frac{\omega^2 r^2}{2g}.$$

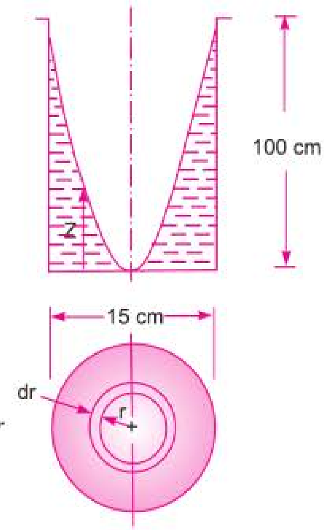


Fig. 5.19

Hydrostatic force on ring at the bottom,

$$dF = \rho g \times \text{Area of ring} \times Z$$

$$= 1000 \times 9.81 \times 2\pi r dr \times \frac{\omega^2 r^2}{2g}$$

$$= 9810 \times 2 \times \pi r \times \frac{\omega^2 r^2}{2g} \times dr$$

$\therefore$ Total pressure force at the bottom

$$= \int dF = \int_0^R 9810 \times 2 \times \pi r \times \frac{\omega^2 r^2}{2g} dr$$

$$= \int_0^{0.075} 19620 \times \pi \times \frac{\omega^2}{2g} r^3 \, dr$$

From Problem 5.24, $\omega = 59.05$ rad/s

$R = 7.5$ cm $= .075$ m.

Substituting these values, we get total pressure force

$$= \frac{19620 \times \pi \times (59.05)^2}{2 \times 9.81} \left[\frac{r^4}{4}\right]_0^{.075}$$

$$= \frac{19620 \times \pi \times (59.05)^2}{2 \times 9.81} \times \frac{(.075)^4}{4} = 86.62 \text{ N}$$

$\therefore$ Difference in pressure forces at the bottom

$121.35 - 86.62 =$ **34.73 N. Ans.**

(*ii*) Forces on the sides of the cylinder

Before rotation $= \rho g A\bar{h}$

where A = Surface area of the sides of the cylinder upto height of water

$= \pi D \times$ Height of water $= \pi \times .15 \times 0.70$ m^2 $= 0.33$ m^2

$\bar{h}$ = C.G. of the wetted area of the sides

$= \frac{1}{2} \times$ height of water $= \frac{0.70}{2} = 0.35$ m

$\therefore$ Force on the sides before rotation $= 1000 \times 9.81 \times 0.33 \times 0.35 = 1133$ N

After rotation, the water is upto the top of the cylinder and hence force on the sides

$$= 1000 \times 9.81 \times \text{Wetted area of the sides} \times \frac{1}{2} \times \text{Height of water}$$

$$= 9810 \times \pi D \times 1.0 \times \frac{1}{2} \times 1.0 = 9810 \times \pi \times .15 \times \frac{1}{2} = 2311.43 \text{ N}$$

$\therefore$ Difference in pressure on the sides

$2311.43 - 1133 =$ **1178.43 N. Ans.**

5.10.5 Closed Cylindrical Vessels. If a cylindrical vessel is closed at the top, which contains some liquid, the shape of paraboloid formed due to rotation of the vessel will be as shown in Fig. 5.20 for different speed of rotations.

Fig. 5.20 (*a*) shows the initial stage of the cylinder, when it is not rotated. Fig. 5.20 (*b*) shows the shape of the paraboloid formed when the speed of rotation is ω_1. If the speed is increased further say ω_2, the shape of paraboloid formed will be as shown in Fig. 5.20 (*c*). In this case the radius of the parabola at the top of the vessel is unknown. Also the height of the paraboloid formed corresponding to angular speed ω_2 is unknown. Thus to solve the two unknown, we should have two equations. One equation is

$$Z = \frac{\omega_2^2 r^2}{2g}$$

The second equation is obtained from the fact that for closed vessel, volume of air before rotation is equal to the volume of air after rotation.

Volume of air before rotation = Volume of closed vessel – Volume of liquid in vessel

Volume of air after rotation = Volume of paraboloid formed $= \frac{\pi r^2 \times Z}{2}$.

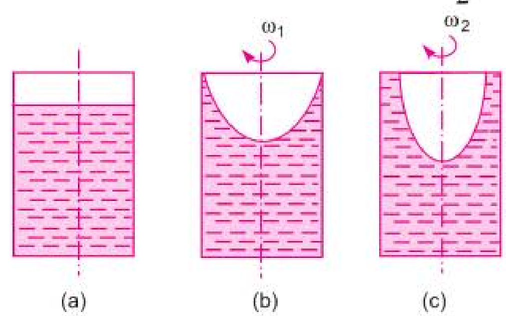


Fig. 5.20

Problem 5.26 *A vessel, cylindrical in shape and closed at the top and bottom, contains water upto a height of 80 cm. The diameter of the vessel is 20 cm and length of vessel is 120 cm. The vessel is rotated at a speed of 400 r.p.m. about its vertical axis. Find the height of paraboloid formed.*

Solution. Given :

Initial height of water $= 80$ cm

Diameter of vessel $= 20$ cm

$\therefore$ Radius, $R = 10$ cm

Length of vessel $= 120$ cm

Speed, $N = 400$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 400}{60} = 41.88 \text{ rad/s}$$

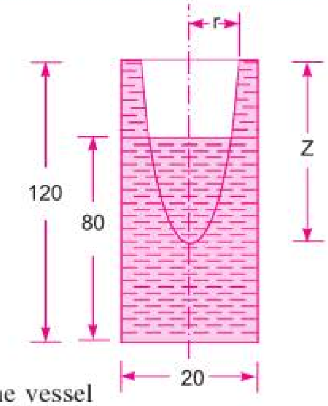


Fig. 5.21

When the vessel is rotated, let Z = Height of paraboloid formed

r = Radius of paraboloid at the top of the vessel

This is the case of closed vessel.

$\therefore$ Volume of air before rotation = Volume of air after rotation

$$\text{or} \quad \frac{\pi}{4} D^2 \times L - \frac{\pi}{4} D^2 \times 80 = \pi r^2 \times \frac{Z}{2}$$

where Z = Height of paraboloid, r = Radius of parabola.

$$\text{or} \quad \frac{\pi}{4} D^2 \times 120 - \frac{\pi}{4} D^2 \times 80 = \pi r^2 \times \frac{Z}{2}$$

$$\text{or} \quad \frac{\pi}{4} \times D^2 \times (120 - 80) = \frac{\pi}{4} D^2 \times 40 = \pi r^2 \times \frac{Z}{2}$$

$$\text{or} \quad \frac{\pi}{4} \times 20^2 \times 40 = 4000 \times \pi = \pi r^2 \times \frac{Z}{2}$$

$$\therefore \quad r^2 \times Z = \frac{4000 \times \pi \times 2}{\pi} = 8000 \qquad \text{...}(i)$$

Using relation $Z = \dfrac{\omega^2 r^2}{2g}$, we get $Z = \dfrac{41.88^2 \times r^2}{2g} = \dfrac{41.88^2 \times r^2}{2 \times 981} = 0.894\ r^2$

$$\therefore \quad r^2 = \frac{z}{0.894}$$

Substituting this value of r^2 in (i), we get

$$\frac{Z}{0.894} \times Z = 8000$$

$$\therefore \quad Z^2 = 8000 \times 0.894 = 7152$$

$$\therefore \quad Z = \sqrt{7152} = \mathbf{84.56\ cm.}$$ **Ans.**

IInd Method

Let Z_1 = Height of paraboloid, if the vessel would not have been closed at the top, corresponding to speed,

$N = 400$ r.p.m.

or $\omega = 41.88$ rad/s

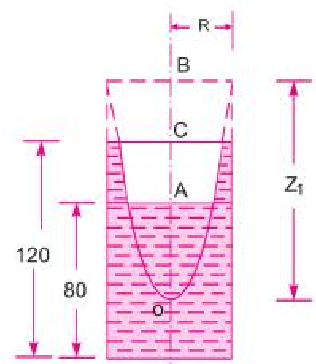


Fig. 5.22

$$\text{Then} \quad Z_1 = \frac{\omega^2 R^2}{2g} = \frac{41.88^2 \times 10^2}{2 \times 981} = 89.34 \text{ cm.}$$

Half of Z_1 will be below the initial height of water in the vessel

i.e., $$AO = \frac{Z_1}{2} = \frac{89.34}{2} = 44.67 \text{ cm}$$

But height of paraboloid for closed vessel

$$= CO = CA + AO = (120 - 80) + 44.67 \text{ cm}$$
$$= 40 + 44.67 = \mathbf{84.67 \text{ cm.}} \text{ Ans.}$$

Problem 5.27 *For the data given in Problem 5.26, find the speed of rotation of the vessel, when axial depth of water is zero.*

Solution. Given :

Diameter of vessel $= 20$ cm

$\therefore$ Radius, $R = 10$ cm

Initial height of water $= 80$ cm

Length of vessel $= 120$ cm

Let ω is the angular speed, when axial depth is zero.

When axial depth is zero, the height of paraboloid is 120 cm and radius of the parabola at the top of the vessel is r.

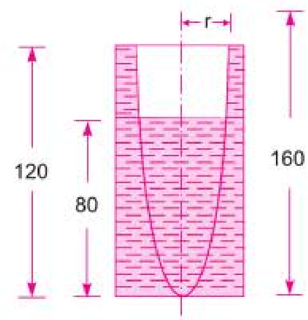


Fig. 5.23

$\therefore$ Using the relation, $$Z = \frac{\omega^2 r^2}{2g} \text{ or } 120 = \frac{\omega^2 \times r^2}{2 \times 980}$$

$\therefore$ $$\omega^2 r^2 = 2 \times 980 \times 120 = 235200 \quad \ldots(i)$$

Volume of air before rotation = Volume of air after paraboloid

$\therefore$ $$\pi R^2 \times (120 - 180) = \text{Volume of paraboloid}$$
$$= \pi r^2 \times \frac{Z}{2}$$

or $$\pi \times 10^2 \times 40 = \frac{\pi r^2 \times Z}{2} = \frac{\pi r^2}{2} \times 120$$

or $$r^2 = \frac{\pi \times 10^2 \times 40 \times 2}{\pi \times 120} = \frac{8000}{120} = 66.67$$

Substituting the value of r^2 in equation (*i*), we get

$$\omega^2 \times 66.67 = 235200$$

$$\omega = \sqrt{\frac{235200}{66.67}} = 59.4 \text{ rad/s}$$

$\therefore$ Speed N is given by $$\omega = \frac{2\pi N}{60}$$

or $$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 59.4}{2\pi} = \mathbf{567.22 \text{ r.p.m.}} \text{ Ans.}$$

Problem 5.28 *The cylindrical vessel of the problem 5.26 is rotated at 700 r.p.m. about its vertical axis. Find the area uncovered at the bottom of the tank.*

Solution. Given :

Initial height of water $= 80$ cm

$\therefore$ Diameter of vessel $= 20$ cm

$\therefore$ Radius, $R = 10$ cm

Length of vessel $= 120$ cm

Speed, $N = 700$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 700}{60} = 73.30 \text{ rad/s.}$$

If the tank is not closed at the top and also is very long, then the height of parabola corresponding to $\omega = 73.3$ will be

$$= \frac{\omega^2 \times R^2}{2 \times g} = \frac{73.3^2 \times 10^2}{2 \times 980} = 274.12 \text{ cm}$$

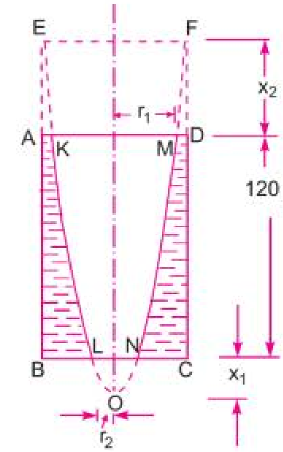


Fig. 5.24

From Fig. 5.24,

$$x_1 + 120 + x_2 = 274.12$$

or $$x_1 + x_2 = 274.12 - 120 = 154.12 \text{ cm} \quad \ldots(i)$$

From the parabola, *KOM*, we have

$$(120 + x_1) = \frac{\omega^2 r_1^2}{2g} = \frac{73.3^2 \times r_1^2}{2 \times 980} \quad \ldots(ii)$$

For the parabola, *LON*, we have

$$x_1 = \frac{\omega^2 r_2^2}{2g} = \frac{73.3^2 \times r_2^2}{2 \times 980} \quad \ldots(iii)$$

Now, volume of air before rotation = Volume of air after rotation

Volume of air before rotation $= \pi R^2 \times (120 - 80) = \pi \times 10^2 \times 40 = 12566.3 \text{ cm}^3 \quad \ldots(iv)$

Volume of air after rotation = Volume of paraboloid *KOM* – volume of paraboloid *LON*

$$= \pi r_1^2 \times \frac{(120 + x_1)}{2} - \pi r_2^2 \times \frac{x_1}{2} \quad \ldots(v)$$

Equating (*iv*) and (*v*), we get

$$12566.3 = \frac{\pi r_1^2 (120 + x_1)}{2} - \frac{\pi r_2^2 \times x_1}{2} \quad \ldots(vi)$$

Substituting the value of r_1^2 from (*ii*) in (*vi*), we get

$$12566.3 = \pi \times \frac{(120 + x_1) \times 2 \times 980}{73.3^2} \times \frac{(120 + x_1)}{2} - \frac{\pi r_2^2 + x_1}{2}$$

$$\left\{ \because \text{ From } (ii),\ r_1^2 = \frac{2 \times 980 \times (120 + x_1)}{(73.3)^2} \right\}$$

or $$12566.3 = 0.573\,(120 + x_1)^2 - \frac{\pi r_2^2 \times x_1}{2}$$

Substituting the value of x_1 from (*iii*) in the above equation

$$12566.3 = 0.573 \left(120 + \frac{73.3^2 \times r_2^2}{2 \times 980}\right)^2 - \frac{\pi r_2^2}{2} \times \frac{73.3^2\, r_2^2}{2 \times 980}$$

$$= 0.573\,(120 + 2.74\, r_2^2)^2 - 4.3 \times r_2^2 \times r_2^2$$

$$= 0.573\,[120^2 + 2.74^2\, r_2^4 + 2 \times 120 \times 2.74\, r_2^2] - 4.3\, r_2^4$$

$$= 0.573\,[14400 + 7.506\, r_2^4 + 657.6\, r_2^2] - 4.3\, r_2^4$$

$$\frac{12566.3}{0.573} = 21930 = 14400 + 7.506\, r_2^4 + 657.6\, r_2^2 - 4.3\, r_2^4$$

or $$r_2^4\,(7.506 - 4.3) + 657.6\, r_2^2 + 14400 - 21930 = 0$$

or $$3.206\, r_2^4 + 657.6\, r_2^2 - 7530 = 0$$

$\therefore$ $$r_2^2 = \frac{-657.6 \pm \sqrt{657.6^2 - 4 \times (-7530) \times (3.206)}}{2 \times 3.206}$$

$$= \frac{-657.6 \pm \sqrt{432437.76 + 96564.72}}{6.412}$$

$$= \frac{-657.6 \pm 727.32}{6.412} = -215.98 \text{ or } 10.87$$

Negative value is not possible

$\therefore$ $$r_2^2 = 10.87 \text{ cm}^2$$

$\therefore$ Area uncovered at the base $= \pi r_2^2 = \pi \times 10.87 =$ **34.149 cm²**. **Ans.**

Problem 5.29 *A closed cylindrical vessel of diameter 30 cm and height 100 cm contains water upto a depth of 80 cm. The air above the water surface is at a pressure of 5.886 N/cm². The vessel is rotated at a speed of 250 r.p.m. about its vertical axis. Find the pressure head at the bottom of the vessel : (a) at the centre, and (b) at the edge.*

Solution. Given :

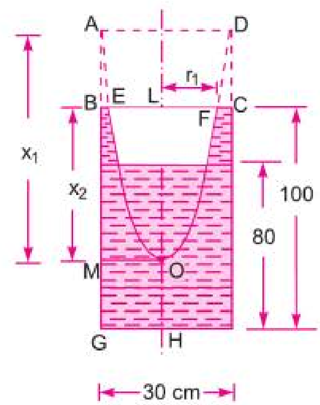


Fig. 5.25

Diameter of vessel $= 30$ cm

$\therefore$ Radius, $R = 15$ cm

Initial height of water, $H = 80$ cm

Length of cylinder, $L = 100$ cm

Pressure of air above water $= 5.886$ N/cm²

or $$p = 5.886 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

Head due to pressure, $h = p/\rho g$

$$= \frac{5.886 \times 10^4}{1000 \times 9.81} = 6 \text{ m of water}$$

Speed, $N = 250$ r.p.m.

$\therefore$ $$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 250}{60} = 26.18 \text{ rad/s}$$

Let x_1 = Height of paraboloid formed, if the vessel is assumed open at the top and it is very long.

Then we have $$x_1 = \frac{\omega^2 R^2}{2g} = \frac{26.18^2 \times 15^2}{2 \times 981} = 78.60 \text{ cm} \quad \ldots(i)$$

Let r_1 is the radius of the actual parabola of height x_2

Then $$x_2 = \frac{\omega^2 r_1^2}{2g} = \frac{26.18^2 \times r_1^2}{2 \times 981} = 0.35\ r_1^2 \qquad \ldots(ii)$$

The volume of air before rotation

$$= \pi R^2\ (100 - 80) = \pi \times 15^2 \times 20 = 14137\ \text{cm}^3$$

Volume of air after rotation = Volume of paraboloid *EOF*

$$= \frac{1}{2} \times \pi r_1^2 \times x_2$$

But volume of air before and after rotation is same.

$$\therefore \qquad 14137 = \frac{1}{2} \times \pi r_1^2 \times x_2$$

But from (*ii*), $$x_2 = 0.35\ r_1^2$$

$$\therefore \qquad 14137 = \frac{1}{2} \times \pi r_1^2 \times 0.35\ r_1^2$$

$$\therefore \qquad r_1^4 = \frac{2 \times 14137}{\pi \times 0.35} = 25714$$

$$r_1 = (25714)^{1/4} = 12.66\ \text{cm}$$

Substituting the value of r_1 in (*ii*), we get

$$x_2 = 0.35 \times 12.66^2 = 56.1\ \text{cm}$$

Pressure head at the bottom of the vessel

(*a*) At the centre. The pressure head at the centre, *i.e.*, at H = Pressure head due to air + OH

$$= 6.0 + (HL - LO) \qquad \{\because\ OH = LH - LO\}$$

$$= 6.0 + (1.0 - 0.561) \qquad \left\{\begin{aligned} \because\ & HL = 100\ \text{cm} = 1\ \text{m} \\ & LO = x_2 = 56.1\ \text{cm} = .561\ \text{m} \end{aligned}\right\}$$

$$= \mathbf{6.439\ m\ of\ water.}$$ **Ans.**

(*b*) At the edge, *i.e.*, at G = Pressure head due to air + height of water above G

$$= 6.0 + AG = 6.0 + (GM + MA) = 6.0 + (HO + x_1)$$

$$= 6.0 + HO + 0.786 \qquad \{\because\ x_1 = 78.6\ \text{cm} = 0.786\ \text{m}\}$$

$$= 6.0 + 0.439 + 0.786 \qquad \left\{\begin{aligned} \because\ HO &= LH - LO = 100 - 56.1 \\ &= 43.9\ \text{cm} = 0.439\ \text{m} \end{aligned}\right\}$$

$$= \mathbf{7.225\ m\ of\ water.}$$ **Ans.**

Problem 5.30 *A closed cylinder of radius R and height H is completely filled with water. It is rotated about its vertical axis with a speed of ω radians/s. Determine the total pressure exerted by water on the top and bottom of the cylinder.*

Solution. Given :

Radius of cylinder $= R$

Height of cylinder $= H$

Angular speed $= \omega$

As the cylinder is closed and completely filled with water, the rise of water level at the ends and depression of water at the centre due to rotation of the vessel, will be prevented. Thus the water will exert force on the complete top of the vessel. Also the pressure will be exerted at the bottom of the cylinder.

Total Pressure exerted on the top of cylinder. The top of cylinder is in contact with water and is in horizontal plane. The pressure variation at any radius in horizontal plane is given by equation (5.21)

or $$\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} = \frac{\rho \omega^2 r^2}{r} = \rho\omega^2 r \quad \{\because \ v = \omega \times r\}$$

Integrating, we get

$$\int dp = \int \rho\omega^2 r dr \quad \text{or} \quad p = \frac{\rho\omega^2 r^2}{2} = \frac{\rho}{2}\,\omega^2 r^2$$

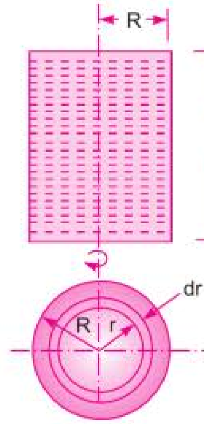


Fig. 5.26

Consider an elementary circular ring of radius r and width dr on the top of the cylinder as shown in Fig. 5.26.

Area of circular ring $= 2\pi r dr$

$\therefore$ Force on the elementary ring = Intensity of pressure × Area of ring

$$= p \times 2\pi r dr$$

$$= \frac{\rho}{2}\,\omega^2 r^2 \times 2\pi r dr. \qquad \left\{\because \ p = \frac{\rho}{2}\,\omega^2 r^2\right\}$$

$\therefore$ Total force on the top of the cylinder is obtained by integrating the above equation between the limits 0 and R.

$$\therefore \qquad \text{Total force or } F_T = \int_0^R \frac{\rho}{2}\,\omega^2 r^2 \times 2\pi r dr = \frac{\rho}{2}\,\omega^2 \times 2\pi \int_0^R r^3 dr$$

$$= \frac{\rho}{2}\,\omega^2 \times 2\pi \left[\frac{r^4}{4}\right]_0^R = \frac{\rho}{2}\,\omega^2 \times 2\pi \times \frac{R^4}{4}$$

$$= \frac{\rho\omega^2}{4} \times \pi R^4 \qquad \text{...(5.25)}$$

Total pressure force on the bottom of cylinder, F_B

= Weight of water in cylinder + total force on the top of cylinder

$$= \rho g \times \pi R^2 \times H + \frac{\rho}{4}\,\omega^2 \times \pi R^4 = \rho g \times \pi R^2 \times H + F_T \qquad \text{...(5.26)}$$

ρ = Density of water.

Problem 5.31 *A closed cylinder of diameter 200 mm and height 150 mm is completely filled with water. Calculate the total pressure force exerted by water on the top and bottom of the cylinder, if it is rotated about its vertical axis at 200 r.p.m.*

Solution. Given :

Dia. of cylinder $= 200$ mm $= 0.20$ m

Radius, $R = 0.1$ m

Height of cylinder, $H = 150$ mm $= 0.15$ m

Speed, $N = 200$ r.p.m.

$\therefore$ Angular speed, $$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.94 \text{ rad/s}$$

Total pressure force on the top of the cylinder is given by equation (5.25)

$$F_T = \frac{\rho}{4} \times \omega^2 \times \pi \times R^4 = \frac{1000}{4} \times 20.94^2 \times \pi \times (0.1)^4 = \mathbf{34.44\ N.\ Ans.}$$

Now total pressure force on the bottom of the cylinder is given by equation (5.26) as

$$\begin{aligned} F_B &= \rho g \times \pi R^2 \times H + F_T \\ &= 1000 \times 9.81 \times \pi \times (0.1)^2 \times 0.15 + 34.44 \\ &= 46.22 + 34.44 = \mathbf{80.66\ N.\ Ans.} \end{aligned}$$

5.10.6 Equation of Free Vortex Flow. For the free vortex, from equation (5.20), we have

$$v \times r = \text{Constant} = \text{say } c$$

or

$$v = \frac{c}{r}$$

Substituting the value of v in equation (5.23), we get

$$dp = \rho \frac{v^2}{r} dr - \rho g\, dz = \rho \times \frac{c^2}{r^2 \times r} dr - \rho g\, dz = \rho \times \frac{c^2}{r^3} dr - \rho g\, dz$$

Consider two points 1 and 2 in the fluid having radius r_1 and r_2 from the central axis respectively as shown in Fig. 5.27. The heights of the points from bottom of the vessel is z_1 and z_2.

Integrating the above equation for the points 1 and 2, we get

$$\int_1^2 dp = \int_1^2 \frac{\rho c^2}{r^3} dr - \int_1^2 \rho g dz$$

or

$$p_2 - p_1 = \rho c^2 \int_1^2 r^{-3}\, dr - \rho g \int_1^2 dz$$

$$= \rho c^2 \left[\frac{r^{-3+1}}{-2}\right]_1^2 - \rho g\,[z_2 - z_1] = \frac{\rho c^2}{-2}\,[r_2^{-2} - r_1^{-2}] - \rho g\,[z_2 - z_1]$$

$$= -\frac{\rho c^2}{2}\left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] - \rho g[z_2 - z_1] = -\frac{\rho}{2}\left[\frac{c^2}{r_2^2} - \frac{c^2}{r_1^2}\right] - \rho g[z_2 - z_1]$$

$$= -\frac{\rho}{2}\left[v_2^2 - v_1^2\right] - \rho g\left[z_2 - z_1\right] \qquad \left\{\because\ v_2 = \frac{c}{r_2},\ v_1 = \frac{c}{r_1}\right\}$$

$$= \frac{\rho}{2}\left[v_1^2 - v_2^2\right] - \rho g\left[z_2 - z_1\right]$$

Dividing by ρg, we get

$$\frac{p_2 - p_1}{\rho g} = \frac{v_1^2 - v_2^2}{2g} - \left[z_2 - z_1\right]$$

or

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 \qquad \text{...(5.27)}$$

Equation (5.27) is Bernoulli's equation. Hence in case of free vortex flow, Bernoulli's equation is applicable.

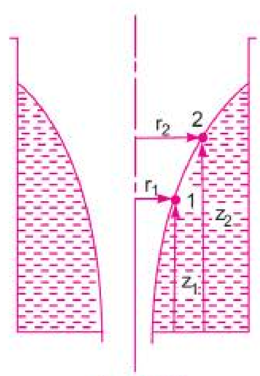


Fig. 5.27

Problem 5.32 *In a free cylindrical vortex flow, at a point in the fluid at a radius of 200 mm and at a height of 100 mm, the velocity and pressures are 10 m/s and 117.72 kN/m² absolute. Find the pressure at a radius of 400 mm and at a height of 200 mm. The fluid is air having density equal to 1.24 kg/m³.*

Solution. **At Point 1 :** Given :

Radius, $r_1 = 200 \text{ mm} = 0.20 \text{ m}$

Height, $z_1 = 100 \text{ mm} = 0.10 \text{ m}$

Velocity, $v_1 = 10 \text{ m/s}$

Pressure, $p_1 = 117.72 \text{ kN/m}^2 = 117.72 \times 10^3 \text{ N/m}^2$

At Point 2 : $r_2 = 400 \text{ mm} = 0.4 \text{ m}$

$z_2 = 200 \text{ mm} = 0.2 \text{ m}$

$p_2 = \text{pressure at point 2}$

$\rho = 1.24 \text{ kg/m}^3$

For the free vortex from equation (5.20), we have

$$v \times r = \text{constant or } v_1 r_1 = v_2 r_2$$

$$v_2 = \frac{v_1 \times r_1}{r_2} = \frac{10 \times 0.2}{0.4} = 5 \text{ m/s}$$

Now using equation (5.27), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $\rho = 1.24 \text{ kg/m}^3$

$$\therefore \quad \frac{117.72 \times 10^3}{1.24 \times 9.81} + \frac{10^2}{2 \times 9.81} + 0.1 = \frac{p_2}{\rho g} + \frac{5^2}{2 \times 9.81} + 0.2$$

or

$$\frac{p_2}{\rho g} = \frac{117.72 \times 10^3}{1.24 \times 9.81} + \frac{10^2}{2 \times 9.81} + 0.1 - \frac{5^2}{2 \times 9.81} - 0.2$$

$$= 9677.4 + 5.096 + 0.1 - 1.274 - 0.2 = 9676.22$$

$$p_2 = 9676.22 \times \rho g = 9676.22 \times 1.24 \times 9.81$$

$$= 117705 \text{ N/m}^2 = 117.705 \times 10^3 \text{ N/m}^2$$

$= 117.705 \text{ kN/m}^2$ (abs.) $=$ **117.705 kN/m². Ans.**

(B) IDEAL FLOW (POTENTIAL FLOW)

▶ 5.11 INTRODUCTION

Ideal fluid is a fluid which is incompressible and inviscid. Incompressible fluid is a fluid for which density (ρ)remains constant. Inviscid fluid is a fluid for which viscosity (μ) is zero. Hence a fluid for which density is constant and viscosity is zero, is known as an ideal fluid.

The shear stress is given by, $\tau = \mu \dfrac{du}{dy}$. Hence for ideal fluid the shear stress will be zero as $\mu = 0$ for ideal fluid. Also the shear force (which is equal to shear stress multiplied by area) will be zero in case of ideal or potential flow. The ideal fluids will be moving with uniform velocity. All the fluid particles will be moving with the same velocity.

The concept of ideal fluid simplifies the typical mathematical analysis. Fluids such as water and air have low viscosity. Also when the speed of air is appreciably lower than that of sound in it, the compressibility is so low that air is assumed to be incompressible. Hence under certain conditions, certain real fluids such as water and air may be treated like ideal fluids.

▶ 5.12 IMPORTANT CASES OF POTENTIAL FLOW

The following are the important cases of potential flow :

(*i*) Uniform flow,
(*ii*) Source flow,
(*iii*) Sink flow,
(*iv*) Free-vortex flow,
(*v*) Superimposed flow.

▶ 5.13 UNIFORM FLOW

In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity. The uniform flow may be :

(*i*) Parallel to x-axis
(*ii*) Parallel to y-axis.

5.13.1 Uniform Flow Parallel to x-Axis. Fig. 5.27 (*a*) shows the uniform flow parallel to x-axis. In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity.

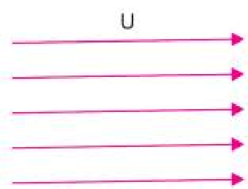


Fig. 5.27 (*a*)

Let U = Velocity which is uniform or constant along x-axis

u and v = Components of uniform velocity U along x and y-axis.

For the uniform flow, parallel to x-axis, the velocity components u and v are given as

$$u = U \text{ and } v = 0 \quad \text{...(5.28)}$$

But the velocity u in terms of stream function is given by,

$$u = \frac{\partial \psi}{\partial y}$$

and in terms of velocity potential the velocity u is given by,

$$u = \frac{\partial \phi}{\partial x}$$

$$\therefore \quad u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \text{...(5.29)}$$

Similarly, it can be shown that $v = -\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial y}$ ...(5.29*A*)

But $u = U$ from equation (5.28). Substituting $u = U$ in equation (5.29), we have

$$\therefore \qquad U = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \text{...(5.30)}$$

or
$$U = \frac{\partial \psi}{\partial y} \text{ and also } U = \frac{\partial \phi}{\partial x}$$

First part gives $d\psi = U\,dy$ whereas second part gives $d\phi = U\,dx$.

Integration of these parts gives as

$$\psi = Uy + C_1 \text{ and } \phi = Ux + C_2$$

where C_1 and C_2 are constant of integration.

Now let us plot the stream lines and potential lines for uniform flow parallel to x-axis.

Plotting of Stream lines. For stream lines, the equation is

$$\psi = U \times y + C_1$$

Let $\psi = 0$, where $y = 0$. Substituting these values in the above equation, we get

$$0 = U \times 0 + C_1 \text{ or } C_1 = 0$$

Hence the equation of stream lines becomes as

$$\psi = U.\,y \qquad \text{...(5.31)}$$

The stream lines are straight lines parallel to x-axis and at a distance y from the x-axis as shown in Fig. 5.28. In equation (5.31), $U.\,y$ represents the volume flow rate (*i.e.*, m^3/s) between x-axis and that stream line at a distance y.

Note. The thickness of the fluid stream perpendicular to the plane is assumed to be unity. Then $y \times 1$ or y represents the area of flow. And $U\,.\,y$ represents the product of velocity and area. Hence $U.\,y$ represents the volume flow rate.

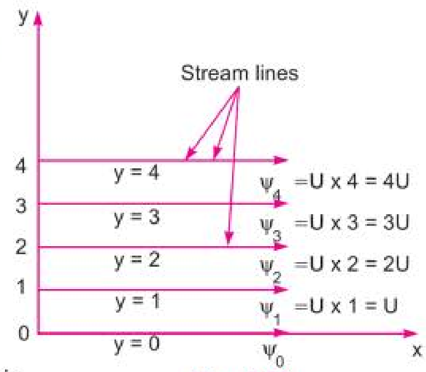


Fig. 5.28

Plotting of potential lines. For potential lines, the equation is

$$\phi = U\,.\,x + C_2 \qquad \text{...(5.32)}$$

Let $\phi = 0$, where $x = 0$. Substituting these values in the above equation, we get $C_2 = 0$.

Hence equation of potential lines becomes as

$$\phi = U.\,x$$

The above equation shows that potential lines are straight lines parallel to y-axis and at a distance of x from y-axis as shown in Fig. 5.29.

Fig. 5.30 shows the plot of stream lines and potential lines for uniform flow parallel to x-axis. The stream lines and potential lines intersect each other at right angles.

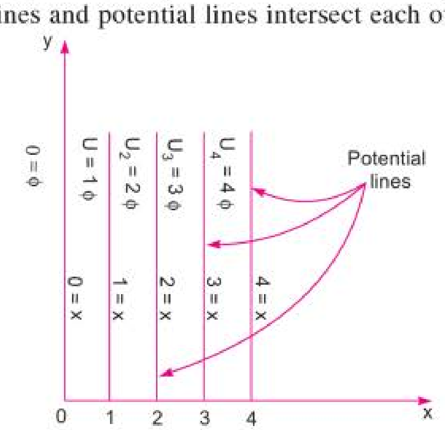


Fig. 5.29

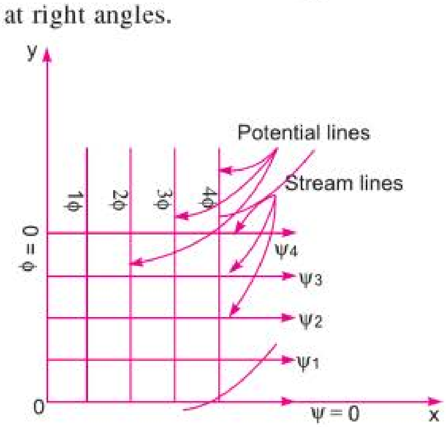


Fig. 5.30

5.13.2 Uniform Potential Flow Parallel to y-Axis.

Fig. 5.31 shows the uniform potential flow parallel to y-axis in which U is the uniform velocity along y-axis.

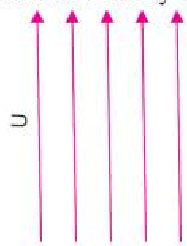


Fig. 5.31

The velocity components u, v along x-axis and y-axis are given by

$$u = 0 \text{ and } v = U \quad \ldots(5.33)$$

These velocity components in terms of stream function (ψ) and velocity potential function (ϕ) are given as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \ldots(5.34)$$

and

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \quad \ldots(5.35)$$

But from equation (5.33), $v = U$. Substituting $v = U$ in equation (5.35), we get

$$U = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \quad \text{or} \quad U = -\frac{\partial \psi}{\partial x} \text{ and also } U = \frac{\partial \phi}{\partial y}$$

First part gives $d\psi = -U\,dx$ whereas second part gives $d\phi = U\,dy$.

Integration of these parts gives as

$$\psi = -U \,.\, x + C_1 \text{ and } \phi = U.y + C_2 \quad \ldots(5.36)$$

where C_1 and C_2 are constant of integration. Let us now plot the stream lines and potential lines.

Plotting of Stream lines. For stream lines, the equation is $\psi = U.x + C_1$

Let $\psi = 0$, where $x = 0$. Then $C_1 = 0$.

Hence the equation of stream lines becomes as $\psi = -U.x$ ...(5.37)

The above equation shows that stream lines are straight lines parallel to y-axis and at a distance of x from the y-axis as shown in Fig. 5.32. The –ve sign shows that the stream lines are in the downward direction.

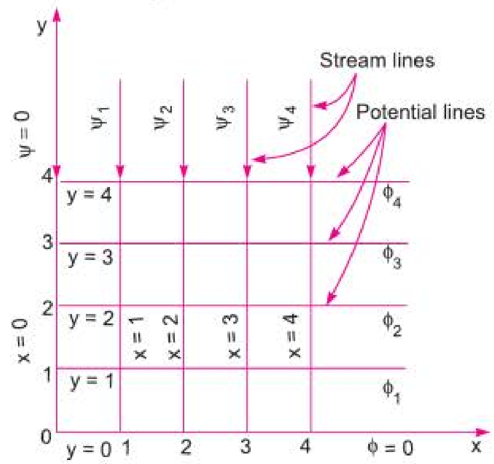


Fig. 5.32

Plotting of Potential lines. For potential lines, the equation is $\phi = U.y + C_2$

Let $\phi = 0$, where $y = 0$. Then $C_2 = 0$.

Hence equation of potential lines becomes as $\phi = U.y$ ...(5.38)

The above equation shows that potential lines are straight lines parallel to x-axis and at a distance of y from the x-axis as shown in Fig. 5.32.

▶ 5.14 SOURCE FLOW

The source flow is the flow coming from a point (source) and moving out radially in all directions of a plane at uniform rate. Fig. 5.33 shows a source flow in which the point O is the source from which the fluid moves radially outward. The strength of a source is defined as the volume flow rate per unit depth. The unit of strength of source is m^2/s. It is represented by q.

Let u_r = radial velocity of flow at a radius r from the source O

q = volume flow rate per unit depth

r = radius

The radial velocity u_r at any radius r is given by,

$$u_r = \frac{q}{2\pi r} \quad ...(5.39)$$

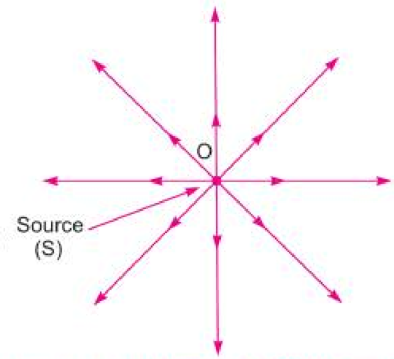


Fig. 5.33 *Source flow (Flow away from source)*

The above equation shows that with the increase of r, the radial velocity decreases. And at a large distance away from the source, the velocity will be approximately equal to zero. The flow is in radial direction, hence the tangential velocity $u_\theta = 0$.

Let us now find the equation of stream function and velocity potential function for the source flow. As in this case, $u_\theta = 0$, the equation of stream function and velocity potential function will be obtained from u_r.

Equation of Stream Function

By definition, the radial velocity and tangential velocity components in terms of stream function are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \quad \text{[See equation (5.12}A\text{)]}$$

But
$$u_r = \frac{q}{2\pi r} \quad \text{[See equation (5.39)]}$$

$$\therefore \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{q}{2\pi r}$$

or
$$d\psi = r.\frac{q}{2\pi r}.d\theta = \frac{q}{2\pi}\,d\theta$$

Integrating the above equation w.r.t. θ, we get

$$\psi = \frac{q}{2\pi} \times \theta + C_1, \text{ where } C_1 \text{ is constant of integration.}$$

Let $\psi = 0$, when $\theta = 0$, then $C_1 = 0$.

Hence the equation of stream function becomes as

$$\psi = \frac{q}{2\pi}.\ \theta \qquad \ldots(5.40)$$

In the above equation, q is constant.

The above equation shows that stream function is a function of θ. For a given value of θ, the stream function ψ will be constant. And this will be a radial line. The stream lines can be plotted by having different values of θ. Here θ is taken in radians.

Plotting of stream lines

When $\theta = 0$, $\psi = 0$

$$\theta = 45^\circ = \frac{\pi}{4} \text{ radians, } \psi = \frac{q}{2\pi}.\frac{\pi}{4} = \frac{q}{8} \text{ units}$$

$$\theta = 90^\circ = \frac{\pi}{2} \text{ radians, } \psi = \frac{q}{2\pi}.\frac{\pi}{2} = \frac{q}{4} \text{ units}$$

$$\theta = 135^\circ = \frac{3\pi}{4} \text{ radians, } \psi = \frac{q}{2\pi}.\frac{3\pi}{4} = \frac{3q}{8} \text{ units}$$

The stream lines will be radial lines as shown in Fig. 5.34.

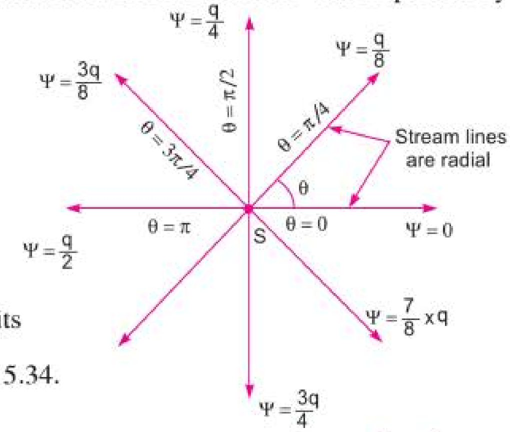


Fig. 5.34 *Stream line for source flow.*

Equation of Potential Function

By definition, the radial and tangential components in terms of velocity function are given by

$$u_r = \frac{\partial \phi}{\partial r} \text{ and } u_\theta = \frac{1}{r}.\frac{\partial \phi}{\partial \theta} \qquad \text{[See equation (5.9A)]}$$

But from equation (5.39), $u_r = \dfrac{q}{2\pi r}$

Equating the two values of u_r, we get

$$\frac{\partial \phi}{\partial r} = \frac{q}{2\pi r} \text{ or } d\phi = \frac{q}{2\pi r}\, dr$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{q}{2\pi r}.\, dr$$

or

$$\phi = \frac{q}{2\pi}\int \frac{1}{r}\, dr \quad \left[\because \frac{q}{2\pi} \text{ is a constant term}\right]$$

$$= \frac{q}{2\pi} \log_e r \qquad \ldots(5.41)$$

In the above equation, q is constant.

The above equation shows, that the velocity potential function is a function of r. For a given value of r, the velocity function ϕ will be constant. Hence it will be a circle with origin at the source. The velocity potential lines will be circles with origin at the source as shown in Fig. 5.35.

Let us now find an expression for the pressure in terms of radius.

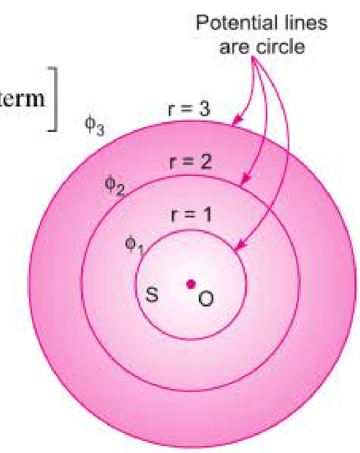


Fig. 5.35 *Potential lines for source.*

Pressure distribution in a plane source flow

The pressure distribution in a plane source flow can be obtained with the help of Bernoulli's equation. Let us assume that the plane of the flow is horizontal. In that case the datum head will be same for two points of flow.

Let p = pressure at a point 1 which is at a radius r from the source at point 1
u_r = velocity at point 1
p_0 = pressure at point 2, which is at a large distance away from the source. The velocity will be zero at point 2. [Refer to equation (5.39)]

Applying Bernoulli's equation, we get

$$\frac{p}{\rho g}+\frac{u_r^2}{2g}=\frac{p_0}{\rho g}+0 \quad \text{or} \quad \frac{(p-p_0)}{\rho g}=-\frac{u_r^2}{2g}$$

or
$$(p-p_0)=-\frac{\rho . u_r^2}{2}$$

But from equation(5.39), $u_r=\dfrac{q}{2\pi r}$

Substituting the value of u_r in the above equation, we get

$$(p-p_0)=-\left(\frac{\rho}{2}\right).\left(\frac{q}{2\pi r}\right)^2$$

$$=-\frac{\rho q^2}{8\pi^2 r^2} \qquad \ldots(5.42)$$

In the above equation, ρ and q are constants.

The above equation shows that the pressure is inversely proportional to the square of the radius from the source.

▶ 5.15 SINK FLOW

The sink flow is the flow in which fluid moves radially inwards towards a point where it disappears at a constant rate. This flow is just opposite to the source flow. Fig. 5.36 shows a sink flow in which the fluid moves radially inwards towards point O, where it disappears at a constant rate. The pattern of stream lines and equipotential lines of a sink flow is the same as that of a source flow. All the equations derived for a source flow shall hold to good for sink flow also except that in sink flow equations, q is to be replaced by $(-q)$.

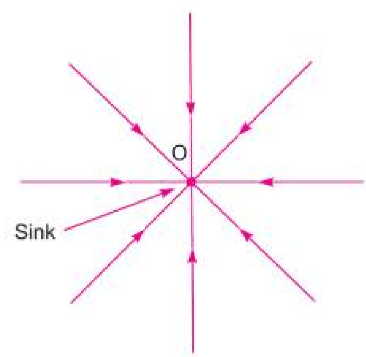


Fig. 5.36 *Sink flow (Flow toward centre)*

Problem 5.33 *Plot the stream lines for a uniform flow of :*
(i) 5 m/s parallel to the positive direction of the x-axis and
(ii) 10 m/s parallel to the positive direction of the y-axis.

Solution. (*i*) The stream function for a uniform flow parallel to the positive direction of the x-axis is given by equation (5.31) as

$$\psi = U \times y$$

The above equation shows that stream lines are straight lines parallel to the x-axis at a distance y from the x-axis. Here U = 5 m/s and hence above equation becomes as

$$\psi = 5y$$

For $y = 0$, stream function $\psi = 0$

For $y = 0.2$, stream function $\psi = 5 \times 0.2 = 1$ unit

For $y = 0.4$, stream function $\psi = 5 \times 0.4 = 2$ unit

The other values of stream function can be obtained by substituting the different values of y. The stream lines are horizontal as shown in Fig. 5.36 (*a*).

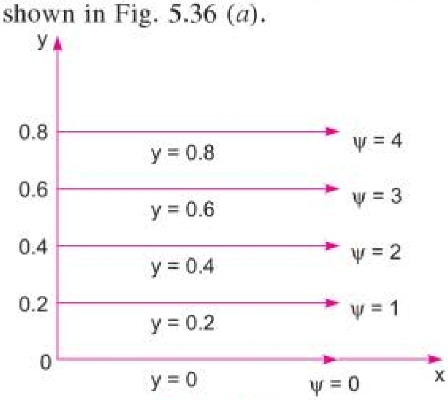


Fig. 5.36 (*a*)

(*ii*) The stream function for a uniform flow parallel to the positive direction of the *y*-axis is given by equation (5.37) as

$$\psi = -U \times x$$

The above equation shows that stream lines are straight lines parallel to the *y*-axis at a distance x from the *y*-axis. Here $U = 10$ m/s and hence the above equation becomes as

$$\psi = -10 \times x$$

The negative sign shows that the stream lines are in the downward direction.

For $x = 0$, the stream function $\psi = 0$

For $x = 0.1$, the stream function $\psi = -10 \times 0.1 = -1.0$ unit

For $x = 0.2$, the stream function $\psi = -10 \times 0.2 = -2.0$ unit

For $x = 0.3$, the stream function $\psi = -10 \times 0.3 = -3.0$ unit

The other values of stream function can be obtained by substituting the different values of x. The stream lines are vertical as shown in Fig. 5.36 (*b*).

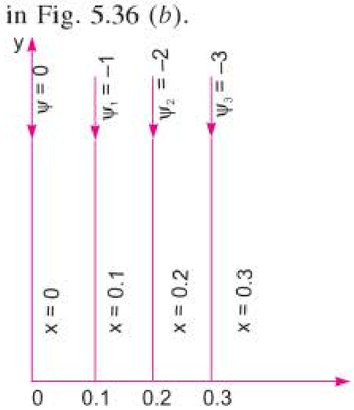


Fig. 5.36 (*b*)

Problem 5.34 *Determine the velocity of flow at radii of 0.2 m, 0.4 m and 0.8 m, when the water is flowing radially outward in a horizontal plane from a source at a strength of 12 m^2/s.*

Solution. Given :

Strength of source, $q = 12\ m^2/s$

The radial velocity u_r at any radius r is given by equation (5.39) as

$$u_r = \frac{q}{2\pi r}$$

When $r = 0.2$ m, $$u_r = \frac{12}{2\pi \times 0.2} = \mathbf{9.55\ m/s.}\ \text{Ans.}$$

When $r = 0.4$ m, $$u_r = \frac{12}{2\pi \times 0.4} = \mathbf{4.77\ m/s.}\ \text{Ans.}$$

When $r = 0.8$ m, $$u_r = \frac{12}{2\pi \times 0.8} = \mathbf{2.38\ m/s.}\ \text{Ans.}$$

Problem 5.35 *Two discs are placed in a horizontal plane, one over the other. The water enters at the centre of the lower disc and flows radially outward from a source of strength 0.628 m^2/s. The pressure, at a radius 50 mm, is 200 kN/m^2. Find :*

(i) pressure in kN/m^2 at a radius of 500 mm and

(ii) stream function at angles of 30° and 60° if $\psi = 0$ at $\theta = 0°$.

Solution. Given :

Source strength, $q = 0.628\ m^2/s$

Pressure at radius 50 mm, $p_1 = 200\ kN/m^2 = 200 \times 10^3\ N/m^2$

(i) Pressure at a radius 500 mm

Let p_2 = pressure at radius 500 mm

$(u_r)_1$ = velocity at radius 50 mm

$(u_r)_2$ = velocity at radius 500 mm

The radial velocity at any radius r is given by equation (5.39) as

$$u_r = \frac{q}{2\pi r}$$

When $r = 50\ mm = 0.05$ m, $$(u_r)_1 = \frac{0.628}{2\pi \times 0.05} = 1.998\ m/s \simeq 2\ m/s$$

When $r = 500\ mm = 0.5$ m, $$(u_r)_2 = \frac{0.628}{2\pi \times 0.5} = 0.2\ m/s$$

Applying Bernoulli's equation at radius 0.05 m and at radius 0.5 m,

$$\frac{p_1}{\rho g} + \frac{(u_r)_1^2}{2g} = \frac{p_2}{\rho g} + \frac{(u_r)_2^2}{2g}$$

or $$\frac{p_1}{\rho} + \frac{(u_r)_1^2}{2} = \frac{p_2}{\rho} + \frac{(u_r)_2^2}{2}$$

or $$\frac{200 \times 10^3}{1000} + \frac{2^2}{2} = \frac{p_2}{1000} + \frac{0.2^2}{2}$$

or $$200 + 2 = \frac{p_2}{1000} + 0.02$$

or $$\frac{p_2}{1000} = 202 - 0.02 = 201.98$$

$\therefore$ $$p_2 = 201.98 \times 1000 \text{ N/m}^2 = \mathbf{201.98 \text{ kN/m}^2}. \textbf{ Ans.}$$

(*ii*) *Stream functions at $\theta = 30°$ and $\theta = 60°$*

For the source flow, the equation of stream function is given by equation (5.40) as

$$\psi = \frac{q}{2\pi}.\theta, \text{ where } \theta \text{ is in radians}$$

When $\theta = 30°$, $$\psi = \frac{0.628}{2\pi} \times \frac{30 \times \pi}{180} \qquad \left(\because \theta = 30° = \frac{30 \times \pi}{180} \text{ radians}\right)$$

$$= \mathbf{0.0523 \text{ m}^2/\text{s}}. \textbf{ Ans.}$$

When $\theta = 60°$, $$\psi = \frac{0.628}{2\pi} \times \frac{60\pi}{180} = \mathbf{0.1046 \text{ m}^2/\text{s}}. \textbf{ Ans.}$$

▶ 5.16 FREE-VORTEX FLOW

Free-vortex flow is a circulatory flow of a fluid such that its stream lines are concentric circles.

For a free-vortex flow, $u_\theta \times r$ = constant (say C)

Also, circulation around a stream line of an irrotation vortex is

$$\Gamma = 2\pi r \times u_\theta = 2\pi \times C \qquad (\because r \times u_\theta = C)$$

where u_θ = tangential velocity at any radius r from the centre.

$\therefore$ $$u_\theta = \frac{\Gamma}{2\pi r}$$

The circulation Γ is taken positive if the free vortex is anticlockwise.

For a free-vortex flow, the velocity components are

$$u_\theta = \frac{\Gamma}{2\pi r} \quad \text{and} \quad u_r = 0$$

Equation of Stream Function

By definition, the stream function is given by

$$u_\theta = \frac{-\partial \psi}{\partial r} \quad \text{and} \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{[See equation (5.12}A\text{)]}$$

In case of free-vortex flow, the radial velocity (u_r) is zero. Hence equation of stream function will be obtained from tangential velocity, u_θ. The value of u_θ is given by

$$u_\theta = \frac{\Gamma}{2\pi r}$$

Equating the two values of u_θ, we get

$$-\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \quad \text{or} \quad d\psi = -\frac{\Gamma}{2\pi r} dr$$

Integrating the above equation, we get

$$\int d\psi = \int -\frac{\Gamma}{2\pi r} dr = \left(-\frac{\Gamma}{2\pi}\right) \int \frac{1}{r} dr$$

or

$$\psi = \left(-\frac{\Gamma}{2\pi}\right) \log_e r \qquad \left(\because \frac{\Gamma}{2\pi} \text{ is a constant term}\right) \quad \ldots(5.43)$$

The above equation shows that stream function is a function of radius. For a given value of r, the stream function is constant. Hence the stream lines are concentric circles as shown in Fig. 5.37.

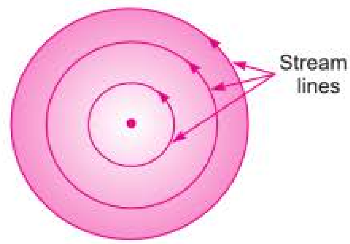


Fig. 5.37

Equation of potential function. By definition, the potential function is given by,

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad \text{and} \quad u_r = \frac{\partial \phi}{\partial r} \qquad \text{[See equation (5.9}A\text{)]}$$

Here $u_r = 0$ and $u_\theta = \frac{\Gamma}{2\pi r}$. Hence, the equation of potential function will be obtained from u_θ.

Equating the two values of u_θ, we get

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \quad \text{or} \quad d\phi = r \cdot \frac{\Gamma}{2\pi r} \cdot d\theta = \frac{\Gamma}{2\pi} d\theta$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{\Gamma}{2\pi} d\theta \quad \text{or} \quad \phi = \frac{\Gamma}{2\pi}\int d\theta = \frac{\Gamma}{2\pi} \cdot \theta \quad \ldots(5.44)$$

The above equation shows that velocity potential function is a function of θ. For a given value of θ, potential function is a constant. Hence equipotential lines are radial as shown in Fig. 5.38.

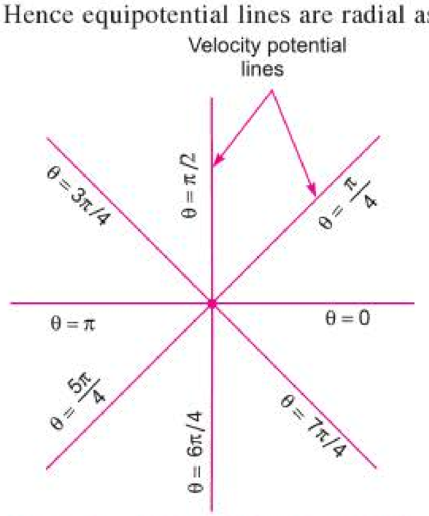


Fig. 5.38 *Potential lines are radial.*

▶ 5.17 SUPER-IMPOSED FLOW

The flow patterns due to uniform flow, a source flow, a sink flow and a free vortex flow can be super-imposed in any linear combination to get a resultant flow which closely resembles the flow around bodies. The resultant flow will still be potential and ideal. The following are the important super-imposed flow :

(*i*) Source and sink pair
(*ii*) Doublet (special case of source and sink combination)
(*iii*) A plane source in a uniform flow (flow past a half body)
(*iv*) A source and sink pair in a uniform flow
(*v*) A doublet in a uniform flow.

5.17.1 Source and Sink Pair. Fig. 5.39 shows a source and a sink of strength q and $(-q)$ placed at A and B respectively at equal distance from the point O on the x-axis. Thus the source and sink are placed symmetrically on the x-axis. The source of strength q is placed at A and sink of strength $(-q)$ is placed at B. The combination of the source and the sink would result in a flownet where stream lines will be circular arcs starting from point A and ending at point B as shown in Fig. 5.40.

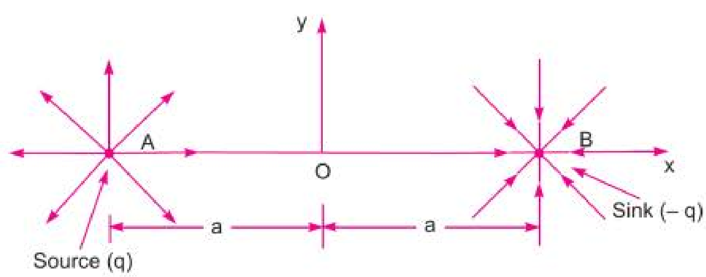


Fig. 5.39 *Source and sink pair.*

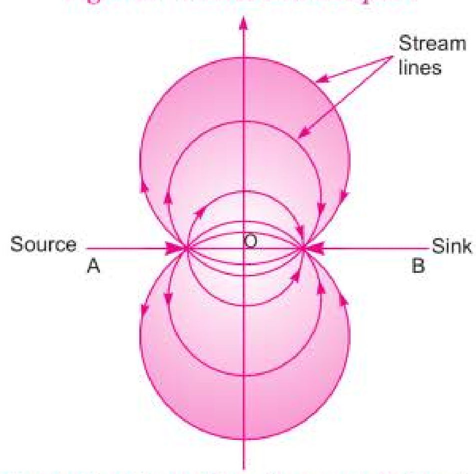


Fig. 5.40 *Stream lines for source-sink pair.*

Equation of stream function and potential function

Let P be any point in the resultant flownet of source and sink as shown in Fig. 5.41.

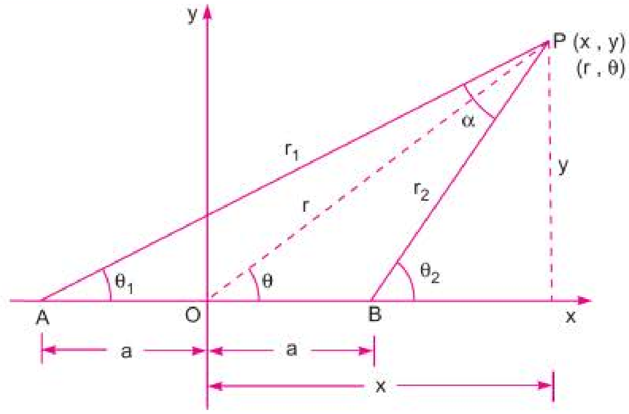


Fig. 5.41

Let r, θ = Cylindrical co-ordinates of point P with respect to origin O

x, y = Corresponding co-ordinates of point P

r_1, θ_1 = Position of point P with respect to source placed at A

r_2, θ_2 = Position of point P with respect to sink placed at B

α = Angle subtended at P by the join of source and sink *i.e.*, angle APB.

Let us find the equation for the resultant stream function and velocity potential function. The equation for stream function due to source is given by equation (5.40) as $\psi_1 = \frac{q\,.\,\theta_1}{2\pi}$ whereas due to sink it is given by $\psi_2 = \frac{(-q\theta_2)}{2\pi}$. The equation for resultant stream function (ψ) will be the sum of these two stream function.

$$\therefore \quad \psi = \psi_1 + \psi_2$$

$$= \frac{q\theta_1}{2\pi} + \left(\frac{-q\theta_2}{2\pi}\right) = \frac{-q}{2\pi}(\theta_2 - \theta_1)$$

$$= \frac{-q}{2\pi}\,.\,\alpha \quad [\because \quad \alpha = \theta_2 - \theta_1. \text{ In triangle } ABP,\ \theta_1 + \alpha + (180^\circ - \theta_2) = 180^\circ \quad \therefore \quad \alpha = \theta_2 - \theta_1]$$

$$= \frac{-q\,.\,\alpha}{2\pi} \qquad \text{...(5.45)}$$

The equation for potential function due to source is given by equation (5.41) as $\phi_1 = \frac{q}{2\pi}\log_e r_1$ and due to sink it is given as $\phi_2 = \frac{-q}{2\pi}\log_e r_2$. The equation for resultant potential function (ϕ) will be the sum of these two potential function.

$$\therefore \quad \phi = \phi_1 + \phi_2$$

$$= \frac{q}{2\pi}\log_e r_1 + \left(\frac{-q}{2\pi}\right)\log_e r_2$$

$$= \frac{q}{2\pi} [\log_e r_1 - \log_e r_2] = \frac{q}{2\pi} \log_e \left(\frac{r_1}{r_2}\right) \qquad \text{...(5.46)}$$

To prove that resultant stream lines will be circular arc passing through source and sink

The resultant stream function is given by equation (5.45) as

$$\psi = \frac{-q \cdot \alpha}{2\pi} \quad \text{or} \quad \frac{-q}{2\pi}(\theta_2 - \theta_1) \qquad (\because \alpha = \theta_2 - \theta_1)$$

For a given stream line ψ = constant. In the above equation the term $\frac{q}{2\pi}$ is also constant. This means that $(\theta_2 - \theta_1)$ or angle α will also be constant for various positions of P in the plane.

To satisfy this, the locus of P must be a circle with AB as chord, having its centre on y-axis, as shown in Fig. 5.40.

Consider the equation (5.45) again as

$$\psi = \frac{-q}{2\pi}\alpha = \frac{-q}{2\pi}(\theta_2 - \theta_1) \qquad (\because \alpha = \theta_2 - \theta_1)$$

$$= \frac{q}{2\pi}(\theta_1 - \theta_2)$$

or
$$(\theta_1 - \theta_2) = \frac{2\pi\psi}{q}$$

Taking tangent to both sides, we get

$$\tan(\theta_1 - \theta_2) = \tan\left(\frac{2\pi\psi}{q}\right) \quad \text{or} \quad \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \cdot \tan\theta_2} = \tan\left(\frac{2\pi\psi}{q}\right) \qquad \text{...}(i)$$

But
$$\tan\theta_1 = \frac{y}{x+a} \quad \text{and} \quad \tan\theta_2 = \frac{y}{x-a} \qquad \text{...(5.46}A)$$

Substituting the values of $\tan\theta_1$ and $\tan\theta_2$ in equation (i),

$$\frac{\frac{y}{(x+a)} - \frac{y}{(x-a)}}{1 + \frac{y}{(x+a)} \cdot \frac{y}{(x-a)}} = \tan\left(\frac{2\pi\psi}{q}\right)$$

or
$$\frac{y(x-a) - y(x+a)}{x^2 - a^2 + y^2} = \tan\left(\frac{2\pi\psi}{q}\right)$$

or
$$\frac{-2ay}{x^2 - a^2 + y^2} = \tan\left(\frac{2\pi\psi}{q}\right)$$

or
$$\frac{-2ay}{x^2 - a^2 + y^2} = \frac{1}{\cot\left(\frac{2\pi\psi}{q}\right)}$$

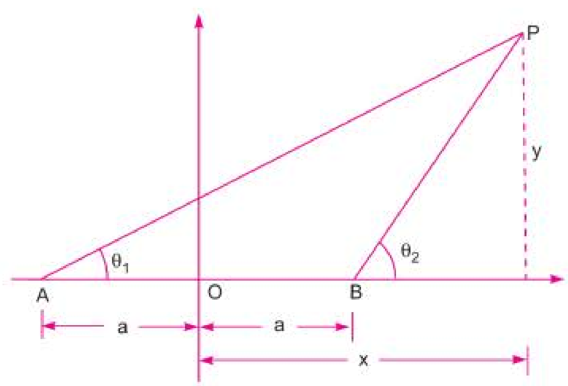


Fig. 5.41 (*a*)

or
$$x^2 - a^2 + y^2 = -2ay\cot\left(\frac{2\pi\psi}{q}\right)$$

or $$x^2 - a^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) = 0$$

or $$x^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) - a^2 = 0$$

or $$x^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) + a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) - a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) - a^2 = 0$$

$$\left[\text{Adding and subtracting } a^2 \cot^2\left(\frac{2\pi\psi}{q}\right)\right]$$

or $$x^2 + \left[y + a\cot\left(\frac{2\pi\psi}{q}\right)\right]^2 = a^2 + a^2 \cot^2\left(\frac{2\pi\psi}{q}\right)$$

$$= a^2\left[1 + \cot^2\left(\frac{2\pi\psi}{q}\right)\right]$$

$$= a^2 \operatorname{cosec}^2\left(\frac{2\pi\psi}{q}\right) \qquad \left[\because\ 1 + \cot^2\left(\frac{2\pi\psi}{q}\right) = \operatorname{cosec}^2\left(\frac{2\pi\psi}{q}\right)\right]$$

or $$x^2 + \left[y + a\cot\left(\frac{2\pi\psi}{q}\right)\right]^2 = \left[a \operatorname{cosec}\left(\frac{2\pi\psi}{q}\right)\right]^2 \qquad \text{...(5.47)}$$

The above is the equation of a circle* with centre on y-axis at a distance of $\pm\, a \cot\left(\frac{2\pi\psi}{q}\right)$ from the origin. The radius of the circle will be $a \operatorname{cosec}\left(\frac{2\pi\psi}{q}\right)$.

Similarly, it can be shown that the potential lines for the source-sink pair will be eccentric non-intersecting circles with their centres on the x-axis as shown in Fig. 5.41 (b).

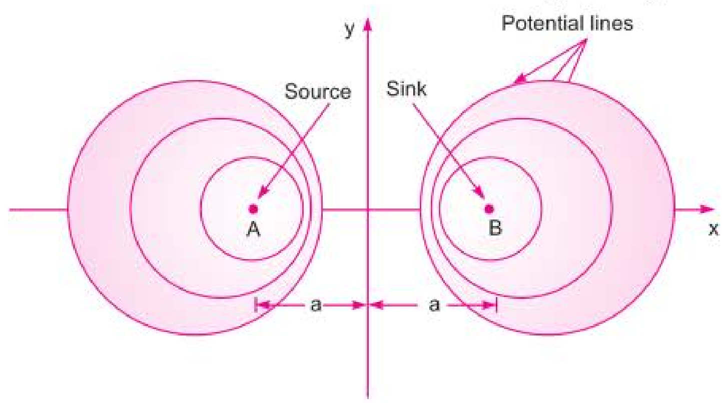


Fig. 5.41 (*b*) *Potential lines for source sink pair (Potential lines are eccentric non-intersecting circles with their centres on x-axis).*

*The equation $x^2 + y^2 = a^2$ is the equation of a circle with centre at origin and of radius 'a'.

Problem 5.36 *A source and a sink of strength 4 m²/s and 8 m²/s are located at (– 1, 0) and (1, 0) respectively. Determine the velocity and stream function at a point P (1, 1) which is lying on the flownet of the resultant stream line.*

Solution. Given :

Source strength, $q_1 = 4 \text{ m}^2/\text{s}$

Sink strength, $q_2 = 8 \text{ m}^2/\text{s}$

Distance of the source and sink from origin, $a = 1$ unit.

The position of the source, sink and point P in the flow field is shown in Fig. 5.42.

From Fig. 5.42, it is clear that angle θ_2 will be 90° and angle θ_1 can be calculated from right angled triangle ABP.

The equation for stream function due to source is given by equation (5.40) as $\psi_1 = \dfrac{q_1 \times \theta_1}{2\pi}$, whereas due to sink it is given by $\psi_2 = \dfrac{-q_2 \times \theta_2}{2\pi}$. The resultant stream function ψ is given as

$$\psi = \psi_1 + \psi_2$$

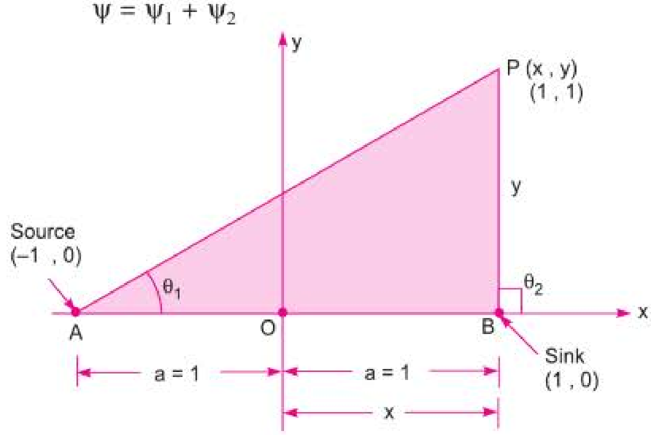


Fig. 5.42

$$= \frac{q_1 \times \theta_1}{2\pi} + \left(\frac{-q_2 \times \theta_2}{2\pi}\right) = \frac{q_1 \times \theta_1}{2\pi} - \frac{q_2 \times \theta_2}{2\pi} \quad \ldots(i)$$

Let us find the values of θ_1 and θ_2 in radians. From the geometry, it is clear that the traingle ABP is a right angled triangle with angle $\theta_2 = 90° = \dfrac{90}{180} \times \pi = \dfrac{\pi}{2}$ radians.

Also $$\tan \theta_1 = \frac{BP}{AB} = \frac{1}{2} = 0.5$$

or $$\theta_1 = \tan^{-1} 0.5 = 26.56° = 26.56 \times \frac{\pi}{180} \text{ radians} = 0.463$$

Substituting these values in equation (i),

$$\psi = \frac{q_1}{2\pi} \times 0.463 - \frac{q_2}{2\pi} \times \frac{\pi}{2}$$

$$= \frac{\pi}{2\pi} \times 0.463 - \frac{8}{2\pi} \times \frac{\pi}{2} \qquad (\because q_1 = 4 \text{ m}^2/\text{s}, q_2 = 8 \text{ m}^2/\text{s})$$

$$= 0.294 - 2.0 = \mathbf{-1.706 \text{ m}^2/\text{s}}. \textbf{ Ans.}$$

To find the velocity at the point P, let us first find the stream function in terms of x and y co-ordinates. The stream function in terms of θ_1 and θ_2 is given by equation (*i*) above as

$$\psi = \frac{q_1 \times \theta_1}{2\pi} - \frac{q_2 \times \theta_2}{2\pi}$$

The values of θ_1 and θ_2 in terms of x, y and a are given by equation (5.46*A*) as

$$\tan \theta_1 = \frac{y}{x+a} \quad \text{and} \quad \tan \theta_2 = \frac{y}{(x-a)}$$

or

$$\theta_1 = \tan^{-1} \frac{y}{x+a} \quad \text{and} \quad \theta_2 = \tan^{-1} \frac{y}{(x-a)}$$

Substituting these values of θ_1 and θ_2 in equation (*i*), we get

$$\psi = \frac{q_1}{2\pi} \tan^{-1} \frac{y}{x+a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x-a}$$

The velocity component $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$.

$$\therefore \qquad u = \frac{\partial \psi}{\partial y}$$

$$= \frac{\partial}{\partial y}\left[\frac{q_1}{2\pi} \tan^{-1} \frac{y}{x+a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x-a}\right]$$

$$= \frac{q_1}{2\pi} \times \frac{1}{1+\left(\dfrac{y}{x+a}\right)^2} \times \frac{1}{(x+a)} - \frac{q_2}{2\pi} \times \frac{1}{1+\left(\dfrac{y}{x-a}\right)^2} \times \frac{1}{(x-a)}$$

$$= \frac{q_1}{2\pi} \frac{(x+a)^2}{(x+a)^2 + y^2} \times \frac{1}{(x+a)} - \frac{q_2}{2\pi} \times \frac{(x-a)^2}{(x-a)^2 + y^2} \times \frac{1}{(x-a)}$$

$$= \frac{q_1}{2\pi} \frac{(x+a)}{(x+a)^2 + y^2} - \frac{q_2}{2\pi} \frac{(x-a)}{(x-a)^2 + y^2}$$

At the point $P(1, 1)$, the component u is obtained by substituting $x = 1$ and $y = 1$ in the above equation. The value of a is also equal to one.

$$\therefore \qquad u = \frac{q_1}{2\pi} \frac{1+1}{(1+1)^2 + 1^2} - \frac{q_2}{2\pi} \frac{(1-1)}{(1-1)^2 + 1^2}$$

$$= \frac{q_1}{2\pi} \frac{2}{5} - \frac{q_2}{2\pi} \times 0 = \frac{q_1}{2\pi} \times \frac{2}{5} = \frac{4}{2\pi} \times \frac{2}{5} = 0.2544 \text{ m/s}$$

Now $$v = -\frac{\partial\psi}{\partial x}$$

$$= -\frac{\partial}{\partial x}\left[\frac{q_1}{2\pi}\tan^{-1}\frac{y}{x+a} - \frac{q_2}{2\pi}\tan^{-1}\frac{y}{x-a}\right]$$

$$= -\left[\frac{q_1}{2\pi}\frac{1}{1+\left(\dfrac{y}{x+a}\right)^2}\times\frac{y(-1)}{(x+a)^2}\times 1 - \frac{q_2}{2\pi}\times\frac{1}{1+\left(\dfrac{y}{x-a}\right)^2}\times\frac{y(-1)}{(x-a)^2}\times 1\right]$$

$$= -\left[\frac{q_1}{2\pi}\frac{(x+a)^2}{(x+a)^2+y^2}\times\frac{(-y)}{(x+a)^2} - \frac{q_2}{2\pi}\frac{(x-a)^2}{(x-a)^2+y^2}\times\frac{(-y)}{(x-a)^2}\right]$$

$$= \frac{q_1}{2\pi}\frac{y}{(x+a)^2+y^2} - \frac{q_2}{2\pi}\frac{y}{(x-a)^2+y^2}$$

At the point $P(1, 1)$,

$$v = \frac{q_1}{2\pi}\times\frac{1}{(1+1)^2+1^2} - \frac{q_2}{2\pi}\times\frac{1}{(1-1)^2+1^2} \qquad (\because a = 1)$$

$$= \frac{q_1}{2\pi}\times\frac{1}{5} - \frac{q_2}{2\pi}\times\frac{1}{1}$$

$$= \frac{q_1}{2\pi}\times\frac{1}{5} - \frac{q_2}{2\pi} = \frac{4}{2\pi}\times\frac{1}{5} - \frac{8}{2\pi} = 0.1272 - 1.272 = -\ 1.145 \text{ m/s}^2$$

$\therefore$ The resultant velocity, $V = \sqrt{u^2+v^2} = \sqrt{0.2544^2+(-1.145)^2}$ = **1.174 m/s. Ans.**

Problem 5.37 *For the above problem, determine the pressure at P(1, 1) if the pressure at infinity is zero and density of fluid is 1000 kg/m³.*

Solution. Given :

Pressure at infinity, $p_0 = 0$

Density of fluid, $\rho = 1000 \text{ kg/m}^3$

The velocity* of fluid at infinity will be zero. If V_0 = velocity at infinity, then $V_0 = 0$.

The resultant velocity of fluid at $P(1, 1)$ = 1.174 m/s (calculated above)

or $V = 1.174$ m/s.

Let p = pressure at $P(1, 1)$

Applying Bernoulli's theorem at point at infinity and at point P, we get

$$\frac{p_0}{\rho g} + \frac{V_0^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g}$$

or $$0 + 0 = \frac{p}{\rho g} + \frac{V^2}{2g} \quad \text{or} \quad 0 = \frac{p}{\rho g} + \frac{V^2}{2g} \quad \text{or} \quad 0 = \frac{p}{\rho} + \frac{V^2}{2}$$

or $$\frac{p}{\rho} = -\frac{V^2}{2} = -\frac{1.174^2}{2} \qquad (\because V = 1.174 \text{ m/s})$$

---

* From equation (5.39), the velocity at a distance 'r' from source or sink is given by $u_r = \dfrac{q}{2\pi r}$. At infinity, r is very very large hence velocity is zero.

or
$$p = -\frac{1.174^2}{2} \times \rho = -\frac{1.174^2 \times 1000}{2} = \mathbf{-689.14\ N/m^2.\ Ans.}$$

5.17.2 Doublet. It is a special case of a source and sink pair (both of them are of equal strength) when the two approach each other in such a way that the distance $2a$ between them approaches zero and the product $2a\ .\ q$ remains constant. This product $2a\ .\ q$ is known as doublet strength and is denoted by μ.

$\therefore$ Doublet strength, $\mu = 2a\ .\ q$ ...(5.48)

Let q and $(-q)$ may be the strength of the source and the sink respectively as shown in Fig. 5.43. Let $2a$ be the distance between them and P be any point in the combined field of source and sink.

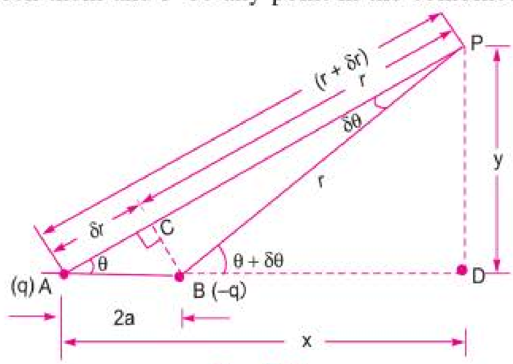


Fig. 5.43

Let θ is the angle made by P at A whereas $(\theta + \delta\theta)$ is the angle at B.

Now the stream function at P,

$$\psi = \frac{q\theta}{2\pi} - \frac{q}{2\pi}(\theta + \delta\theta) = -\frac{q}{2\pi}\delta\theta \qquad \text{...(5.49)}$$

From B, draw $BC \perp$ on AP. Let $AC = \delta r$, $CP = r$ and $AP = r + \delta r$. Also angle $BPC = \delta\theta$. The angle $\delta\theta$ is very small. The distance BC can be taken equal to $r \times \delta\theta$. In triangle ABC, angle $BCA = 90°$ and hence distance BC is also equal to $2a\ .\ \sin\theta$. Equating the two values of BC, we get

$$r \times \delta\theta = 2a\ .\ \sin\theta$$

$$\therefore \qquad \delta\theta = \frac{2a\ .\sin\theta}{r}$$

Substituting the value of $\delta\theta$ in equation (5.49), we get

$$\psi = -\frac{q}{2\pi} \times \frac{2a\sin\theta}{r}$$

$$= -\frac{\mu}{2\pi} \times \frac{\sin\theta}{r} \qquad [\because\ 2a\ .\ q = \mu \text{ from equation (5.48)}] \qquad \text{...(5.50)}$$

In Fig. 5.43, when $2a \rightarrow 0$, the angle $\delta\theta$ subtended by point P with A and B becomes very small. Also $\delta r \rightarrow 0$ and AP becomes equal to r. Then

$$\sin\theta = \frac{PD}{AP} = \frac{y}{r}$$

Also $\qquad AP^2 = AD^2 + PD^2 \quad \text{or} \quad r^2 = x^2 + y^2$

Substituting the value of $\sin\theta$ in equation (5.50), we get

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{r} \times \frac{1}{r} = -\frac{\mu y}{2\pi r^2} = -\frac{\mu y}{2\pi\left(x^2 + y^2\right)} \qquad (\because \; r^2 = x^2 + y^2)$$

...(5.50A)

or $$x^2 + y^2 = -\frac{\mu y}{2\pi\psi} \quad \text{or} \quad x^2 + y^2 + \frac{\mu y}{2\pi\psi} = 0$$

The above equation can be written as

$$x^2 + y^2 + 2 \times y \times \frac{\mu}{4\pi\psi} + \left(\frac{\mu}{4\pi\psi}\right)^2 - \left(\frac{\mu}{4\pi\psi}\right)^2 = 0 \qquad \left[\text{Adding and subtracting } \left(\frac{\mu}{4\pi\psi}\right)^2\right]$$

or $$x^2 + \left(y + \frac{\mu}{4\pi\psi}\right)^2 = \left(\frac{\mu}{4\pi\psi}\right)^2 \qquad ...(5.51)$$

The above is the equation of a circle with centre $\left(0, \frac{\mu}{4\pi\psi}\right)$ and radius $\frac{\mu}{4\pi\psi}$. The centre of the circle lies on y-axis at a distance of $\frac{\mu}{4\pi\psi}$ from x-axis. As the radius of the circle is also equal to $\frac{\mu}{4\pi\psi}$, hence the circle will be tangent to the x-axis. Hence stream lines of the doublet will be the family of circles tangent to the x-axis as shown in Fig. 5.44.

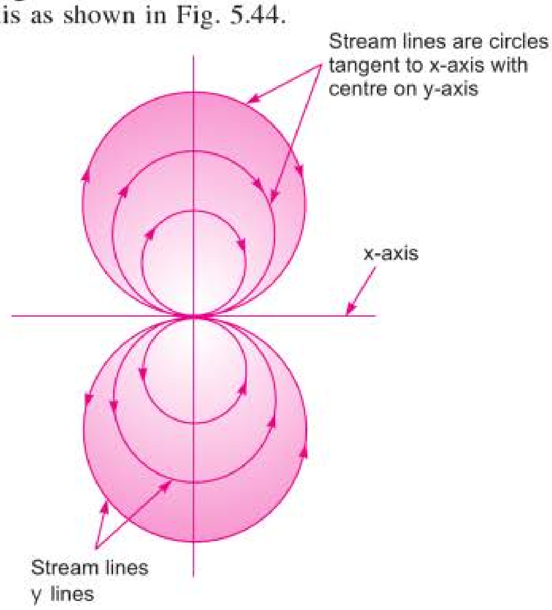


Fig. 5.44 *Stream lines for a doublet.*

Potential function at P

Refer to Fig. 5.43. The potential function at P is given by

$$\phi = \frac{q}{2\pi} \log_e (r + \delta r) + \left(-\frac{q}{2\pi}\right) \log_e r \qquad \text{[Refer to equation (5.41)]}$$

$$= \frac{q}{2\pi}\log_e(r+\delta r) - \frac{q}{2\pi}\log_e r = \frac{q}{2\pi}\log_e\left(\frac{r+\delta r}{r}\right) = \frac{q}{2\pi}\log_e\left(1+\frac{\delta r}{r}\right)^*$$

$$= \frac{q}{2\pi}\left[\frac{\delta r}{r} + \left(\frac{\delta r}{r}\right)^2 \times \frac{1}{2} + ...\right]$$

$$= \frac{q}{2\pi}\cdot\frac{\delta r}{r} \qquad \left[\text{As } \frac{\delta r}{r} \text{ is a small quantity. Hence } \left(\frac{\delta r}{r}\right)^2 \text{ becomes negligible}\right]$$

But in Fig. 5.43, from triangle ABC, we get $\frac{\delta r}{2a} = \cos\theta$

$\therefore \qquad \delta r = 2a\cos\theta$

Substituting the value of δr, we get

$$\phi = \frac{q}{2\pi} \times \frac{2a\cos\theta}{r}$$

$$= \frac{\mu}{2\pi} \times \frac{\cos\theta}{r} \qquad [\because 2a \times q = \mu \text{ from equation } (i)] \quad ...(5.52)$$

In Fig. 5.43, when $2a \to 0$, the angle $\delta\theta$ becomes very small.

Also $\delta r \to 0$ and AP becomes equal to r. Then

$$\cos\theta = \frac{AD}{AP} = \frac{x}{r}$$

Also $AP^2 = AD^2 + PD^2$ or $r^2 = x^2 + y^2$

Substituting the value of $\cos\theta$ in equation (5.52), we get

$$\phi = \frac{\mu}{2\pi} \times \left(\frac{x}{r}\right) \times \frac{1}{r} = \frac{\mu}{2\pi} \times \frac{x}{r^2}$$

$$= \frac{\mu}{2\pi} \times \frac{x}{(x^2+y^2)} \qquad [\because r^2 = x^2 + y^2]$$

or $$x^2 + y^2 = \frac{\mu}{2\pi} \times \frac{x}{\phi} \quad \text{or} \quad x^2 + y^2 - \frac{\mu}{2\pi} \times \frac{x}{\phi} = 0$$

The above equation can be written as

$$x^2 - \frac{\mu}{2\pi}\frac{x}{\phi} + \left(\frac{\mu}{4\pi\phi}\right)^2 - \left(\frac{\mu}{4\pi\phi}\right)^2 + y^2 = 0 \qquad \left[\text{Adding and subtracting } \left(\frac{\mu}{4\pi\phi}\right)^2\right]$$

or $$\left(x - \frac{\mu}{4\pi\phi}\right)^2 + y^2 = \left(\frac{\mu}{4\pi\phi}\right)^2 \qquad ...(5.53)$$

The above is the equation of a circle with centre $\left(\frac{\mu}{4\pi\phi}, 0\right)$ and radius $\left(\frac{\mu}{4\pi\phi}\right)$. The centre of the circle lies on x-axis at a distance of $\frac{\mu}{4\pi\phi}$ from y-axis. As the radius of the circle is equal to the distance of the centre of the circle from the y-axis, hence the circle will be tangent to the y-axis.

* Expansion of $\log_e(1+x) = x + \frac{x^2}{2} + ...$

Hence the potential lines of a doublet will be a family of circles tangent to the y-axis with their centres on the x-axis as shown in Fig. 5.45.

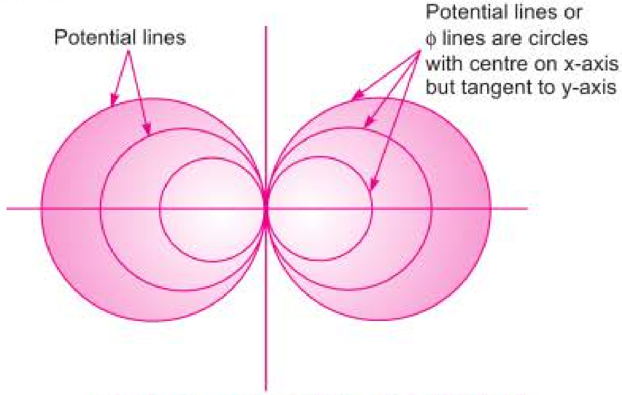


Fig. 5.45 *Potential lines for a doublet.*

Problem 5.38 *A point P(0.5, 1) is situated in the flow field of a doublet of strength 5 m²/s. Calculate the velocity at this point and also the value of the stream function.*

Solution. Given : Point $P(0.5, 1)$. This means $x = 0.5$ and $y = 1.0$

Strength of doublet, $\mu = 5 \text{ m}^2/\text{s}$

(*i*) *Velocity at point P*

The velocity at the given point can be obtained if we know the stream function (ψ). But stream function is given by equation (5.50A) as

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{\left(x^2 + y^2\right)}$$

The velocity components u and v are obtained from the stream function as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{\mu}{2\pi} \times \frac{y}{\left(x^2 + y^2\right)}\right]$$

$$= -\frac{\mu}{2\pi}\frac{\partial}{\partial y}\left[\frac{y}{\left(x^2 + y^2\right)}\right] \qquad \left(\because \frac{\mu}{2\pi} \text{ is a constant term}\right)$$

$$= -\frac{\mu}{2\pi}\left[\frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\right]$$

$$\left[\because \frac{\partial}{\partial y}\left[y\left(x^2 + y^2\right)^{-1}\right] = y[-1]\left[x^2 + y^2\right]^{-2}[2y] + \left(x^2 + y^2\right)^{-1}.1\right.$$

$$\left. = \frac{-2y^2}{\left(x^2 + y^2\right)^2} + \frac{1}{\left(x^2 + y^2\right)} = \frac{-2y^2 + x^2 + y^2}{\left(x^2 + y^2\right)^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\right]$$

and $$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[-\frac{\mu}{2\pi} \times \frac{y}{\left(x^2 + y^2\right)}\right]$$

$$= \frac{\mu}{2\pi}\frac{\partial}{\partial x}\left[\frac{y}{\left(x^2+y^2\right)}\right] = \frac{\mu}{2\pi}\left[\frac{-2xy}{\left(x^2+y^2\right)^2}\right]$$

Substituting the values of $\mu = 5\ m^2/s$, $x = 0.5$ and $y = 1.0$, we get the velocity components as

$$u = -\frac{\mu}{2\pi}\left[\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right] = -\frac{5}{2\pi}\left[\frac{0.5^2-1^2}{\left(0.5^2+1^2\right)^2}\right] = -\frac{5}{2\pi}\frac{0.75}{1.25^2} = -0.382$$

and $$v = \frac{\mu}{2\pi}\left[\frac{-2xy}{\left(x^2+y^2\right)^2}\right] = \frac{5}{2\pi}\left[\frac{-2\times 0.5\times 1}{\left(0.5^2+1^2\right)^2}\right] = \frac{5}{2\pi}\left[\frac{-1}{1.25^2}\right] = -0.509$$

$\therefore$ Resultant velocity, $V = \sqrt{u^2+v^2} = \sqrt{(-0.382)^2+(-0.509)^2}$ = **0.636 m/s. Ans.**

(*ii*) *Value of stream function at point P*

$$\psi = -\frac{\mu}{2\pi}\frac{y}{\left(x^2+y^2\right)} = -\frac{5}{2\pi}\times\frac{1.0}{\left(0.5^2+1^2\right)} = -\frac{5}{2\pi}\times\frac{1}{1.25}$$

$$= \mathbf{-0.636\ m^2/s.}$$ **Ans.**

Solution in polar co-ordinates

The above question can also be done in r, θ (*i.e.*, polar) co-ordinates. The stream function in r, θ co-ordinates is given by equation (5.50) as

$$\psi = -\frac{\mu}{2\pi}\times\frac{\sin\theta}{r} \quad ...(i)$$

and velocity components in radial and tangential directions are given as

$$u_r = \frac{1}{r}\times\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\frac{\mu}{2\pi}\frac{\sin\theta}{r}\right]$$

$$= \frac{1}{r}\times\left(-\frac{\mu}{2\pi}\right)\times\frac{1}{r}\frac{\partial}{\partial\theta}(\sin\theta)$$

$$\left[\because \frac{\mu}{2\pi} \text{ is a constant term and also } r \text{ is constant w. r. t. } \theta\right]$$

$$= -\frac{\mu}{2\pi}\times\frac{1}{r^2}\cos\theta \quad ...(ii)$$

and $$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[-\frac{\mu}{2\pi}\frac{\sin\theta}{r}\right]$$

$$= -\left(-\frac{\mu}{2\pi}\sin\theta\right)\frac{\partial}{\partial r}\left[\frac{1}{r}\right] = \frac{\mu}{2\pi}\sin\theta\,(-1)\,.\,\frac{1}{r^2}$$

$$\left[\because \frac{\mu\sin\theta}{2\pi} \text{ is a constant w. r. t. } r\right]$$

$$= -\frac{\mu}{2\pi}\times\frac{\sin\theta}{r^2} \quad ...(iii)$$

Now $$r = \sqrt{x^2 + y^2} = \sqrt{0.5^2 + 1^2} = \sqrt{1.25}$$

$\therefore$ $$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{1.25}} = 0.894 \text{ and } \cos\theta = \frac{x}{r} = \frac{0.5}{\sqrt{1.25}} = 0.447$$

Substituting the values of r, $\sin\theta$ and $\cos\theta$ in above equations (*i*), (*ii*) and (*iii*), we get

$$\psi = -\frac{\mu}{2\pi}\frac{\sin\theta}{r} = -\frac{5}{2\pi}\times\frac{0.894}{\sqrt{1.25}} = \mathbf{-\ 0.636\ m^2/s.}\ \textbf{Ans.}$$

$$u_r = -\frac{\mu}{2\pi}\times\frac{1}{r^2}\times\cos\theta = -\frac{5}{2\pi}\times\frac{1}{(1.25)}\times 0.447 = -\ 0.2845 \text{ m/s}$$

and $$u_\theta = -\frac{\mu}{2\pi}\times\frac{\sin\theta}{r^2} = -\frac{5}{2\pi}\times\frac{0.894}{1.25} = -\ 0.569 \text{ m/s}$$

$\therefore$ Resultant velocity, $$V = \sqrt{u_r^2 + u_\theta^2}$$

$$= \sqrt{(-\ 0.2845)^2 + (-\ 0.569)^2} = \mathbf{0.636\ m/s.}\ \textbf{Ans.}$$

5.17.3 A Plane Source in a Uniform Flow (Flow Past a Half-Body). Fig. 5.46 (*a*) shows a uniform flow of velocity U and Fig. 5.46 (*b*) shows a source flow of strength q. When this uniform flow is flowing over the source flow, a resultant flow will be obtained as shown in Fig. 5.46. This resultant flow is also known as the flow past a half-body. Let the source is placed on the origin O. Consider a point $P(x, y)$ lying in the resultant flow field with polar co-ordinates r and θ as shown in Fig. 5.46.

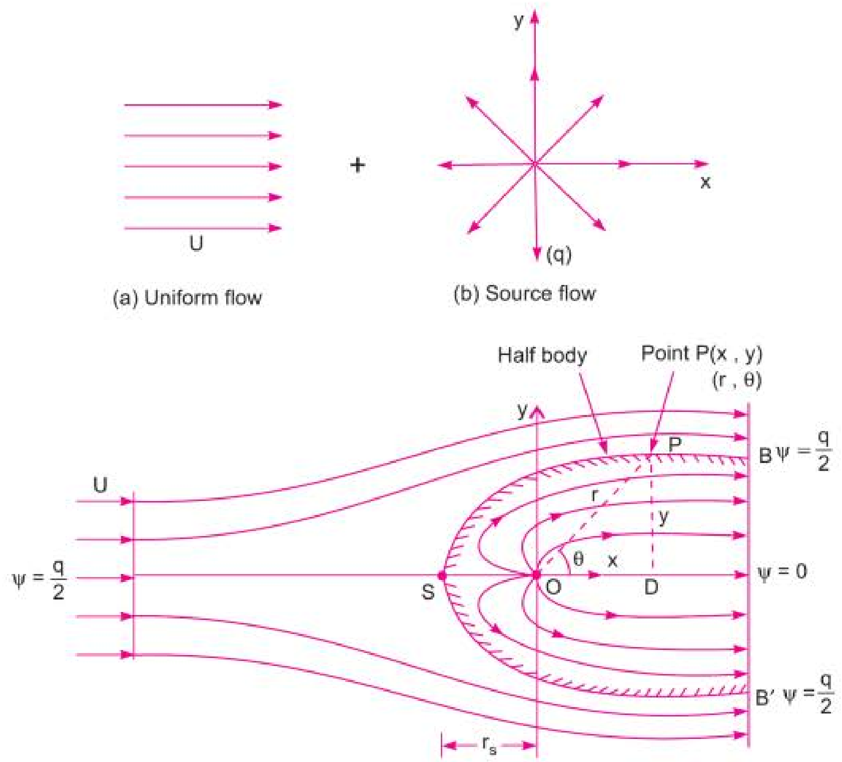


Fig. 5.46 Flow pattern resulting from the combination of a uniform flow and a source.

The stream function (ψ) and potential function (ϕ) for the resultant flow are obtained as given below :

ψ = Stream function due to uniform flow + stream function due to source

$$= U \,.\, y + \frac{q}{2\pi}\,\theta \qquad ...(5.54)$$

$$= U \,.\, r \sin\theta + \frac{q}{2\pi}\,\theta \qquad (\because \; y = r\sin\theta) \; ...(5.54A)$$

and ϕ = Velocity potential function due to uniform flow + Velocity potential function due to source

$$= U \,.\, x + \frac{q}{2\pi}\log_e r = U \,.\, r\cos\theta + \frac{q}{2\pi}\log_e r \qquad ...(5.54B)$$

The following are the important points for the resultant flow pattern :

(*i*) *Stagnation point.* On the left side of the source, at the point S lying on the x-axis, the velocity of uniform flow and that due to source are equal and opposite to each other. Hence the net velocity of the combined flow field is zero. This point is known as stagnation point and is denoted by S. The polar co-ordinates of the stagnation point S are r_S and π, where r_S is radial distance of point S from O.

The net velocity (or resultant velocity) is zero at the stagnation point S.

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(U.r\sin\theta + \frac{q}{2\pi}\theta\right) \qquad \left[\because \; \psi = U.r\sin\theta + \frac{q}{2\pi}\theta\right]$$

$$\therefore \qquad = \frac{1}{r}\left[U.r\cos\theta + \frac{q}{2\pi}\right] = U \,.\, \cos\theta + \frac{q}{2\pi r}$$

At the stagnation point, $\theta = \pi$ radians (180°) and $r = r_S$ and net velocity is zero. This means $u_r = 0$ and $v_\theta = 0$. Substituting these values in the above equation, we get

$$0 = U \,.\, \cos 180^\circ + \frac{q}{2\pi r_S} \qquad [\because \; u_r = 0,\ \theta = 180^\circ \text{ and } r = r_S]$$

$$= -U + \frac{q}{2\pi r_S} \quad \text{or} \quad U = \frac{q}{2\pi r_S}$$

or

$$r_S = \frac{q}{2\pi U} \qquad ...(5.55)$$

From the above equation it is clear that position of stagnation point depends upon the free stream velocity U and source strength q. At the stagnation point, the value of stream function is obtained from equation (5.54A) as

$$\psi = U \,.\, r\sin\theta + \frac{q}{2\pi}\,.\,\theta$$

For the stagnation point, the above equation becomes as

$$\therefore \qquad \psi_s = U.\, r_s \sin 180^\circ + \frac{q}{2\pi}\times\theta$$

$$[\because \quad \text{At stagnation point, } \theta = \pi \text{ radians} = 180^\circ \text{ and } r = r_S]$$

$$= 0 + \frac{q}{2} = \frac{q}{2} \qquad ...(5.56)$$

The above relation gives the equation of stream line passing through stagnation point. We know that no fluid mass crosses a stream line. Hence a stream line is a *virtual solid surface*.

(*ii*) *Shape of resultant flow.* At the stagnation point S, the net velocity is zero. The fluid particles that issue from the source cannot proceed further to the left of stagnation point. They are carried along the contour BSB' that separates the source flow from uniform flow. The curve BSB' can be regarded as the **solid boundary** of a round nosed body such as a bridge pier around which the uniform flow is forced to pass. The contour BSB' is called the half body, because it has only the leading point, it trails to infinity at down stream end.

The value of stream function of the stream line passing through stagnation point S and passing over the solid boundary (*i.e.*, curve BSB') is $\psi_S = \frac{q}{2}$.

Thus the composite flow consists of :

(1) flow over a plane half-body (*i.e.*, flow over curve BSB') outside $\psi = \frac{q}{2}$ and

(2) source flow within the plane half-body.

The plane half-body is described by the dividing stream line, $\psi = \frac{q}{2}$.

But the stream function at any point in the combined flow field is given by equation (5.54) as

$$\psi = U \,.\, y + \frac{q}{2\pi}\,\theta$$

If we take $\psi = \frac{q}{2}$ in the above equation, we will get the equation of the dividing stream line.

$\therefore$ Equation of the dividing stream line (*i.e.*, equation of curve BSB') will be

$$\frac{q}{2} = U \,.\, y + \frac{q}{2\pi} \,.\, \theta \text{ or } U \,.\, y = \frac{q}{2} - \frac{q}{2\pi}\,\theta = \frac{q}{2}\left(1 - \frac{\theta}{\pi}\right)$$

or
$$y = \frac{q}{2U}\left(1 - \frac{\theta}{\pi}\right) \qquad \text{...(5.57)}$$

From the above equation, the main dimensions of the plane half-body may be obtained. From this equation, it is clear that y is maximum, when $\theta = 0$.

Hence At $\theta = 0$, y is maximum and $y_{\max} = \frac{q}{2U}$ ... the maximum ordinate

At $\theta = \frac{\pi}{2}$, $y = \frac{q}{2U}\left(1 - \frac{\pi}{2}\cdot\frac{1}{\pi}\right) = \frac{q}{4U}$ ... the ordinate above the origin

At $\theta = \pi$, $y = \frac{q}{2U}\left(1 - \frac{\pi}{\pi}\right) = 0$ ... the leading point of the half-body

At $\theta = \frac{3\pi}{2}$, $y = \frac{q}{2U}\left(1 - \frac{3\pi}{2\pi}\right) = -\frac{q}{4U}$ ... the ordinate below the origin.

The main dimensions are shown in Fig. 5.47.

(*iii*) *Resultant velocity at any point*

The velocity components at any point in the flow field are given by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{d}{d\theta}\left[U.r\sin\theta + \frac{q}{2\pi}\,\theta\right]$$

$$= \frac{1}{r}\left[U.r\cos\theta + \frac{q}{2\pi}\right] = U.\cos\theta + \frac{q}{2\pi r}$$

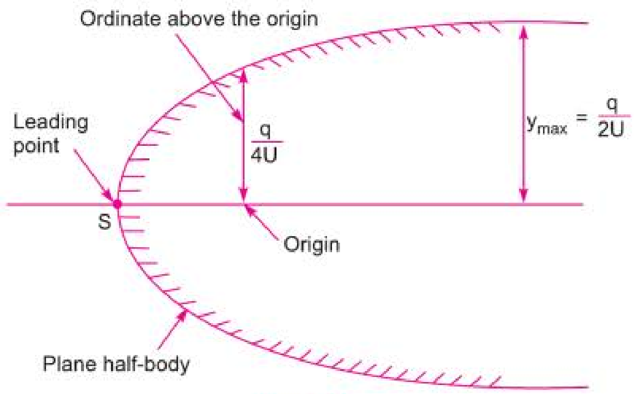


Fig. 5.47

The above equation gives the radial velocity at any point in the flow field. This radial velocity is due to uniform flow and due to source. Due to source the radial velocity is $\frac{q}{2\pi r}$. Hence the velocity due to source diminishes with increase in radial distance from the source. At large distance from the source the contribution of source is negligible and hence free stream uniform flow is not influenced by the presence of source.

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[U.r\sin\theta + \frac{q}{2\pi}\theta\right]$$

$$= -[U\,.\,\sin\theta + 0] = -U\sin\theta \qquad \left[\because \frac{q}{2\pi}\theta \text{ is constant w.r.t. } r\right]$$

$\therefore$ Resultant velocity, $V = \sqrt{u_r^2 + u_\theta^2}$

(*iv*) *Location of stagnation point*

At the stagnation point, the velocity components are zero. Hence equating the radial and tangential velocity components to zero, we get

$$u_r = 0 \quad \text{or} \quad U\cos\theta + \frac{q}{2\pi r} = 0 \quad \text{or} \quad U\cos\theta = -\frac{q}{2\pi r}$$

or $$r\cos\theta = -\frac{q}{2\pi U} \quad \text{But} \quad r\cos\theta = x$$

$\therefore$ $$x = -\frac{q}{2\pi U}$$

When $u_\theta = 0$ or $-U\sin\theta = 0$ or $\sin\theta = 0$ as U cannot be zero

or $\theta = 0$ or π But $y = r\sin\theta$ $\therefore$ $y = 0$

Hence stagnation point is at $\left(-\frac{q}{2\pi U}, 0\right)$, the leading point of the half-body.

(*v*) *Pressure at any point in flow field*

Let p_0 = pressure at infinity where velocity is U

p = pressure at any point P in the flow field, where velocity is V

Now applying the Bernoulli's equation at a point at infinity and at a point P in the flow field, we get

$$\frac{p_0}{\rho g}+\frac{U^2}{2g}=\frac{p}{\rho g}+\frac{V^2}{2g} \quad \text{or} \quad \frac{U^2}{2g}-\frac{V^2}{2g}=\frac{p}{\rho g}-\frac{p_0}{\rho g}=\frac{p-p_0}{\rho g}$$

The pressure co-efficient is defined as

$$C_p=\frac{p-p_0}{\frac{1}{2}\rho U^2}$$

$$=\frac{\rho g\left[\frac{U^2}{2g}-\frac{V^2}{2g}\right]}{\frac{1}{2}\rho U^2} \qquad \left[\because \; p-p_0=\rho g\left(\frac{U^2}{2g}-\frac{V^2}{2g}\right)\right]$$

$$=\frac{U^2-V^2}{U^2}=1-\left(\frac{V}{U}\right)^2 \qquad \text{...(5.58)}$$

Problem 5.39 *A uniform flow with a velocity of 3 m/s is flowing over a plane source of strength 30 m²/s. The uniform flow and source flow are in the same plane. A point P is situated in the flow field. The distance of the point P from the source is 0.5 m and it is at an angle of 30° to the uniform flow. Determine : (i) stream function at point P, (ii) resultant velocity of flow at P and (iii) location of stagnation point from the source.*

Solution. Given : Uniform velocity, U = 3 m/s ; source strength, q = 30 m²/s ; co-ordinates of point P are r = 0.5 m and θ = 30°.

(*i*) *Stream function at point P*

The stream function at any point in the combined flow field is given by equation (5.54*A*)

$$\psi = U \,.\, r\sin\theta + \frac{q}{2\pi}\theta$$

at point P, r = 0.5 m and θ = 30° or $\frac{30}{180}\times\pi$ radians.

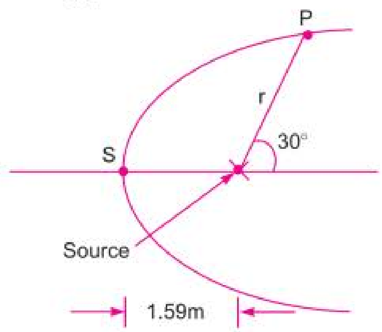


Fig. 5.48

∴ Stream function at point P,

$$\psi = 3\times 0.5\times \sin 30° + \frac{30}{2\pi}\times\left(\frac{30}{180}\times\pi\right)$$

$$= 0.75 + 2.5 = \mathbf{3.25\ m^2/s.}\ \textbf{Ans.}$$

(*ii*) *Resultant velocity at P*

The velocity components anywhere in the flow are given by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[U.r\sin\theta + \frac{q}{2\pi}\theta\right]$$

$$= \frac{1}{r}\left[U.r\cos\theta + \frac{q}{2\pi}\right] = U.\cos\theta + \frac{q}{2\pi r}$$

$$= 3 \times \cos 30° + \frac{30}{2\pi \times 0.5} \qquad (\because \text{At } P,\ r = 0.5,\ \theta = 30°,\ q = 30)$$

$$= 2.598 + 9.55 = 12.14$$

and $$u_\theta = \frac{-\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[U.r\sin\theta + \frac{q}{2\pi}.\theta\right]$$

$$= -U\sin\theta + 0 = -U\sin\theta$$

$$= -3 \times \sin 30° = -1.5$$

$\therefore$ Resultant velocity, $$V = \sqrt{u_r^2 + u_\theta^2}$$

$$= \sqrt{12.14^2 + (-1.5)^2} = \mathbf{12.24\ m/s.}\ \textbf{Ans.}$$

(*iii*) *Location of stagnation point*

The horizontal distance of the stagnation point *S* from the source is given by equation (5.55) as

$$r_S = \frac{q}{2\pi U} = \frac{30}{2\pi \times 3} = \mathbf{1.59\ m.}\ \textbf{Ans.}$$

The stagnation point will be at a distance of 1.59 m to the left side of the source on the *x*-axis.

Problem 5.40 *A uniform flow with a velocity of 20 m/s is flowing over a source of strength 10 m^2/s. The uniform flow and source flow are in the same plane. Obtain the equation of the dividing stream line and sketch the flow pattern.*

Solution. Given : Uniform velocity, U = 20 m/s ; Source strength, q = 10 m^2/s

(*i*) *Equation of the dividing stream line*

The stream function at any point in the combined flow field is given by equation (5.54*A*)

$$\psi = U\ .\ r\sin\theta + \frac{q}{2\pi}\theta$$

$$= 20 \times r\sin\theta + \frac{10}{2\pi}\theta \qquad (\because\ U = 20 \text{ m/s and } q = 10 \text{ m}^2\text{/s})$$

The value of the stream function for the dividing stream line is $\psi = \frac{q}{2}$. Hence substituting $\psi = \frac{q}{2}$ in the above equation, we get the equation of the dividing stream line.

$\therefore$ $$\frac{q}{2} = 20r\sin\theta + \frac{10}{2\pi}\theta$$

or $$\frac{10}{2} = 20r\sin\theta + \frac{10}{2\pi}\theta \qquad (\because\ q = 10)$$

or $$5 = 20r \sin\theta + \frac{10}{2\pi}\theta = 20y + \frac{10}{2\pi}\theta \qquad (\because \quad r\sin\theta = y)$$

$\therefore$ $$20y = 5 - \frac{10}{2\pi}\theta$$

or $$y = \frac{5}{20} - \frac{10}{2\pi} \times \frac{\theta}{20} = 0.25 - \frac{\theta}{4\pi} \qquad ...(i)$$

The above relation gives the equation of the dividing stream line.

From the above equation, for different values of θ the value of y is obtained as :

Value of θ	*Value of y from (i)*	*Remarks*
0	0.25 m	Max. half width of body
$\frac{\pi}{2}$	0.125 m	The +ve ordinate above the origin
π	0	The leading point
$\frac{3\pi}{2}$	– 0.125 m	The –ve ordinate below the origin
2π	– 0.25 m	The max. –ve ordinate

(*ii*) *Sketch of flow pattern*

For sketching the flow pattern, let us first find the location of the stagnation point. The horizontal distance of the stagnation point S from the source is given by the equation,

$$r_s = \frac{q}{2\pi U} = \frac{10}{2\pi \times 20} = 0.0795 \text{ m}$$

Hence the stagnation point lies on the x-axis at a distance of 0.0795 m or 79.5 mm from the source towards left of the source. The flow pattern is shown in Fig. 5.49.

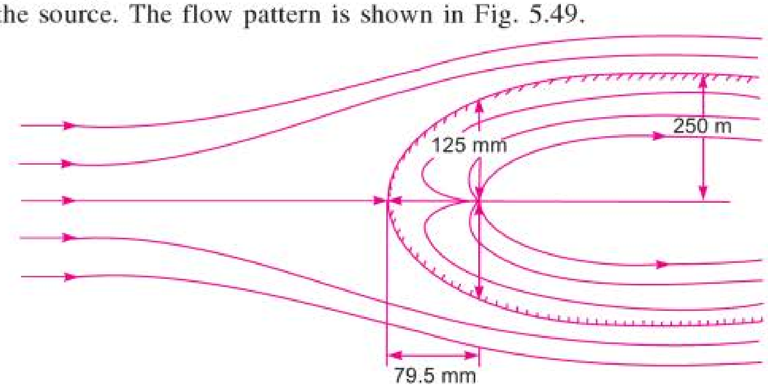


Fig. 5.49

Problem 5.41 *A uniform flow with a velocity of 2 m/s is flowing over a source placed at the origin. The stagnation point occurs at (– 0.398, 0). Determine :*

(i) Strength of the source, *(ii) Maximum width of Rankine half-body and*

(iii) Other principal dimensions of the Rankine half-body.

Solution. Given :

Uniform velocity, $U = 2$ m/s

Co-ordinates of stagnation point = (– 0.398, 0)

This means r_s = 0.398 and stagnation point lies on x-axis at a distance of 0.398 m towards left of origin. The source is placed at origin.

(*i*) *Strength of the source*

Let q = strength of the source

We know that
$$r_s = \frac{q}{2\pi U}$$

or
$$q = 2\pi U \times r_s = 2\pi \times 2 \times 0.398 = 5.0014 \text{ m}^2/\text{s} \simeq \mathbf{5\ m^2/s.}\ \textbf{Ans.}$$

(*ii*) *Maximum width of Rankine half-body*

The main dimensions of the Rankine half-body are obtained from equation (5.57) as

$$y = \frac{q}{2U}\left(1 - \frac{\theta}{\pi}\right) \qquad \ldots(i)$$

The value of y is maximum, when $\theta = 0$.

$$\therefore \qquad y_{\max} = \frac{q}{2U}\left(1 - \frac{0}{\pi}\right) = \frac{q}{2U} = \frac{5}{2 \times 2} = 1.25 \text{ m}$$

$\therefore$ Maximum width of Rankine body = $2 \times y_{\max} = 2 \times 1.25$ = **2.5 m. Ans.**

(*iii*) *Other Principal dimensions of Rankine half-body*

Using equation (5.57), we get

$$y = \frac{q}{2U}\left(1 - \frac{\theta}{\pi}\right)$$

At $\theta = \frac{\pi}{2}$,
$$y = \frac{q}{2U}\left[1 - \frac{\left(\frac{\pi}{2}\right)}{\pi}\right] = \frac{q}{2U}\left[1 - \frac{1}{2}\right] = \frac{q}{4U} = \frac{5}{4 \times 2} = 0.625 \text{ m}$$

The above value gives the upper ordinate at the origin, where source is placed.

$\therefore$ Width of body at origin = $2 \times 0.625 = 1.25$ m

At the stagnation point, the width of the body is zero.

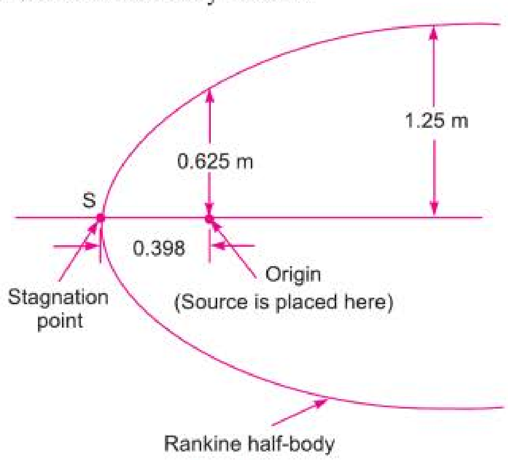


Fig. 5.50

5.17.4 A Source and Sink Pair in a Uniform Flow (Flow Past a Rankine Oval Body). Fig. 5.51 (*a*) shows a uniform flow of velocity U and Fig. 5.51 (*b*) shows a source sink pair of equal strength. When this uniform flow is flowing over the source sink pair, a resultant flow will be obtained as shown in Fig. 5.51 (*c*). This resultant flow is also known as the flow past a Rankine oval body.

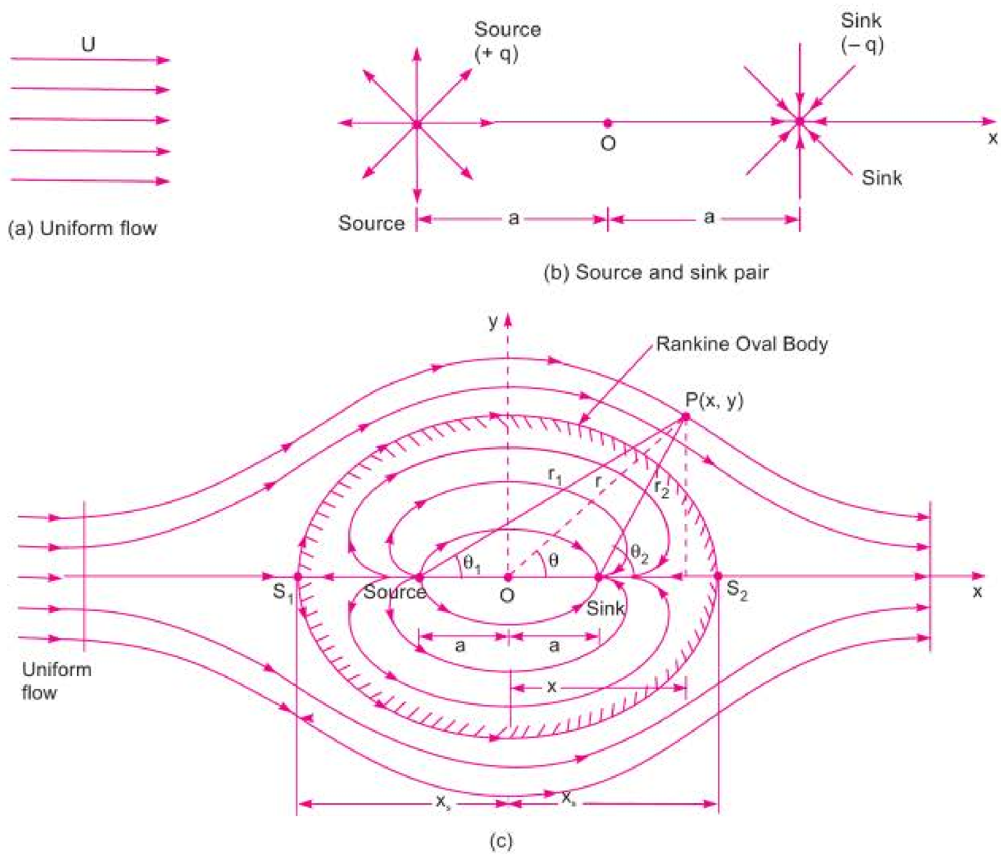


Fig. 5.51

Let U = Velocity of uniform flow along x-axis

q = Strength of source

$(-q)$ = Strength of sink

$2a$ = Distance between source and sink which is along x-axis.

The origin O of the x-y co-ordinates is mid-way between source and sink. Consider a point $P(x, y)$ lying in the resultant flow field. The stream function (ψ) and velocity potential function (ϕ) for the resultant flow field are obtained as given below :

ψ = Stream function due to uniform flow + stream function due to source + stream function due to sink

$$= \psi_{\text{uniform flow}} + \psi_{\text{source}} + \psi_{\text{sink}}$$

$$= U \times y + \frac{q}{2\pi}\,\theta_1 + \frac{(-q)}{2\pi} \times \theta_2$$

(where θ_1 is the angle made by P with source along x-axis and θ_2 with sink)

$$= U \times y + \frac{q\theta_1}{2\pi} - \frac{q\theta_2}{2\pi} = U \times y + \frac{q}{2\pi}(\theta_1 - \theta_2)$$

$$= U \times r \sin\theta + \frac{q}{2\pi}(\theta_1 - \theta_2) \qquad (\because \quad y = r \sin\theta) \quad ...(5.59)$$

and ϕ = potential function due to uniform flow + potential function due to source + potential function due to sink

$$= \phi_{\text{uniform flow}} + \phi_{\text{source}} + \phi_{\text{sink}}$$

$$= U \times x + \frac{q}{2\pi}\log_e r_1 + \frac{(-q)}{2\pi}\log_e r_2$$

$$= U \times r\cos\theta + \frac{q}{2\pi}[\log_e r_1 - \log_e r_2] \qquad (\because \quad x = r\cos\theta)$$

$$= U \times r\cos\theta + \frac{q}{2\pi}\left[\log_e \frac{r_1}{r_2}\right] \qquad ...(5.60)$$

The following are the important points for the resultant flow pattern :

(*a*) There will be two stagnation points S_1 and S_2, one to the left of the source and other to the right of the sink. At the stagnation points, the resultant velocity (*i.e.*, velocity due to uniform flow, velocity due to source and velocity due to sink) will be zero. The stagnation point S_1 is to the left of the source and stagnation point S_2 will be to the right of the sink on the x-axis.

Let x_S = Distance of the stagnation points from origin O along x-axis.

Let us calculate this distance x_S.

For the stagnation point S_1,

(*i*) Velocity due to uniform flow = U

(*ii*) Velocity due to source = $\dfrac{q}{2\pi(x_S - a)}$ $\left[\begin{array}{l}\because \text{ The velocity at any radius due to source} = \dfrac{q}{2\pi r} \\ \text{For } S_1\text{, the radius from source} = (x_S - a)\end{array}\right]$

(*iii*) Velocity due to sink = $\dfrac{-q}{2\pi(x_S + a)}$ [$\because$ At S_1, the radius from sink = $(x_S + a)$]

At point S_1, the velocity due to uniform flow is in the positive x-direction whereas due to source and sink are in the –ve x-direction.

$\therefore$ The resultant velocity at $S_1 = U - \dfrac{q}{2\pi(x_S - a)} - \dfrac{(-q)}{2\pi(x_S + a)}$

But the resultant velocity at stagnation point S_1 should be zero.

$$\therefore \qquad U - \frac{q}{2\pi(x_S - a)} + \frac{q}{2\pi(x_S + a)} = 0$$

or

$$U = \frac{q}{2\pi(x_S - a)} - \frac{q}{2\pi(x_S + a)}$$

$$= \frac{q}{2\pi}\left[\frac{1}{(x_S - a)} - \frac{1}{(x_S + a)}\right] = \frac{q}{2\pi}\left[\frac{(x_S + a) - (x_S - a)}{(x_S - a)(x_S + a)}\right] = \frac{q}{2\pi}\frac{2a}{(x_S^2 - a^2)}$$

or $$x_S^2 - a^2 = \frac{q \cdot a}{\pi U}$$

or $$x_S^2 = a^2 + \frac{qa}{\pi U} = a^2\left[1 + \frac{q}{\pi a U}\right]$$

$\therefore$ $$x_S = a\sqrt{\left(1 + \frac{q}{\pi a U}\right)} \qquad \text{...(5.61)}$$

The above equation gives the location of the stagnation point on the x-axis.

(*b*) The stream line passing through the stagnation points is having zero velocity and hence can be replaced by a solid body. This solid body is having a shape of oval as shown in Fig. 5.51. There will be two flow fields, one within the oval contour and the other outside the solid body. The flow field within the oval contour will be due to source and sink whereas the flow field outside the body will be due to uniform flow only.

The shape of solid body is obtained from the stream line having stream function equal to zero. But the stream function is given by equation as

$$\psi = U \times r \sin\theta + \frac{q}{2\pi}(\theta_1 - \theta_2)$$

For the shape of solid body, $\psi = 0$

$\therefore$ $$0 = U \times r \sin\theta + \frac{q}{2\pi}(\theta_1 - \theta_2)$$

or $$U \times r \sin\theta = -\frac{q}{2\pi}(\theta_1 - \theta_2) = \frac{q}{2\pi}(\theta_2 - \theta_1)$$

$\therefore$ $$r = \frac{q}{2\pi}\frac{(\theta_2 - \theta_1)}{U \sin\theta} \qquad \text{...(5.62)}$$

From the above equation, the distances of the surface of the solid body from the origin can be obtained or the shape of the solid body can be obtained. The maximum width of the body (y_{max}) will be equal to *OM* as shown in Fig. 5.52.

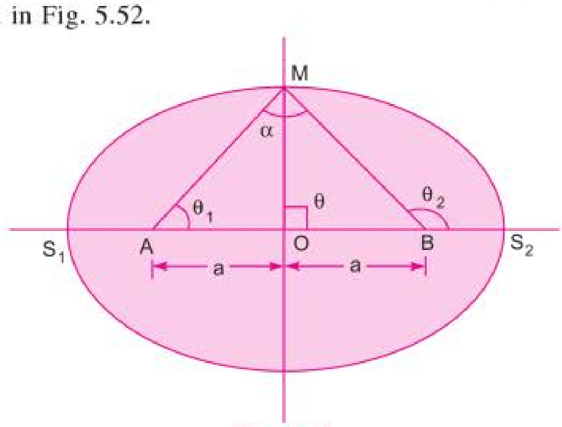


Fig. 5.52

From triangle AOM, we have

$$\tan \theta_1 = \frac{OM}{AO}$$

or $$OM = AO \tan \theta_1 = a \tan \theta_1$$

or $$y_{max} = a \tan \theta_1 \qquad (\because OM = y_{max}) \quad ...(5.63)$$

Let us find the value of θ_1.

When the point P lies on M, then $r = OM$, $\theta = 90° = \frac{\pi}{2}$

and $$\theta_2 = 180° - \theta_1 = \pi - \theta_1 \qquad \text{[Refer to Fig. 5.52]}$$

[$\because$ $AM = BM$ $\therefore$ Angle ABM = Angle $BAM = \theta_1$]

Substituting these values in equation (5.62), we get

$$OM = \frac{q}{2\pi} \frac{\left((\pi - \theta_1) - \theta_1\right)}{U \sin \frac{\pi}{2}} = \frac{q}{2\pi} \frac{(\pi - 2\theta_1)}{U}$$

or $$y_{max} = \frac{q(\pi - 2\theta_1)}{2\pi U} \qquad \text{[where } OM = y_{max}]$$

or $$2\pi U y_{max} = q(\pi - 2\theta_1) \qquad \text{or} \qquad \frac{2\pi U y_{max}}{q} = \pi - 2\theta_1$$

or $$2\theta_1 = \pi - \frac{2\pi U y_{max}}{q} \qquad \text{or} \qquad \theta_1 = \frac{\pi}{2} - \frac{\pi U y_{max}}{q}$$

Substituting this value of θ_1 in equation (5.63), we get

$$y_{max} = a \tan\left[\frac{\pi}{2} - \frac{\pi U y_{max}}{q}\right] = a \cot\left[\frac{\pi U y_{max}}{q}\right] \quad ...(5.64)$$

From the above equation, the value of y_{max} is obtained by hit and trial method till L.H.S. = R.H.S. In this equation $\left(\frac{\pi U y_{max}}{q}\right)$ is in radians.

The length and width of the Rankine oval is obtained as :

Length, $$L = 2 \times x_S$$

$$= 2 \times a \sqrt{\left(1 + \frac{q}{\pi a U}\right)} \qquad \left[\because x_S = a\sqrt{\left(1 + \frac{q}{\pi a U}\right)}\right] \quad ...(5.65)$$

and Width, $$B = 2 \times y_{max}$$

$$= 2a \cot\left(\frac{\pi U y_{max}}{q}\right). \quad ...(5.66)$$

Problem 5.42 *A uniform flow of velocity 6 m/s is flowing along x-axis over a source and a sink which are situated along x-axis. The strength of source and sink is 15 m^2/s and they are at a distance of 1.5 m apart. Determine :*

(i) Location of stagnation points, *(ii) Length and width of the Rankine oval*
(iii) Equation of profile of the Rankine body.

Solution. Given : Uniform flow velocity, $U = 6$ m/s
Strength of source and sink, $q = 15\ \text{m}^2/\text{s}$
Distance between source and sink, $2a = 1.5$ m

$$\therefore \quad a = \frac{1.5}{2} = 0.75 \text{ m}$$

(*i*) *Location of stagnation points* (Refer to Fig. 5.51)

For finding the location of the stagnation points, the equation (5.61) is used.

$$\therefore \quad x_s = a\sqrt{\left(1 + \frac{q}{\pi a U}\right)} = 0.75\sqrt{\left(1 + \frac{15}{\pi \times 0.75 \times 6}\right)} = 1.076 \text{ m}$$

The above equation gives the distance of the stagnation points from the origin. There will be two stagnation points.

The distance of stagnation points from the source and sink = $x_s - a = 1.076 - 0.75 =$ **0.326 m. Ans.**

(*ii*) *Length and width of the Rankine oval*

Length, $L = 2 \times x_s = 2 \times 1.076 = 2.152$ m.

Width, $B = 2 \times y_{max}$ ...(*i*)

Let us now find the value of y_{max}

Using equation (5.64), we get

$$y_{max} = a \cot\left(\frac{\pi U y_{max}}{q}\right) = 0.75 \cot\left(\frac{\pi \times 6 \times y_{max}}{15}\right) = 0.75 \cot\,(0.4\pi\, y_{max})$$

$$= 0.75 \cot\left(0.4\pi\, y_{max} \times \frac{180}{\pi}\right)^\circ$$

$$\left[\because \ (0.4\pi\, y_{max}) \text{ is in radians and hence } (0.4\pi\, y_{max}) \times \frac{180}{\pi} \text{ will be in degrees}\right]$$

$$= 0.75 \cot\,(72 \times y_{max})^\circ$$

The above equation will be solved by hit and trial method. The value of $x_s = 1.076$. But x_s is equal to length of major axis of Rankine body and y_{max} is the length of minor axis of the Rankine body. The length of minor axis will be less than length of major axis. Let us first assume $y_{max} = 0.8$ m. Then

y_{max}	*L.H.S.*	*R.H.S.*
0.8	0.8	$0.75 \cot (72 \times 0.8)^\circ = 0.75 \cot 51.6^\circ = 0.475$
0.7	0.7	$0.75 \cot (72 \times 0.7)^\circ = 0.75 \cot 50.4^\circ = 0.577$
0.6	0.6	$0.75 \cot (72 \times 0.6)^\circ = 0.75 \cot 43.2^\circ = 0.798$
0.65	0.65	$0.75 \cot (72 \times 0.65)^\circ = 0.75 \cot 46.8^\circ = 0.704$
0.67	0.67	$0.75 \cot (72 \times 0.67)^\circ = 0.75 \cot 48.24^\circ = 0.669 \simeq 0.67$

From above it is clear that, when $y_{max} = 0.67$, then L.H.S. = R.H.S.

$$\therefore \quad y_{max} = 0.67 \text{ m}$$

Substituting this value in equation (*i*), we get

Width, $B = 2 \times y_{max} = 2 \times 0.67 =$ **1.34 m. Ans.**

(*iii*) *Equation of profile of the Rankine body*

The equation of profile of the Rankine body is given by equation (5.62) as

$$r = \frac{q}{2\pi} \frac{(\theta_2 - \theta_1)}{U \sin\theta} = \frac{15}{2\pi} \frac{(\theta_2 - \theta_1)}{6 \times \sin\theta} = \frac{\mathbf{0.398}\,(\boldsymbol{\theta_2 - \theta_1})}{\mathbf{sin}\,\boldsymbol{\theta}}.\ \textbf{Ans.}$$

5.17.5 A Doublet in a Uniform Flow (Flow Past a Circular Cylinder). Fig. 5.53 (*a*) shows a uniform flow of velocity U in the positive x-direction and Fig. 5.53 (*b*) shows a doublet at the origin. Doublet is a special case of a source and a sink combination in which both of equal strength approach each other such that distance between them tends to be zero. When the uniform flow is flowing over the doublet, a resultant flow will be obtained as shown in Fig. 5.53 (*c*). This resultant flow is **known as the flow past a Rankine oval of equal axes or flow past a circular cylinder.**

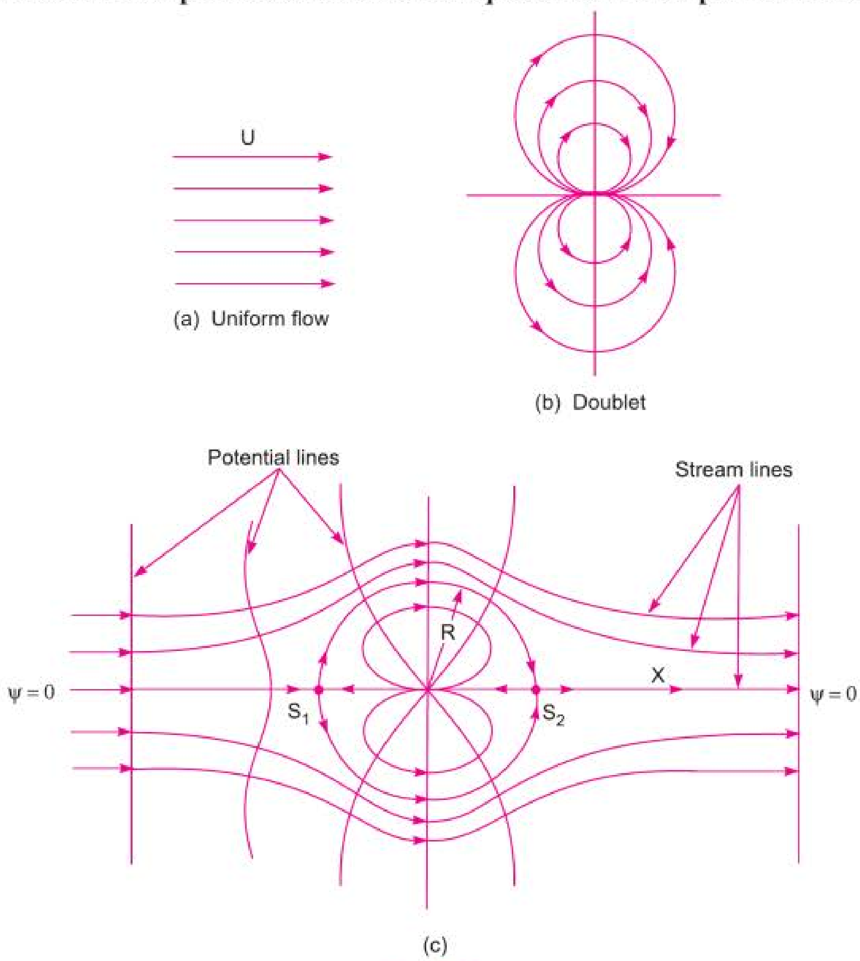


Fig. 5.53

The stream function (ψ) and velocity potential function (ϕ) for the resultant flow is obtained as given below :

$$\psi = \text{stream function due to uniform flow} + \text{stream function due to doublet}$$

$$= U \times y + \left(\frac{-\mu}{2\pi r}\sin\theta\right)$$

$$\left[\text{Stream function due to doublet is given by equation (5.50) as} = -\frac{\mu}{2\pi r}\sin\theta\right]$$

$$= U \times r \times \sin\theta - \frac{\mu}{2\pi r}\sin\theta \qquad (\because \quad y = r\sin\theta)$$

$$= \left(U \times r - \frac{\mu}{2\pi r}\right)\sin\theta \qquad ...(5.67)$$

and ϕ = Potential function due to uniform flow + potential function due to doublet

$$= U \times x + \frac{\mu}{2\pi} \times \frac{\cos\theta}{r}$$

$\left[\text{From equation (5.52), potential function due to doublet} = \frac{\mu}{2\pi} \times \frac{\cos\theta}{r}\right]$

$$= U \times r\cos\theta + \frac{\mu}{2\pi} \times \frac{\cos\theta}{r} \qquad (\because x = r\cos\theta)$$

$$= \left(U \times r + \frac{\mu}{2\pi r}\right)\cos\theta \qquad ...(5.68)$$

Shape of Rankine oval of equal axes

To get the profile of the Rankine oval of equal axes, the stream line ψ is taken as zero. Hence substituting $\psi = 0$ in equation (5.67), we get

$$0 = \left(U \times r - \frac{\mu}{2\pi r}\right)\sin\theta$$

This means either $\quad \sin\theta = 0 \quad$ or $\quad U \times r - \frac{\mu}{2\pi r} = 0$

(*i*) If $\sin\theta = 0$, then $\theta = 0$ and $\pm\pi$ *i.e.*, a horizontal line through the origin of the doublet. This horizontal line is the x-axis.

(*ii*) If $\quad U \times r - \frac{\mu}{2\pi r} = 0$, then $U \times r = \frac{\mu}{2\pi r}$ or $r^2 = \frac{\mu}{2\pi U}$

or $\quad r = \sqrt{\frac{\mu}{2\pi U}}$ = a constant as μ and U are constant.

Let this constant is equal to R.

$$\therefore \quad r = \sqrt{\frac{\mu}{2\pi U}} = R$$

This gives that the closed body profile is a circular cylinder of radius R with centre on doublet. The dividing stream line corresponds to $\psi = 0$. This stream line is a circle of radius R. The stream line $\psi = 0$ has two stagnation points S_1 and S_2. At S_1, the uniform flow splits into two streams that flow along the circle with radius $R = \sqrt{\frac{\mu}{2\pi U}}$, the two branches meet again at the stagnation point S_2 and the flow continues in the downward direction. The uniform flow occurs outside the circle whereas the flow field due to doublet lies entirely within the circle. The stream function for the composite flow is given by equation (5.67) as

$$\psi = \left(U \times r - \frac{\mu}{2\pi r}\right) \sin\theta = U\left(r - \frac{\mu}{2\pi U r}\right) \sin\theta$$

$$= U\left(r - \frac{R^2}{r}\right) \sin\theta \qquad \left(\because \frac{\mu}{2\pi U} = R^2\right) \quad ...(5.69)$$

Velocity Components (u_r and u_θ)

The velocity components at any point in the flow field are given by,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[U\left(r - \frac{R^2}{r}\right)\sin\theta\right] = \frac{1}{r} U\left(r - \frac{R^2}{r}\right) \cos\theta$$

$$= U\left(1 - \frac{R^2}{r^2}\right) \cos\theta \quad ...(5.70)$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[U\left(r - \frac{R^2}{r}\right)\sin\theta\right] = -U\left(1 + \frac{R^2}{r^2}\right) \sin\theta$$

$$= -U\left(1 + \frac{R^2}{r^2}\right) \sin\theta \quad ...(5.71)$$

$\therefore$ Resultant velocity, $\quad V = \sqrt{u_r^2 + u_\theta^2} \quad ...(5.72)$

On the surface of the cylinder, $r = R$

$$u_r = U\left[1 - \frac{R^2}{R^2}\right] \cos\theta \qquad [\because \text{ In equation (5.70), } r = R]$$

$$= 0$$

and

$$u_\theta = -U\left[1 + \frac{R^2}{R^2}\right] \sin\theta = -2U\sin\theta \quad ...(5.73)$$

–ve sign shows the clockwise direction of tangential velocity at that point. The value of u_θ is maximum, when $\theta = 90°$ and $270°$.

At $\theta = 0°$ or $180°$, the value of $u_\theta = 0$. Hence on the surface of the cylinder, the resultant velocity is zero, when $\theta = 0°$ or $180°$. These two points on the surface of cylinder [*i.e.,* at $\theta = 0°$ and $180°$] where resultant velocity is zero, are known as stagnation points. They are denoted by S_1 and S_2. Stagnation point S_1 corresponds to $\theta = 180°$ and S_2 corresponds to $\theta = 0°$.

Pressure distribution on the surface of the cylinder

Let p_0 = pressure at a point in the uniform flow far away from the cylinder and towards the left of the cylinder [*i.e.,* approaching uniform flow]

U = velocity of uniform flow at that point

p = pressure at a point on the surface of the cylinder

V = resultant velocity at that point on the surface of the cylinder. This velocity is equal to u_θ as u_r is zero on the surface of the cylinder.

$\therefore$ $$V = u_\theta = -2U \sin\theta$$

Applying Bernoulli's equation at the above two points,

$$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g}$$

or $$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{[-2U\sin\theta]^2}{2g} \qquad [\because \quad V = u_\theta = -2U \sin\theta]$$

or $$\frac{p_0}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{4U^2\sin^2\theta}{2}$$

or $$\frac{p - p_0}{\rho} = \frac{U^2}{2} - \frac{4U^2\sin^2\theta}{2} = \frac{1}{2}U^2(1 - 4\sin^2\theta)$$

or $$\frac{p - p_0}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta$$

But $\dfrac{p - p_0}{\frac{1}{2}\rho U^2}$ is a dimensionless term and is known as dimensionless pressure co-efficient and is denoted by C_p.

$\therefore$ $$C_p = \frac{p - p_0}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta$$

Value of pressure co-efficient for different values of θ

Value of θ	*Value of C_p*
0	$1 - 4\sin^2\theta = 1 - 0 = 1$
30°	$1 - 4\sin^2 30° = 1 - 4 \times \left(\frac{1}{2}\right)^2 = 1 - \frac{4}{4} = 1 - 1 = 0$
90°	$1 - 4\sin^2 90° = 1 - 4 \times 1 = 1 - 4 = -3$
150°	$1 - 4\sin^2 150° = 1 - 4 \times \frac{1}{4} = 1 - 1 = 0$
180°	$1 - 4\sin^2 180° = 1 - 0 = 1$

At $\theta = 0$ and 180°, there are stagnation points S_2 and S_1 respectively.

At $\theta = 30°$ and 150°, the pressure co-efficient is zero.

At $\theta = 90°$, the pressure co-efficient is -3 (*i.e.*, least pressure)

The variation of pressure co-efficient along the surface of the cylinder for different values of θ are shown in Fig. 5.54.

The positive pressure is acting normal to the surface and towards the surface of the cylinder whereas the negative pressure is acting normal to the surface and away from the surface of the cylinder as shown in Fig. 5.55.

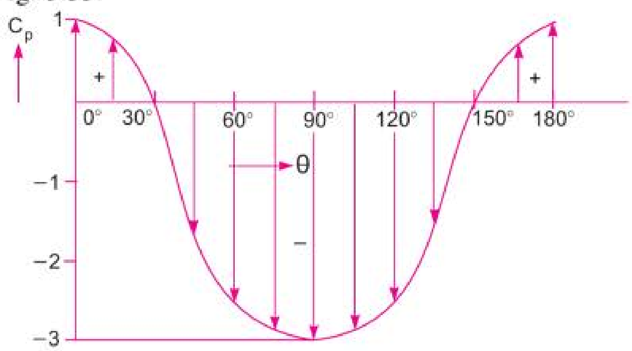


Fig. 5.54

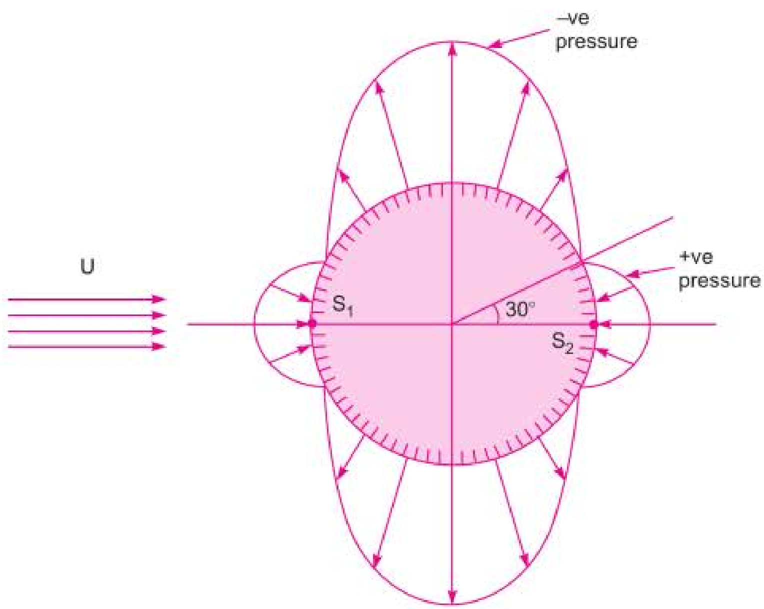


Fig. 5.55

Problem 5.43 *A uniform flow of 12 m/s is flowing over a doublet of strength 18 m²/s. The doublet is in the line of the uniform flow. Determine :*

(i) shape of the Rankine oval *(ii) radius of the Rankine circle*

(iii) value of stream line function at Rankine circle

(iv) resultant velocity at a point on the Rankine circle at an angle of 30° from x-axis

(v) value of maximum velocity on the Rankine circle and location of the point where velocity is max.

Solution. Given : $U = 12$ m/s ; $\mu = 18$ m²/s

(i) Shape of the Rankine oval

When a uniform flow is flowing over a doublet and doublet and uniform flow are in line, then the shape of the Rankine oval will be a circle of radius $= \sqrt{\dfrac{\mu}{2\pi U}}$. **Ans.**

(*ii*) *Radius of the Rankine circle*

$$R = r = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{18}{2\pi \times 12}} = \mathbf{0.488\ m.\ Ans.}$$

(*iii*) *Value of stream line function at the Rankine circle*

The value of stream line function (ψ) at the Rankine circle is zero *i.e.*, $\psi = 0$.

(*iv*) *Resultant velocity on the surface of the circle, when* $\theta = 30°$

On the surface of the cylinder, the radial velocity (u_r) is zero. The tangential velocity (u_θ) is given by equation (5.73) as

$$u_\theta = -2U \sin\theta = -2 \times 12 \times \sin 30° = \mathbf{-12\ m/s.\ Ans.}$$

–ve sign shows the clockwise direction of tangential velocity at that point.

$\therefore$ Resultant velocity, $\quad V = \sqrt{u_r^2 + u_\theta^2} = \sqrt{0^2 + (-12)^2} = \mathbf{12\ m/s.\ Ans.}$

(*v*) *Maximum velocity and its location*

The resultant velocity at any point on the surface of the cylinder is equal to u_θ. But u_θ is given by,

$$u_\theta = -2U \sin\theta$$

This velocity will be maximum, when $\theta = 90°$.

$\therefore$ Max. velocity $\quad = -2U = -2 \times 12 = \mathbf{-24\ m/s.\ Ans.}$

Problem 5.44 *A uniform flow of 10 m/s is flowing over a doublet of strength 15 m²/s. The doublet is in the line of the uniform flow. The polar co-ordinates of a point P in the flow field are 0.9 m and 30°. Find : (i) stream line function and (ii) the resultant velocity at the point.*

Solution. Given : $U = 10$ m/s ; $\mu = 15$ m²/s ; $r = 0.9$ m and $\theta = 30°$.

Let us first find the radius (R) of the Rankine circle. This is given by

$$R = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{15}{2\pi \times 10}} = 0.488 \text{ m}$$

The polar co-ordinates of the point P are 0.9 m and 30°.

Hence $\quad r = 0.9$ m and $\theta = 30°$.

As the value of r is more than the radius of the Rankine circle, hence point P lies outside the cylinder.

(*i*) *Value of stream line function at the point P*

The stream line function for the composite flow at any point is given by equation (5.69) as

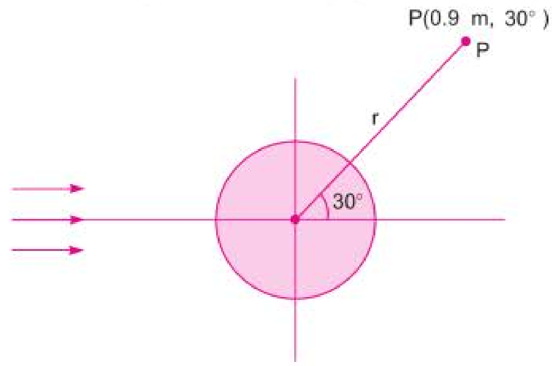


Fig. 5.56

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta$$

$$= 10\left(0.9 - \frac{0.488^2}{0.9}\right)\sin 30^\circ (\because\ r = 0.9 \text{ m},\ R = 0.488 \text{ and } \theta = 30^\circ)$$

$$= 10(0.9 - 0.2646) \times \frac{1}{2} = \mathbf{3.177\ m^2/s.}\ \textbf{Ans.}$$

(*ii*) *Resultant velocity at the point P*

The radial velocity and tangential velocity at any point in the flow field are given by equations (5.70) and (5.71) respectively.

$\therefore$
$$u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta = 10\left(1 - \frac{0.488^2}{0.9^2}\right)\cos 30^\circ = 611 \text{ m/s}$$

+ve sign shows the radial velocity is outward.

and
$$u_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta = -10\left(1 + \frac{0.488^2}{0.9^2}\right)\sin 30^\circ = -6.47 \text{ m/s}$$

–ve sign shows the clockwise direction of tangential velocity.

$\therefore$ Resultant velocity,

$$V = \sqrt{u_r^2 + u_\theta^2}$$

$$= \sqrt{6.11^2 + (-6.47)^2} = \sqrt{37.33 + 44.86}$$

$$= \mathbf{8.89\ m/s.}\ \textbf{Ans.}$$

HIGHLIGHTS

1. If the fluid characteristics like velocity, pressure, density etc. do not change at a point with respect to time, the fluid flow is called steady flow. If they change w.r.t. time, the fluid flow is called unsteady flow.

   Or $\left(\frac{\partial v}{\partial t}\right) = 0$ for steady flow and $\left(\frac{\partial v}{\partial t}\right) \neq 0$ for unsteady flow.

2. If the velocity in a fluid flow does not change with respect to space (length of direction of flow), the flow is said uniform otherwise non-uniform. Thus,

   $\left(\frac{\partial v}{\partial s}\right) = 0$ for uniform flow and $\left(\frac{\partial v}{\partial s}\right) \neq 0$ for non-uniform flow.

3. If the Reynolds number in a pipe is less than 2000, the flow is said to be laminar and if Reynold number is more than 4000, the flow is said to be turbulent.
4. For compressible flow, $\rho \neq$ constant
   For incompressible flow, $\rho =$ constant.
5. Rate of discharge for incompressible fluid (liquid), $Q = A \times v$.
6. Continuity equation is written as $A_1v_1 = A_2v_2 = A_3v_3$.

7. Continuity equation in differential form,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ for three-dimensional flow}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \text{ for two-dimensional flow.}$$

8. The components of acceleration in x, y and z direction are

$$a_x = u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}.$$

9. The components of velocity in x, y and z direction in terms of velocity potential (ϕ) are

$$u=-\frac{\partial \phi}{\partial x},\ v=-\frac{\partial \phi}{\partial y} \text{ and } w=-\frac{\partial \phi}{\partial z}.$$

10. The stream function (ψ) is defined only for two-dimensional flow. The velocity components in x and y directions in terms of stream function are $u=-\dfrac{\partial \psi}{\partial y}$ and $v=\dfrac{\partial \psi}{\partial x}$.

11. Angular deformation or shear strain rate is given as

$$\text{Shear strain rate} = \frac{1}{2}\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]$$

12. Rotational components of a fluid particle are

$$\omega_z=\frac{1}{2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right];\ \omega_x=\frac{1}{2}\left[\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right];\ \omega_y=\frac{1}{2}\left[\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right]$$

13. Vorticity is two times the value of rotation.
14. Flow of a fluid along a curved path is known as vortex flow. If the particles are moving round in curved path with the help of some external torque the flow is called forced vortex flow. And if no external torque is required to rotate the fluid particles, the flow is called free-vortex flow.
15. The relation between tangential velocity and radius :
    for forced vortex, $v=\omega\times r$,
    for free vortex, $v\times r=$ constant.
16. The pressure variation along the radial direction for vortex flow along a horizontal plane, $\dfrac{\partial p}{\partial r}=\rho\dfrac{v^2}{r}$ and pressure variation in the vertical plane $\dfrac{\partial p}{\partial z}=-\rho g$.
17. For the forced vortex flow, $Z=\dfrac{v^2}{2g}=\dfrac{\omega^2 r^2}{2g}=\dfrac{\omega^2 R^2}{2g}$

where Z = height of paraboloid formed
ω = angular velocity.

18. For a forced vortex flow in a open tank.
Fall of liquid level at centre = Rise of liquid level at the ends.

19. In case of closed cylinder, the volume of air before rotation is equal to the volume of air after rotation.

20. If a close cylindrical vessel completely filled with water is rotated about its vertical axis, the total pressure forces acting on the top and bottom are

$$F_T = \frac{\rho}{4}\,\omega^2 \pi R^4$$

and $$F_B = F_T + \text{weight of water in cylinder}$$

where F_T = Pressure force on top of cylinder
F_B = Pressure force on the bottom of cylinder
ω = Angular velocity
R = Radius of the vessel
ρ = Density of fluid = $\dfrac{w}{g}$.

21. For a free vortex flow the equation is $\dfrac{p_1}{\rho g} + \dfrac{v_1^2}{2g} + z_1 = \dfrac{p_2}{\rho g} + \dfrac{v_2^2}{2g} + z_2$.

EXERCISE

(A) THEORETICAL PROBLEMS

1. What are the methods of describing fluid flow ?
2. Explain the terms :
   (*i*) Path line, (*ii*) Streak line,
   (*iii*) Stream line, and (*iv*) Stream tube.
3. Distinguish between :
   (*i*) Steady flow and un-steady flow, (*ii*) Uniform and non-uniform flow,
   (*iii*) Compressible and incompressible flow,
   (*iv*) Rotational and irrotational flow, (*v*) Laminar and turbulent flow.
4. Define the following and give one practical example for each :
   (*i*) Laminar flow, (*ii*) Turbulent flow,
   (*iii*) Steady flow, and (*iv*) Uniform flow.
5. Define the equation of continuity. Obtain an expression for continuity equation for a three-dimensional flow. *(R.G.P.V, S 2002)*
6. What do you understand by the terms : (*i*) Total acceleration, (*ii*) Convective acceleration, and (*iii*) Local acceleration ? *(Delhi University, Dec. 2002)*
7. (*a*) Define the terms :
   (*i*) Velocity potential function, and (*ii*) Stream function.
   (*b*) What are the conditions for flow to be irrotational ?
8. What do you mean by equipotential line and a line of constant stream function ?
9. (*a*) Describe the use and limitations of the flow nets.
   (*b*) Under what conditions can one draw flow net ?
10. Define the terms :
    (*i*) Vortex flow, (*ii*) Forced vortex flow, and (*iii*) Free vortex flow.
11. Differentiate between forced vortex and free vortex flow.

12. Derive an expression for the depth of paraboloid formed by the surface of a liquid contained in a cylindrical tank which is rotated at a constant angular velocity ω about its vertical axis.
13. Derive an expression for the difference of pressure between two points in a free vortex flow. Does the difference of pressure satisfy Bernoulli's equation ? Can Bernoulli's equation be applied to a forced vortex flow ?
14. Derive, from first principles, the condition for irrotational flow. Prove that, for potential flow, both the stream function and velocity potential function satisfy the Laplace equation.
15. Define velocity potential function and stream function.
16. Under what conditions can one treat real fluid flow as irrotational (as an approximation).
17. Define the following :
    (*i*) Steady flow, (*ii*) Non-uniform flow,
    (*iii*) Laminar flow, and (*iv*) Two-dimensional flow.
18. (*a*) Distinguish between rotational flow and irrotational flow. Give one example of each
    (*b*) Cite two examples of unsteady, non-uniform flow. How can the unsteady flow be transformed to steady flow ? *(J.N.T. University, S 2002)*
19. Explain uniform flow with source and sink. Obtain expressions for stream and velocity potential functions.
20. A point source is a point where an incompressible fluid is imagined to be created and sent out evenly in all directions. Determine its velocity potential and stream function.
21. (*i*) Explain doublet and define the strength of the doublet
    (*ii*) Distinguish between a source and a sink.
22. Sketch the flow pattern of an ideal fluid flow past a cylinder with circulation.
23. Show that in case of forced vortex flow, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.
24. Differentiate between :
    (*i*) Stream function and velocity potential function
    (*ii*) Stream line and streak line and
    (*iii*) Rotational and irrotational flows.

(B) NUMERICAL PROBLEMS

1. The diameters of a pipe at the sections 1 and 2 are 15 cm and 20 cm respectively. Find the discharge through the pipe if velocity of water at section 1 is 4 m/s. Determine also the velocity at section 2. [**Ans.** 0.07068 m^3/s, 2.25 m/s]
2. A 40 cm diameter pipe, conveying water, branches into two pipes of diameters 30 cm and 20 cm respectively. If the average velocity in the 40 cm diameter pipe is 3 m/s. Find the discharge in this pipe. Also determine the velocity in 20 cm pipe if the average velocity in 30 cm diameter pipe is 2 m/s. [**Ans.** 0.3769 m^3/s, 7.5 m/s]
3. A 30 cm diameter pipe carries oil of sp. gr. 0.8 at a velocity of 2 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil. [**Ans.** 4.5 m/s, 113 kg/s]
4. The velocity vector in a fluid flow is given by $V = 2x^3\mathbf{i} - 5x^2y\mathbf{j} + 4t\mathbf{k}$.
   Find the velocity and acceleration of a fluid particle at (1, 2, 3) at time, $t = 1$. [**Ans.** 10.95 units, 16.12 units]
5. The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :

(*i*) $u = 4x^2, v = 4xyz$ (*ii*) $u = 4x^2 + 3xy, w = z^3 - 4xy - 2yz.$

$$\left[\textbf{Ans.}\ (i)\ w = -8xz - 2xz^2 + f(x, y) \quad (ii)\ v = -8xy - \frac{y^2}{2} + 3yz^2 + f(x, z)\right]$$

Calculate the unknown velocity components so that they satisfy the following equations :
(*i*) $u = 2x^2, v = 2xyz, w = ?$ (*ii*) $u = 2x^2 + 2xy, w = z^3 - 4xz + 2yz, v = ?$ [**Ans.** (*i*) $w = -1xz - x^2z$]

6. A fluid flow is given by : $V = xy^2i - 2yz^2j - \left(zy^2 - \frac{2z^3}{3}\right)k.$

Prove that it is a case of possible steady incompressible fluid flow.
Calculate the velocity and acceleration at the point [1, 2, 3]. [**Ans.** 36.7 units, 874.50 units]

7. Find the convective acceleration at the middle of a pipe which converges uniformly from 0.6 m diameter to 0.3 m diameter over 3 m length. The rate of flow is 40 lit/s. If the rate of flow changes uniformly from 40 lit/s to 80 lit/s in 40 seconds, find the total acceleration at the middle of the pipe at 20th second. [**Ans.** .0499 m/s^2 ; .11874 m/s^2]

8. The velocity potential function, ϕ, is given by $\phi = x^2 - y^2$. Find the velocity components in x and y direction. Also show that ϕ represents a possible case of fluid flow. [**Ans.** $u = 2x$ and $v = -2y$]

9. For the velocity potential function, $\phi = x^2 - y^2$, find the velocity components at the point (4, 5). [**Ans.** $u = 8$, $v = -10$ units]

10. A stream function is given by : $\psi = 2x - 5y$. Calculate the velocity components and also magnitude and direction of the resultant velocity at any point. [**Ans.** $u = 5$, $v = 2$, Resultant = 5.384 and $\theta = 21° 48'$]

11. If for a two-dimensional potential flow, the velocity potential is given by : $\phi = 4x(3y - 4)$, determine the velocity at the point (2, 3). Determine also the value of stream function ψ at the point (2, 3).

$$\left[\textbf{Ans.}\ 40 \text{ units}, \psi = 6x^2 - 4\left(\frac{3}{2}y^2 - 4y\right), -18\right]$$

12. The stream function for a two-dimensional flow is given by $\psi = 8xy$, calculate the velocity at the point $p(4, 5)$. Find the velocity potential function ϕ. [**Ans.** $u = -32$ units, $v = 40$ units, $\phi = 4y^2 - 4x^2$]

13. Sketch the stream lines represented by $\psi = xy$. Also find out the velocity and its direction at point (2, 3). [**Ans.** 3.60 units and $\theta = 56° 18.6'$ or $123° 42'$]

14. For the velocity components given as : $u = ay \sin xy$, $v = ax \sin xy$.
Obtain an expression for the velocity potential function. [**Ans.** $\phi = a \cos xy$]

15. A fluid flow is given by : $V = 10x^3i - 8x^3yj$.
Find the shear strain rate and state whether the flow is rotational or irrotational. [**Ans.** $-8xy$, rotational]

16. The velocity components in a two-dimensional flow are :

$$u = 8x^2y - \frac{8}{3}y^3 \text{ and } v = -8xy^3 + \frac{8}{3}x^3.$$

Show that these velocity components represent a possible case of an irrotational flow.

$$\left[\textbf{Ans.}\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\ \omega_z = 0\right]$$

17. An open circular cylinder of 20 cm diameter and 100 cm long contains water upto a height of 80 cm. It is rotated about its vertical axis. Find the speed of rotation when :
(*i*) no water spills, (*ii*) axial depth is zero. [**Ans.** (*i*) 267.51 r.p.m., (*ii*) 422.98 r.p.m.]

18. A cylindrical vessel 15 cm in diameter and 40 cm long is completely filled with water. The vessel is open at the top. Find the quantity of water left in the vessel, when it is rotated about its vertical axis with a speed of 300 r.p.m. [**Ans.** 4566.3 cm^2]

19. An open circular cylinder of 20 cm diameter and 120 cm long contains water upto a height of 80 cm. It is rotated about its vertical axis at 400 r.p.m. Find the difference in total pressure force (*i*) at the bottom of the cylinder, and (*ii*) at the sides of the cylinder due to rotation. [**Ans.** (*i*) 14.52 N, (*ii*) 2465.45 N]

20. A closed cylindrical vessel of diameter 15 cm and length 100 cm contains water upto a height of 80 cm. The vessel is rotated at a speed of 500 r.p.m. about its vertical axis. Find the height of paraboloid formed. [**Ans.** 56.06 cm]

21. For the data given in question 20, find the speed of rotation of the vessel, when axial depth is zero. [**Ans.** 891.7 r.p.m.]

22. If the cylindrical vessel of question 20, is rotated at 950 r.p.m. about its vertical axis, find the area uncovered at the base of the tank. [**Ans.** 20.4 cm^2]

23. A closed cylindrical vessel of diameter 20 cm and height 100 cm contains water upto a height of 70 cm. The air above the water surface is at a pressure of 78.48 kN/m^2. The vessel is rotated at a speed of 300 r.p.m. about its vertical axis. Find the pressure head at the bottom of the vessel ; (*a*) at the centre, and (*b*) at the edge. [**Ans.** (*a*) 8.4485 m (*b*) 8.9515 m]

24. A closed cylinder of diameter 30 cm and height 20 cm is completely filled with water. Calculate the total pressure force exerted by water on the top and bottom of the cylinder, if it is rotated about its vertical axis at 300 r.p.m. [**Ans.** $F_T = 392.4$ N, $F_B = 531$ N]

25. In a free cylindrical vortex flow of water, at a point at a radius of 150 mm the velocity and pressure are 5 m/s and 14.715 N/cm^2. Find the pressure at a radius of 300 mm. [**Ans.** 15.65 N/cm^2]

26. Do the following velocity components represent physically possible flows ?

$$u = x^2 + z^2 + 5,\ v = y^2 + z^2,\ w = 4xyz.$$ [**Ans.** No.]

27. State if the flow represented by $u = 3x + 4y$ and $v = 2x - 3y$ is rotational or irrotational. [**Ans.** Rotational]

28. A vessel, cylindrical in shape and closed at the top and bottom, contains water upto a height of 700 mm. The diameter of the vessel is 200 mm and length of vessel is 1.1 m. Find the speed of rotation of the vessel if the axial depth of water is zero.

29. Define rotational and irrotational flow. The stream function and velocity potential for a flow are given by :

$$\psi = 2xy,\ \phi = x^2 - y^2.$$

Show that the conditions of continuity and irrotational flow are satisfied.

30. For the steady incompressible flow, are the following values of u and v possible ?

(*i*) $u = 4xy + y^2$, $v = 6xy + 3x$ and (*ii*) $u = 2x^2 + y^2$, $v = -4xy$. [**Ans.** (*i*) No, (*ii*) Yes]

31. Define two-dimensional stream function and velocity potential. Show that following stream function :

$$\psi = 6x - 4y + 7xy + 9$$

represents an irrotational flow. Find its velocity potential. [**Ans.** $\phi = 4x + 6y - 3.5x^2 + 3.5y^2 + C$]

32. Check if $\phi = x^2 - y^2 + y$ represents the velocity potential for 2-dimensional irrotational flow. If it does, then determine the stream function ψ. [**Ans.** Yes, $\psi = -2xy + x$]

33. If stream function for steady flow is given by $\psi = (y^2 - x^2)$, determine whether the flow is rotational or irrotational. Then determine the velocity potential ϕ. [**Ans.** Irrotational, $\phi = -2xy + C$]

34. A pipe (1) 450 mm in diameter branches into two pipes (2) and (3) of diameters 300 mm and 200 mm respectively as shown in Fig. 5.57. If the average velocity in 450 mm diameter pipe is 3 m/s, find : (*i*) discharge through 450 mm dia. pipe and (*ii*) velocity in 200 mm diameter pipe if the average velocity in 300 mm pipe is 2.5 m/s. *(J.N.T.U., Hyderabad, S 2002)*

[**Hint.** Given : $d_1 = 450$ mm $= 0.45$ m, $d_2 = 300$ mm $= 0.3$ m
$d_3 = 200$ mm $= 0.2$ m, $V_1 = 3$ m/s, $V_2 = 2.5$ m/s

(*i*) $$Q_1 = A_1V_1 = \frac{\pi}{4}(0.45^2) \times 3 = \mathbf{0.477\ m^3/s.}$$

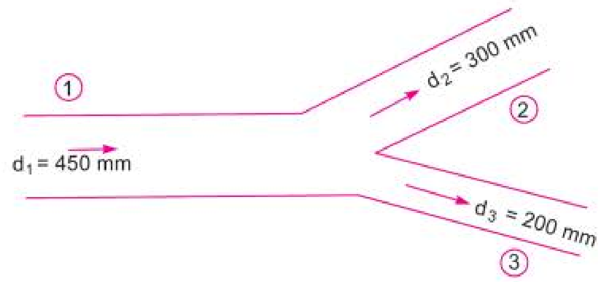


Fig. 5.57

(*ii*) $$Q_2 = A_2V_2 = \frac{\pi}{4}(.3^2) \times 2.5 = 0.176\ \text{m}^3/\text{s}$$

But $$Q_1 = Q_2 + Q_3 \quad \therefore \quad Q_3 = Q_1 - Q_2 = 0.477 - 0.176 = 0.301$$

Also $$Q_3 = A_3 \times V_3 = \frac{\pi}{4}(0.2^2) \times V_3$$

$\therefore$ $$V_3 = \frac{Q_3}{\frac{\pi}{4}(0.2^2)} = \frac{0.301}{0.0314} = \mathbf{9.6\ m/s.]}$$

6

CHAPTER

DYNAMICS OF FLUID FLOW

▶ 6.1 INTRODUCTION

In the previous chapter, we studied the velocity and acceleration at a point in a fluid flow, without taking into consideration the forces causing the flow. This chapter includes the study of forces causing fluid flow. Thus dynamics of fluid flow is the study of fluid motion with the forces causing flow. The dynamic behaviour of the fluid flow is analysed by the Newton's second law of motion, which relates the acceleration with the forces. The fluid is assumed to be incompressible and non-viscous.

▶ 6.2 EQUATIONS OF MOTION

According to Newton's second law of motion, the net force F_x acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a_x in the x-direction. Thus mathematically,

$$F_x = m.a_x \quad \text{...(6.1)}$$

In the fluid flow, the following forces are present :

(*i*) F_g, gravity force.

(*ii*) F_p, the pressure force.

(*iii*) F_v, force due to viscosity.

(*iv*) F_t, force due to turbulence.

(*v*) F_c, force due to compressibility.

Thus in equation (6.1), the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x.$$

(*i*) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion.**

(*ii*) For flow, where (F_t) is negligible, the resulting equations of motion are known as **Navier-Stokes Equation.**

(*iii*) If the flow is assumed to be ideal, viscous force (F_v) is zero and equation of motions are known as **Euler's equation of motion.**

6.3 EULER'S EQUATION OF MOTION

This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as :

Consider a stream-line in which flow is taking place in s-direction as shown in Fig. 6.1. Consider a cylindrical element of cross-section dA and length ds. The forces acting on the cylindrical element are:

1. Pressure force pdA in the direction of flow.
2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow.
3. Weight of element $\rho g dA ds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element × acceleration in the direction s.

$$\therefore \quad pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos\theta$$

$$= \rho dA ds \times a_s \qquad ...(6.2)$$

where a_s is the acceleration in the direction of s.

Now $\quad a_s = \dfrac{dv}{dt}$, where v is a function of s and t.

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\dfrac{\partial v}{\partial t} = 0$

$$\therefore \quad a_s = \frac{v\partial v}{\partial s}$$

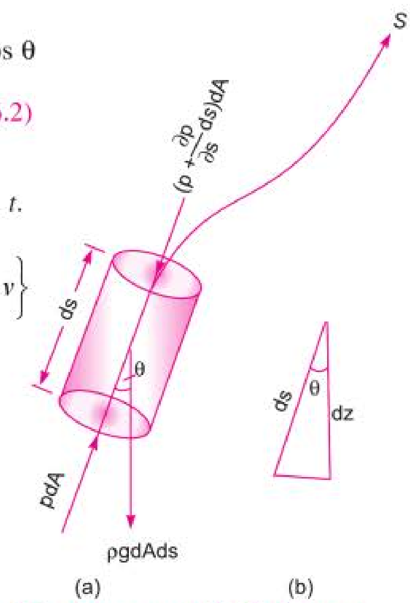


Fig. 6.1 *Forces on a fluid element.*

Substituting the value of a_s in equation (6.2) and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} ds dA - \rho g\, dA ds \cos\theta = \rho dA ds \times \frac{\partial v}{\partial s}$$

Dividing by $\quad \rho ds dA, \; -\dfrac{\partial p}{\rho \partial s} - g\cos\theta = \dfrac{v\partial v}{\partial s}$

or
$$\frac{\partial p}{\rho \partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

But from Fig. 6.1 (b), we have $\cos\theta = \dfrac{dz}{ds}$

$$\therefore \quad \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0 \quad \text{or} \quad \frac{dp}{\rho} + gdz + vdv = 0$$

or
$$\frac{dp}{\rho} + gdz + vdv = 0 \qquad ...(6.3)$$

Equation (6.3) is known as Euler's equation of motion.

▶ 6.4 BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion (6.3) as

$$\int \frac{dp}{\rho} + \int g dz + \int v dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\therefore \quad \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

or
$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

or
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant} \qquad \text{...(6.4)}$$

Equation (6.4) is a Bernoulli's equation in which

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

$v^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.

▶ 6.5 ASSUMPTIONS

The following are the assumptions made in the derivation of Bernoulli's equation :

(*i*) The fluid is ideal, *i.e.*, viscosity is zero (*ii*) The flow is steady

(*iii*) The flow is incompressible (*iv*) The flow is irrotational.

Problem 6.1 *Water is flowing through a pipe of 5 cm diameter under a pressure of 29.43 N/cm² (gauge) and with mean velocity of 2.0 m/s. Find the total head or total energy per unit weight of the water at a cross-section, which is 5 m above the datum line.*

Solution. Given :

Diameter of pipe = 5 cm = 0.5 m

Pressure, $p = 29.43 \text{ N/cm}^2 = 29.43 \times 10^4 \text{ N/m}^2$

Velocity, $v = 2.0$ m/s

Datum head, $z = 5$ m

Total head = pressure head + kinetic head + datum head

Pressure head $= \frac{p}{\rho g} = \frac{29.43 \times 10^4}{1000 \times 9.81} = 30 \text{ m}$ $\quad \left\{ \rho \text{ for water} = 1000 \frac{\text{kg}}{\text{m}^3} \right\}$

Kinetic head $= \frac{v^2}{2g} = \frac{2 \times 2}{2 \times 9.81} = 0.204 \text{ m}$

$\therefore$ Total head $= \frac{p}{\rho g} + \frac{v^2}{2g} + z = 30 + 0.204 + 5 =$ **35.204 m. Ans.**

Problem 6.2 *A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.*

Solution. Given :

$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

∴ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$

$$V_1 = 4.0 \text{ m/s}$$

$$D_2 = 0.1 \text{ m}$$

∴ $A_2 = \frac{\pi}{4} (.1)^2 = .00785 \text{ m}^2$

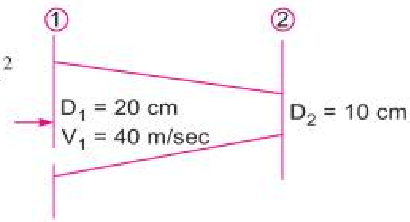


Fig. 6.2

(*i*) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = \textbf{0.815 m. Ans.}$$

(*ii*) Velocity head at section 2 = $V_2^2/2g$

To find V_2, apply continuity equation at 1 and 2

∴ $A_1V_1 = A_2V_2$ or $V_2 = \frac{A_1V_1}{A_2} = \frac{.0314}{.00785} \times 4.0 = 16.0 \text{ m/s}$

∴ Velocity head at section 2 = $\frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81} = \textbf{83.047 m. Ans.}$

(*iii*) Rate of discharge $= A_1V_1$ or A_2V_2

$= 0.0314 \times 4.0 = 0.1256 \text{ m}^3\text{/s}$

$=$ **125.6 litres/s. Ans.** $\{\because 1 \text{ m}^3 = 1000 \text{ litres}\}$

Problem 6.3 *State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's equation from first principle and state the assumptions made for such a derivation.*

Solution. Statement of Bernoulli's Theorem. It states that in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant. The total energy consists of pressure energy, kinetic energy and potential energy or datum energy. These energies per unit weight of the fluid are :

Pressure energy = $\frac{p}{\rho g}$

Kinetic energy = $\frac{v^2}{2g}$

Datum energy = z

Thus mathematically, Bernoulli's theorem is written as

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{Constant.}$$

Derivation of Bernoulli's theorem. For derivation of Bernoulli's theorem, Articles 6.3 and 6.4 should be written.

Assumptions are given in Article 6.5.

Problem 6.4 *The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 litres/s. The section 1 is 6 m above datum and section 2 is 4 m above datum. If the pressure at section 1 is 39.24 N/cm², find the intensity of pressure at section 2.*

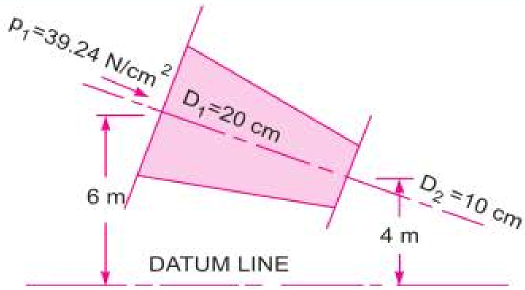


Fig. 6.3

Solution. Given :

At section 1, $D_1 = 20 \text{ cm} = 0.2 \text{ m}$

$$A_1 = \frac{\pi}{4}(.2)^2 = .0314 \text{ m}^2$$

$$p_1 = 39.24 \text{ N/cm}^2 = 39.24 \times 10^4 \text{ N/m}^2$$

$$z_1 = 6.0 \text{ m}$$

At section 2, $D_2 = 0.10 \text{ m}$

$$A_2 = \frac{\pi}{4}(0.1)^2 = .00785 \text{ m}^2$$

$$z_2 = 4 \text{ m}$$

$$p_2 = ?$$

Rate of flow, $Q = 35 \text{ lit/s} = \dfrac{35}{1000} = .035 \text{ m}^3/\text{s}$

Now $Q = A_1V_1 = A_2V_2$

$\therefore$ $$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114 \text{ m/s}$$

and $$V_2 = \frac{Q}{A_2} = \frac{.035}{.00785} = 4.456 \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or $$\frac{39.24 \times 10^4}{1000 \times 9.81} + \frac{(1.114)^2}{2 \times 9.81} + 6.0 = \frac{p_2}{1000 \times 9.81} + \frac{(4.456)^2}{2 \times 9.81} + 4.0$$

or $$40 + 0.063 + 6.0 = \frac{p_2}{9810} + 1.012 + 4.0$$

or $$46.063 = \frac{p_2}{9810} + 5.012$$

$\therefore$ $$\frac{p_2}{9810} = 46.063 - 5.012 = 41.051$$

$\therefore$ $$p_2 = 41.051 \times 9810 \text{ N/m}^2$$

$$= \frac{41.051 \times 9810}{10^4} \text{ N/cm}^2 = \mathbf{40.27 \text{ N/cm}^2}. \textbf{ Ans.}$$

Problem 6.5 *Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm² and the pressure at the upper end is 9.81 N/cm². Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.*

Solution. Given :

Section 1, $D_1 = 300 \text{ mm} = 0.3 \text{ m}$

$p_1 = 24.525 \text{ N/cm}^2 = 24.525 \times 10^4 \text{ N/m}^2$

Section 2, $D_2 = 200 \text{ mm} = 0.2 \text{ m}$

$p_2 = 9.81 \text{ N/cm}^2 = 9.81 \times 10^4 \text{ N/m}^2$

Rate of flow $= 40$ lit/s

or $$Q = \frac{40}{1000} = 0.04 \text{ m}^3/\text{s}$$

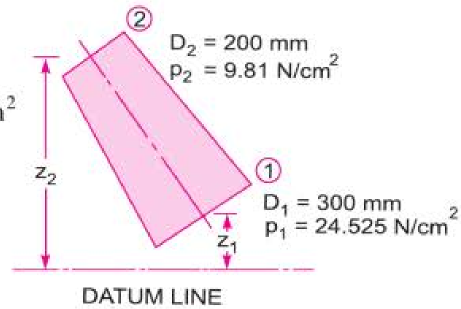


Fig. 6.4

Now $A_1V_1 = A_2V_2 =$ rate of flow $= 0.04$

$$\therefore \quad V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4}D_1^2} = \frac{0.04}{\frac{\pi}{4}(0.3)^2} = 0.5658 \text{ m/s}$$

$$\simeq 0.566 \text{ m/s}$$

$$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4}(D_2)^2} = \frac{0.04}{\frac{\pi}{4}(0.2)^2} = 1.274 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or $$\frac{24.525 \times 10^4}{1000 \times 9.81} + \frac{.566 \times .566}{2 \times 9.81} + z_1 = \frac{9.81 \times 10^4}{1000 \times 9.81} + \frac{(1.274)^2}{2 \times 9.81} + z_2$$

or $$25 + .32 + z_1 = 10 + 1.623 + z_2$$

or $$25.32 + z_1 = 11.623 + z_2$$

$$\therefore \quad z_2 - z_1 = 25.32 - 11.623 = 13.697 = 13.70 \text{ m}$$

$\therefore$ Difference in datum head $= z_2 - z_1 =$ **13.70 m. Ans.**

Problem 6.6 *The water is flowing through a taper pipe of length 100 m having diameters 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 litres/s. The pipe has a slope of 1 in 30. Find the pressure at the lower end if the pressure at the higher level is 19.62 N/cm².*

Solution. Given :

Length of pipe, $L = 100$ m

Dia. at the upper end, $D_1 = 600 \text{ mm} = 0.6 \text{ m}$

$\therefore$ Area, $$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times (.6)^2$$

$$= 0.2827 \text{ m}^2$$

p_1 = pressure at upper end

$= 19.62 \text{ N/cm}^2$

Fig. 6.5

$$= 19.62 \times 10^4 \text{ N/m}^2$$

Dia. at lower end, $D_2 = 300 \text{ mm} = 0.3 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}$

$$Q = \text{rate of flow} = 50 \text{ litres/s} = \frac{50}{1000} = 0.05 \text{ m}^3\text{/s}$$

Let the datum line passes through the centre of the lower end.

Then $z_2 = 0$

As slope is 1 in 30 means $z_1 = \frac{1}{30} \times 100 = \frac{10}{3} \text{ m}$

Also we know $Q = A_1V_1 = A_2V_2$

$\therefore$ $V_1 = \frac{Q}{A} = \frac{0.05}{.2827} = 0.1768 \text{ m/sec} = 0.177 \text{ m/s}$

and $V_2 = \frac{Q}{A_2} = \frac{0.5}{.07068} = 0.7074 \text{ m/sec} = 0.707 \text{ m/s}$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or $$\frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{.177^2}{2 \times 9.81} + \frac{10}{3} = \frac{p_2}{\rho g} + \frac{.707^2}{2 \times 9.81} + 0$$

or $$20 + 0.001596 + 3.334 = \frac{p_2}{\rho g} + 0.0254$$

or $$23.335 - 0.0254 = \frac{p_2}{1000 \times 9.81}$$

or $$p_2 = 23.3 \times 9810 \text{ N/m}^2 = 228573 \text{ N/m}^2 = \mathbf{22.857 \text{ N/cm}^2}. \textbf{ Ans.}$$

▶ 6.6 BERNOULLI'S EQUATION FOR REAL FLUID

The Bernoulli's equation was derived on the assumption that fluid is inviscid (non-viscous) and therefore frictionless. But all the real fluids are viscous and hence offer resistance to flow. Thus there are always some losses in fluid flows and hence in the application of Bernoulli's equation, these losses have to be taken into consideration. Thus the Bernoulli's equation for real fluids between points 1 and 2 is given as

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L \qquad \text{...(6.5)}$$

where h_L is loss of energy between points 1 and 2.

Problem 6.7 *A pipe of diameter 400 mm carries water at a velocity of 25 m/s. The pressures at the points A and B are given as 29.43 N/cm^2 and 22.563 N/cm^2 respectively while the datum head at A and B are 28 m and 30 m. Find the loss of head between A and B.*

Solution. Given :

Dia. of pipe, $D = 400 \text{ mm} = 0.4 \text{ m}$

Velocity, $V = 25 \text{ m/s}$

At point A, $p_A = 29.43 \text{ N/cm}^2 = 29.43 \times 10^4 \text{ N/m}^2$

$z_A = 28 \text{ m}$

$v_A = v = 25 \text{ m/s}$

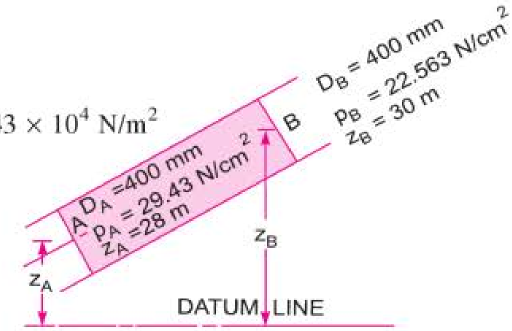


Fig. 6.6

$\therefore$ Total energy at A,
$$E_A = \frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A$$

$$= \frac{29.43 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 28$$

$$= 30 + 31.85 + 28 = 89.85 \text{ m}$$

At point B, $p_B = 22.563 \text{ N/cm}^2 = 22.563 \times 10^4 \text{ N/m}^2$

$z_B = 30 \text{ m}$

$v_B = v = v_A = 25 \text{ m/s}$

$\therefore$ Total energy at B,
$$E_B = \frac{p_B}{\rho g} + \frac{v_B^2}{2g} + z_B$$

$$= \frac{22.563 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 30 = 23 + 31.85 + 30 = 84.85 \text{ m}$$

$\therefore$ Loss of energy $= E_A - E_B = 89.85 - 84.85 =$ **5.0 m. Ans.**

Problem 6.8 *A conical tube of length 2.0 m is fixed vertically with its smaller end upwards. The velocity of flow at the smaller end is 5 m/s while at the lower end it is 2 m/s. The pressure head at the smaller end is 2.5 m of liquid. The loss of head in the tube is $\frac{0.35(v_1 - v_2)^2}{2g}$, where v_1 is the velocity at the smaller end and v_2 at the lower end respectively. Determine the pressure head at the lower end. Flow takes place in the downward direction.*

Solution. Let the smaller end is represented by (1) and lower end by (2)

Given :

Length of tube, $L = 2.0 \text{ m}$

$v_1 = 5 \text{ m/s}$

$p_1/\rho g = 2.5 \text{ m of liquid}$

$v_2 = 2 \text{ m/s}$

Loss of head
$$= h_L = \frac{0.35(v_1 - v_2)^2}{2g}$$

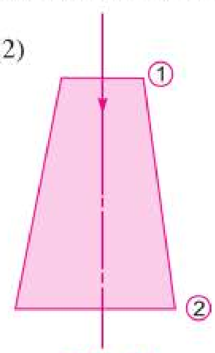


Fig. 6.7

$$= \frac{0.35\,[5-2]^2}{2g} = \frac{0.35 \times 9}{2 \times 9.81} = 0.16 \text{ m}$$

Pressure head, $\dfrac{p_2}{\rho g} = ?$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

Let the datum line passes through section (2). Then $z_2 = 0$, $z_1 = 2.0$

$$\therefore \quad 2.5 + \frac{5^2}{2 \times 9.81} + 2.0 = \frac{p_2}{\rho g} + \frac{2^2}{2 \times 9.81} + 0 + 0.16$$

$$2.5 + 1.27 + 2.0 = \frac{p_2}{\rho g} + 0.203 + .16$$

or $$\frac{p_2}{\rho g} = (2.5 + 1.27 + 2.0) - (.203 + .16)$$

$$= 5.77 - .363 = \mathbf{5.407 \text{ m of fluid.}} \text{ Ans.}$$

Problem 6.9 *A pipeline carrying oil of specific gravity 0.87, changes in diameter from 200 mm diameter at a position A to 500 mm diameter at a position B which is 4 metres at a higher level. If the pressures at A and B are 9.81 N/cm² and 5.886 N/cm² respectively and the discharge is 200 litres/s determine the loss of head and direction of flow.*

Solution. Discharge, $Q = 200 \text{ lit/s} = 0.2 \text{ m}^3\text{/s}$

Sp. gr. of oil $= 0.87$

$\therefore$ ρ for oil $= .87 \times 1000 = 870 \dfrac{\text{kg}}{\text{m}^3}$

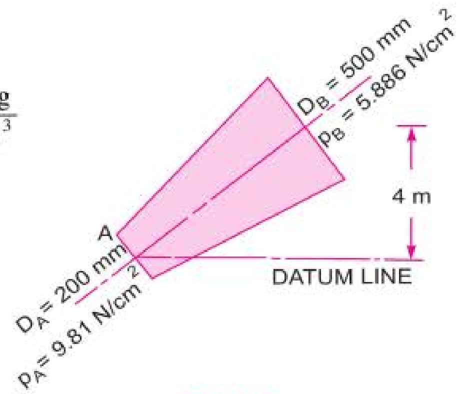


Fig. 6.8

Given : **At section *A*,** $D_A = 200 \text{ mm} = 0.2 \text{ m}$

Area, $A_A = \dfrac{\pi}{4}(D_A)^2 = \dfrac{\pi}{4}(.2)^2$

$= 0.0314 \text{ m}^2$

$p_A = 9.81 \text{ N/cm}^2$

$= 9.81 \times 10^4 \text{ N/m}^2$

If datum line is passing through *A*, then

$Z_A = 0$

$$V_A = \frac{Q}{A_A} = \frac{0.2}{0.0314} = 6.369 \text{ m/s}$$

At section *B*, $D_B = 500 \text{ mm} = 0.50 \text{ m}$

Area, $A_B = \dfrac{\pi}{4} D_B^2 = \dfrac{\pi}{4}(.5)^2 = 0.1963 \text{ m}^2$

$p_B = 5.886 \text{ N/cm}^2 = 5.886 \times 10^4 \text{ N/m}^2$

$$Z_B = 4.0 \text{ m}$$

$$V_B = \frac{Q}{\text{Area}} = \frac{0.2}{.1963} = 1.018 \text{ m/s}$$

Total energy at A $= E_A = \dfrac{p_A}{\rho g} + \dfrac{V_A^2}{2g} + Z_A$

$$= \frac{9.81 \times 10^4}{870 \times 9.81} + \frac{(6.369)^2}{2 \times 9.81} + 0 = 11.49 + 2.067 = 13.557 \text{ m}$$

Total energy at B $= E_B = \dfrac{p_B}{\rho g} + \dfrac{V_B^2}{2g} + Z_B$

$$= \frac{5.886 \times 10^4}{870 \times 9.81} + \frac{(1.018)^2}{2 \times 9.81} + 4.0 = 6.896 + 0.052 + 4.0 = 10.948 \text{ m}$$

(*i*) **Direction of flow.** As E_A is more than E_B and hence flow is taking place from A to B. **Ans.**

(*ii*) Loss of head $= h_L = E_A - E_B = 13.557 - 10.948 =$ **2.609 m. Ans.**

6.7 PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is applied in all problems of incompressible fluid flow where energy considerations are involved. But we shall consider its application to the following measuring devices :

1. Venturimeter.
2. Orifice meter.
3. Pitot-tube.

6.7.1 Venturimeter. A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It consists of three parts :

(*i*) A short converging part, (*ii*) Throat, and (*iii*) Diverging part. It is based on the Principle of Bernoulli's equation.

Expression for rate of flow through venturimeter

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing (say water), as shown in Fig. 6.9.

Let d_1 = diameter at inlet or at section (1),

p_1 = pressure at section (1)

v_1 = velocity of fluid at section (1),

a = area at section (1) $= \dfrac{\pi}{4} d_1^2$

and d_2, p_2, v_2, a_2 are corresponding values at section (2).

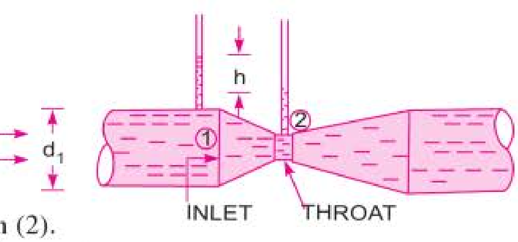


Fig. 6.9 *Venturimeter.*

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence $z_1 = z_2$

$$\therefore \quad \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad \text{or} \quad \frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But $\frac{p_1 - p_2}{\rho g}$ is the difference of pressure heads at sections 1 and 2 and it is equal to h or $\frac{p_1 - p_2}{\rho g} = h$

Substituting this value of $\frac{p_1 - p_2}{\rho g}$ in the above equation, we get

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \quad \text{...(6.6)}$$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2 v_2}{a_1}$$

Substituting this value of v_1 in equation (6.6)

$$h = \frac{v_2^2}{2g} - \frac{\left(\frac{a_2 v_2}{a_1}\right)^2}{2g} = \frac{v_2^2}{2g}\left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{v_2^2}{2g}\left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

or $$v_2^2 = 2gh \frac{a_1^2}{a_1^2 - a_2^2}$$

$\therefore$ $$v_2 = \sqrt{2gh \frac{a_1^2}{a_1^2 - a_2^2}} = \frac{a_1}{\sqrt{a_1^2 - a_2^2}}\sqrt{2gh}$$

$\therefore$ Discharge, $$Q = a_2 v_2$$

$$= a_2 \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \quad \text{...(6.7)}$$

Equation (6.7) gives the discharge under ideal conditions and is called, theoretical discharge. Actual discharge will be less than theoretical discharge.

$\therefore$ $$Q_{\text{act}} = C_d \times \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \quad \text{...(6.8)}$$

where C_d = Co-efficient of venturimeter and its value is less than 1.

Value of 'h' given by differential U-tube manometer

Case I. Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

S_h = Sp. gravity of the heavier liquid
S_o = Sp. gravity of the liquid flowing through pipe
x = Difference of the heavier liquid column in U-tube

Then $$h = x\left[\frac{S_h}{S_o} - 1\right] \quad \text{...(6.9)}$$

Case II. If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x\left[1 - \frac{S_l}{S_o}\right] \qquad \text{...(6.10)}$$

where S_l = Sp. gr. of lighter liquid in U-tube

S_o = Sp. gr. of fluid flowing through pipe

x = Difference of the lighter liquid columns in U-tube.

Case III. Inclined Venturimeter with Differential U-tube manometer. The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right] \qquad \text{...(6.11)}$$

Case IV. Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right] \qquad \text{...(6.12)}$$

Problem 6.10 *A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take C_d = 0.98.*

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area at inlet, $a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \frac{\pi}{4} \times 15^2 = 176.7 \text{ cm}^2$

$C_d = 0.98$

Reading of differential manometer = x = 20 cm of mercury.

$\therefore$ Difference of pressure head is given by (6.9)

or
$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

where S_h = Sp. gravity of mercury = 13.6, S_o = Sp. gravity of water = 1

$$= 20\left[\frac{13.6}{1} - 1\right] = 20 \times 12.6 \text{ cm} = 252.0 \text{ cm of water.}$$

The discharge through venturimeter is given by eqn. (6.8)

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 9.81 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4}$$

$$= 125756 \text{ cm}^3/\text{s} = \frac{125756}{1000} \text{ lit/s} = \mathbf{125.756 \text{ lit/s. Ans.}}$$

Problem 6.11 *An oil of sp. gr. 0.8 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 25 cm. Calculate the discharge of oil through the horizontal venturimeter. Take $C_d = 0.98$.*

Solution. Given :

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_h = 13.6$

Reading of differential manometer, $x = 25$ cm

$\therefore$ Difference of pressure head, $h = x\left[\frac{S_h}{S_o} - 1\right]$

$$= 25\left[\frac{13.6}{0.8} - 1\right] \text{ cm of oil} = 25\,[17 - 1] = 400 \text{ cm of oil.}$$

Dia. at inlet, $d_1 = 20$ cm

$\therefore$ $a_1 = \frac{\pi}{4} d_1^{\,2} = \frac{\pi}{4} \times 20^2 = 314.16 \text{ cm}^2$

$d_2 = 10$ cm

$\therefore$ $a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$

$C_d = 0.98$

$\therefore$ The discharge Q is given by equation (6.8)

or
$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 400}$$

$$= \frac{21421375.68}{\sqrt{98696 - 6168}} = \frac{21421375.68}{304} \text{ cm}^3/\text{s}$$

$$= 70465 \text{ cm}^3/\text{s} = \mathbf{70.465 \text{ litres/s. Ans.}}$$

Problem 6.12 *A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 60 litres/s. Find the reading of the oil-mercury differential manometer. Take $C_d = 0.98$.*

Solution. Given : $d_1 = 20$ cm

$\therefore$ $a_1 = \frac{\pi}{4} 20^2 = 314.16 \text{ cm}^2$

$d_2 = 10$ cm

$\therefore$ $$a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

$$C_d = 0.98$$

$$Q = 60 \text{ litres/s} = 60 \times 1000 \text{ cm}^3\text{/s}$$

Using the equation (6.8), $$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

or $$60 \times 1000 = 9.81 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times h} = \frac{1071068.78\sqrt{h}}{304}$$

or $$\sqrt{h} = \frac{304 \times 60000}{1071068.78} = 17.029$$

$\therefore$ $$h = (17.029)^2 = 289.98 \text{ cm of oil}$$

But $$h = x\left[\frac{S_h}{S_o} - 1\right]$$

where S_h = Sp. gr. of mercury = 13.6

S_o = Sp. gr. of oil = 0.8

x = Reading of manometer

$\therefore$ $$289.98 = x\left[\frac{13.6}{0.8} - 1\right] = 16x$$

$\therefore$ $$x = \frac{289.98}{16} = 18.12 \text{ cm.}$$

$\therefore$ Reading of oil-mercury differential manometer = **18.12 cm. Ans.**

Problem 6.13 *A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of water. The pressure at inlet is 17.658 N/cm² and the vacuum pressure at the throat is 30 cm of mercury. Find the discharge of water through venturimeter. Take C_d = 0.98.*

Solution. Given :

Dia. at inlet, $$d_1 = 20 \text{ cm}$$

$\therefore$ $$a_1 = \frac{\pi}{4} \times (20)^2 = 314.16 \text{ cm}^2$$

Dia. at throat, $$d_2 = 10 \text{ cm}$$

$\therefore$ $$a_2 = \frac{\pi}{4} \times 10^2 = 78.74 \text{ cm}^2$$

$$p_1 = 17.658 \text{ N/cm}^2 = 17.658 \times 10^4 \text{ N/m}^2$$

ρ for water $$= 1000 \frac{\text{kg}}{\text{m}^3} \text{ and } \therefore \frac{p_1}{\rho g} = \frac{17.658 \times 10^4}{9.81 \times 1000} = 18 \text{ m of water}$$

$$\frac{p_2}{\rho g} = -30 \text{ cm of mercury}$$

$$= -0.30 \text{ m of mercury} = -0.30 \times 13.6 = -4.08 \text{ m of water}$$

$\therefore$ Differential head $= h = \dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g} = 18 - (-4.08)$

$= 18 + 4.08 = 22.08$ m of water = 2208 cm of water

The discharge Q is given by equation (6.8)

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.74)^2}} \times \sqrt{2 \times 981 \times 2208}$$

$$= \frac{50328837.21}{304} \times 165555 \text{ cm}^3\text{/s} = \textbf{165.555 lit/s. Ans.}$$

Problem 6.14 *The inlet and throat diameters of a horizontal venturimeter are 30 cm and 10 cm respectively. The liquid flowing through the meter is water. The pressure intensity at inlet is 13.734 N/cm² while the vacuum pressure head at the throat is 37 cm of mercury. Find the rate of flow. Assume that 4% of the differential head is lost between the inlet and throat. Find also the value of C_d for the venturimeter.*

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \dfrac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 10$ cm

$\therefore$ $a_2 = \dfrac{\pi}{4}(10)^2 = 78.54 \text{ cm}^2$

Pressure, $p_1 = 13.734 \text{ N/cm}^2 = 13.734 \times 10^4 \text{ N/m}^2$

$\therefore$ Pressure head, $\dfrac{p_1}{\rho g} = \dfrac{13.734 \times 10^4}{1000 \times 9.81} = 14$ m of water

$\dfrac{p_2}{\rho g} = -37$ cm of mercury

$= \dfrac{-37 \times 13.6}{100}$ m of water $= -5.032$ m of water

Differential head, $h = p_1/\rho g - p_2/\rho g$

$= 14.0 - (-5.032) = 14.0 + 5.032$

$= 19.032$ m of water $= 1903.2$ cm

Head lost, $h_f = 4\%$ of $h = \dfrac{4}{100} \times 19.032 = 0.7613$ m

$\therefore$ $C_d = \sqrt{\dfrac{h - h_f}{h}} = \sqrt{\dfrac{19.032 - .7613}{19.032}} = 0.98$

$\therefore$ Discharge $= C_d \dfrac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$

$$= \frac{0.98 \times 706.85 \times 78.54 \times \sqrt{2 \times 981 \times 1903.2}}{\sqrt{(706.85)^2 - (78.54)^2}}$$

$$= \frac{105132247.8}{\sqrt{499636.9 - 6168}} = 149692.8 \text{ cm}^3/\text{s} = \mathbf{0.14969 \text{ m}^3/\text{s}}. \textbf{ Ans.}$$

PROBLEMS ON INCLINED VENTURIMETER

Problem 6.15 *A 30 cm × 15 cm venturimeter is inserted in a vertical pipe carrying water, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 20 cm. Find the discharge. Take C_d = 0.98.*

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \dfrac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \dfrac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

$$h = x\left[\frac{S_h}{S_o} - 1\right] = 20\left[\frac{13.6}{1.0} - 1.0\right] = 20 \times 12.6 = 252.0 \text{ cm of water}$$

$C_d = 0.98$

Discharge, $Q = C_d \dfrac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.3 - 31222.9}} = \frac{86067593.36}{684.4}$$

$$= 125756 \text{ cm}^3/\text{s} = \mathbf{125.756 \text{ lit/s}}. \textbf{ Ans.}$$

Problem 6.16 *A 20 cm × 10 cm venturimeter is inserted in a vertical pipe carrying oil of sp. gr. 0.8, the flow of oil is in upward direction. The difference of levels between the throat and inlet section is 50 cm. The oil mercury differential manometer gives a reading of 30 cm of mercury. Find the discharge of oil. Neglect losses.*

Solution. Dia. at inlet, $d_1 = 20$ cm

$\therefore$ $a_1 = \dfrac{\pi}{4}(20)^2 = 314.16 \text{ cm}^2$

Dia. at throat, $d_2 = 10$ cm

$\therefore \quad a_2 = \frac{\pi}{4} (10)^2 = 78.54 \text{ cm}^2$

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_g = 13.6$

Differential manometer reading, $x = 30$ cm

$$\therefore \quad h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_g}{S_o} - 1\right]$$

$$= 30\left[\frac{13.6}{0.8} - 1\right] = 30\ [17 - 1] = 30 \times 16 = 480 \text{ cm of oil}$$

$$C_d = 1.0$$

The discharge,

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= \frac{1.0 \times 314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 480} \text{ cm}^3/\text{s}$$

$$= \frac{23932630.7}{304} = 78725.75 \text{ cm}^3/\text{s} = \mathbf{78.725 \text{ litres/s.}} \text{ Ans.}$$

Problem 6.17 *In a vertical pipe conveying oil of specific gravity 0.8, two pressure gauges have been installed at A and B where the diameters are 16 cm and 8 cm respectively. A is 2 metres above B. The pressure gauge readings have shown that the pressure at B is greater than at A by 0.981 N/cm². Neglecting all losses, calculate the flow rate. If the gauges at A and B are replaced by tubes filled with the same liquid and connected to a U-tube containing mercury, calculate the difference of level of mercury in the two limbs of the U-tube.*

Solution. Given :

Sp. gr. of oil, $S_o = 0.8$

$\therefore$ Density, $\rho = 0.8 \times 1000 = 800 \frac{\text{kg}}{\text{m}^3}$

Dia. at A, $D_A = 16 \text{ cm} = 0.16 \text{ m}$

$\therefore$ Area at A, $A_1 = \frac{\pi}{4}(.16)^2 = 0.0201 \text{ m}^2$

Dia. at B, $D_B = 8 \text{ cm} = 0.08 \text{ m}$

$\therefore$ Area at B, $A_2 = \frac{\pi}{4}(.08)^2 = 0.005026 \text{ m}^2$

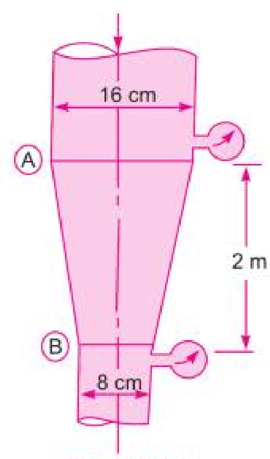


Fig. 6.9 (*a*)

(*i*) Difference of pressures, $p_B - p_A = 0.981 \text{ N/cm}^2$

$$= 0.981 \times 10^4 \text{ N/m}^2 = \frac{9810\text{N}}{\text{m}^2}$$

Difference of pressure head

$$\therefore \quad \frac{p_B - p_A}{\rho g} = \frac{9810}{800 \times 9.81} = 1.25 \qquad (\because \ \rho = 800 \text{ kg/m}^3)$$

Applying Bernoulli's theorem at A and B and taking the reference line passing through section B, we get

$$\frac{p_A}{\rho g}+\frac{V_A^2}{2g}+Z_A=\frac{p_B}{\rho g}+\frac{V_B^2}{2g}+Z_B$$

or
$$\frac{p_A}{\rho g}-\frac{p_B}{p_g}-Z_A-Z_B=\frac{V_B^2}{2g}-\frac{V_A^2}{2g}$$

or
$$\left(\frac{p_A-p_B}{\rho g}\right)+2.0-0.0=\frac{V_B^2}{2g}-\frac{V_A^2}{2g}$$

or
$$-1.25+2.0=\frac{V_B^2}{2g}-\frac{V_A^2}{2g} \qquad \left(\because \frac{p_B-p_A}{\rho g}=1.25\right)$$

$$0.75=\frac{V_B^2}{2g}-\frac{V_A^2}{2g} \qquad ...(i)$$

Now applying continuity equation at A and B, we get

$$V_A\times A_1=V_B\times A_2$$

or
$$V_B=\frac{V_A\times A_1}{A_2}=\frac{V_A\times\frac{\pi}{4}(.16)^2}{\frac{\pi}{4}(.08)^2}=4V_A$$

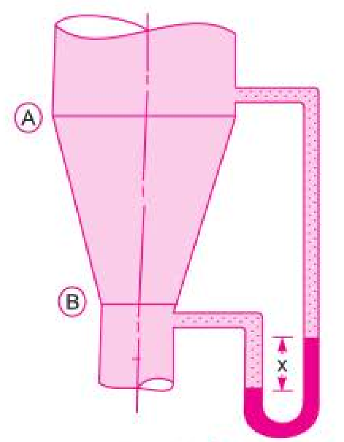


Fig. 6.9 (*b*)

Substituting the value of V_B in equation (*i*), we get

$$0.75=\frac{16V_A^2}{2g}-\frac{V_A^2}{2g}=\frac{15V_A^2}{2g}$$

$$\therefore \qquad V_A=\sqrt{\frac{0.75\times2\times9.81}{15}}=0.99 \text{ m/s}$$

$\therefore$ Rate of flow,
$$Q=V_A\times A_1$$
$$=0.99\times0.0201=\mathbf{0.01989\ m^3/s.}\ \textbf{Ans.}$$

(*ii*) Difference of level of mercury in the U-tube.

Let h = Difference of mercury level.

Then
$$h=x\left(\frac{S_g}{S_o}-1\right)$$

where
$$h=\left(\frac{p_A}{\rho g}+Z_A\right)-\left(\frac{p_B}{\rho g}+Z_B\right)=\frac{p_A-p_B}{\rho g}+Z_A-Z_B$$
$$=-1.25+2.0-0$$
$$=0.75 \qquad \left(\because \frac{p_B-p_A}{\rho g}=1.25\right)$$

$$\therefore \qquad 0.75=x\left[\frac{13.6}{0.8}-1\right]=x\times16$$

$$\therefore \qquad x=\frac{0.75}{16}=0.04687 \text{ m}=\mathbf{4.687\ cm.}\ \textbf{Ans.}$$

Problem 6.18 *Find the discharge of water flowing through a pipe 30 cm diameter placed in an inclined position where a venturimeter is inserted, having a throat diameter of 15 cm. The difference of pressure between the main and throat is measured by a liquid of sp. gr. 0.6 in an inverted U-tube which gives a reading of 30 cm. The loss of head between the main and throat is 0.2 times the kinetic head of the pipe.*

Solution. Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \dfrac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \dfrac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

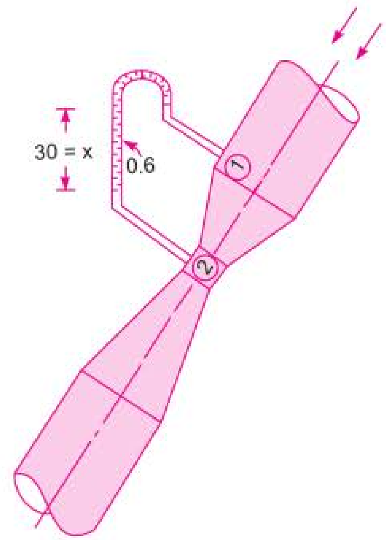


Fig. 6.10

Reading of differential manometer, $x = 30$ cm

Difference of pressure head, h is given by

$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = h$$

Also $$h = x\left[1 - \frac{S_l}{S_o}\right]$$

where $S_l = 0.6$ and $S_o = 1.0$

$$= 30\left[1 - \frac{0.6}{1.0}\right] = 30 \times .4 = 12.0 \text{ cm of water}$$

Loss of head, $h_L = 0.2 \times$ kinetic head of pipe $= 0.2 \times \dfrac{v_1^2}{2g}$

Now applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_L$$

or $$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = h_L$$

But $$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = h = 12.0 \text{ cm of water}$$

and $$h_L = 0.2 \times v_1^2/2g$$

$\therefore$ $$12.0 + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0.2 \times \frac{v_1^2}{2g}$$

$\therefore$ $$12.0 + 0.8\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0 \qquad \text{...(1)}$$

Applying continuity equation at sections (1) and (2), we get

$$a_1 v_1 = a_2 v_2$$

$$\therefore \qquad v_1 = \frac{a_2}{a_1} v_2 = \frac{\frac{\pi}{4}(15)^2 v_2}{\frac{\pi}{4}(30)^2} = \frac{v_2}{4}$$

Substituting this value of v_1 in equation (1), we get

$$12.0 + \frac{0.8}{2g}\left(\frac{v_2}{4}\right)^2 - \frac{v_2^2}{2g} = 0 \text{ or } 12.0 + \frac{v_2^2}{2g}\left[\frac{0.8}{16} - 1\right] = 0$$

or
$$\frac{v_2^2}{2g}[.05 - 1] = -12.0 \text{ or } \frac{0.95\, v_2^2}{2g} = 12.0$$

$$\therefore \qquad v_2 = \sqrt{\frac{2 \times 981 \times 12.0}{0.95}} = 157.4 \text{ cm/s}$$

$\therefore$ Discharge $= a_2 v_2$

$= 176.7 \times 157.4 \text{ cm}^3/\text{s} = 27800 \text{ cm}^3/\text{s} =$ **27.8 litres/s. Ans.**

Problem 6.19 *A 30 cm × 15 cm venturimeter is provided in a vertical pipe line carrying oil of specific gravity 0.9, the flow being upwards. The difference in elevation of the throat section and entrance section of the venturimeter is 30 cm. The differential U-tube mercury manometer shows a gauge deflection of 25 cm. Calculate :*

(i) the discharge of oil, and

(ii) the pressure difference between the entrance section and the throat section. Take the co-efficient of discharge as 0.98 and specific gravity of mercury as 13.6.

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ Area, $a_2 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

Let section (1) represents inlet and section (2) represents throat. Then $z_2 - z_1 = 30$ cm

Sp. gr. of oil, $S_o = 0.9$

Sp. gr. of mercury, $S_g = 13.6$

Reading of diff. manometer, $x = 25$ cm

The differential head, h is given by

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right)$$

$$= x\left[\frac{S_g}{S_o} - 1\right] = 25\left[\frac{13.6}{0.9} - 1\right] = 352.77 \text{ cm of oil}$$

(i) The discharge, Q of oil $= C_d \dfrac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$

$$= \frac{0.98 \times 706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} = \sqrt{2 \times 981 \times 352.77}$$

$$= \frac{101832219.9}{684.4} = 148790.5 \text{ cm}^3\text{/s}$$

$$= \textbf{148.79 litres/s. Ans.}$$

(ii) Pressure difference between entrance and throat section

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 352.77$$

or $\left(\dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g}\right) + z_1 - z_2 = 352.77$

But $z_2 - z_1 = 30$ cm

$\therefore$ $\left(\dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g}\right) - 30 = 352.77$

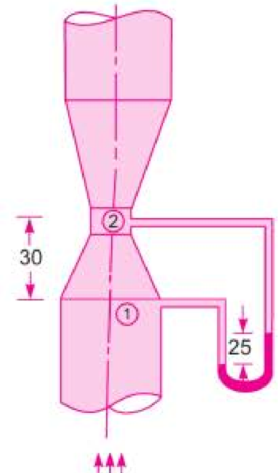


Fig. 6.11

$\therefore$ $\dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g} = 352.77 + 30 = 382.77$ cm of oil $=$ **3.8277 m of oil. Ans.**

or $(p_1 - p_2) = 3.8277 \times \rho g$

But density of oil $=$ Sp. gr. of oil $\times$ 1000 kg/m^3

$= 0.9 \times 1000 = 900$ kg/cm^3

$\therefore$ $(p_1 - p_2) = 3.8277 \times 900 \times 9.81 \ \dfrac{\text{N}}{\text{m}^2}$

$$= \frac{33795}{10^4} \text{ N/cm}^2 = \textbf{3.3795 N/cm}^\mathbf{2}\textbf{. Ans.}$$

Problem 6.20 *Crude oil of specific gravity 0.85 flows upwards at a volume rate of flow of 60 litre per second through a vertical venturimeter with an inlet diameter of 200 mm and a throat diameter of 100 mm. The co-efficient of discharge of the venturimeter is 0.98. The vertical distance between the pressure tappings is 300 mm.*

(i) If two pressure gauges are connected at the tappings such that they are positioned at the levels of their corresponding tapping points, determine the difference of readings in N/cm^2 of the two pressure gauges.

(ii) If a mercury differential manometer is connected, in place of pressure gauges, to the tappings such that the connecting tube upto mercury are filled with oil, determine the difference in the level of the mercury column.

Solution. Given :

Specific gravity of oil, $S_o = 0.85$

$\therefore$ Density, $\rho = 0.85 \times 1000 = 850$ kg/m^3

Discharge, $Q = 60$ litre/s

$$= \frac{60}{1000} = 0.06 \text{ m}^3/\text{s}$$

Inlet dia, $d_1 = 200$ mm $= 0.2$ m

$\therefore$ Area, $a_1 = \frac{\pi}{4}(.2)^2 = 0.0314$ m^2

Throat dia., $d_2 = 100$ mm $= 0.1$ m

$\therefore$ Area, $a_2 = \frac{\pi}{4}(0.1)^2 = 0.00785$ m^2

Value of C_d $= 0.98$

Let section (1) represents inlet and section (2) represents throat. Then

$$z_2 - z_1 = 300 \text{ mm} = 0.3 \text{ m}$$

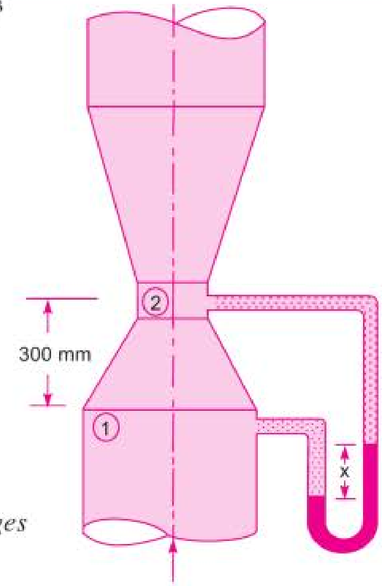


Fig. 6.11 (*a*)

(*i*) *Difference of readings in N/cm^2 of the two pressure gauges*

The discharge Q is given by,

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

or

$$0.06 = \frac{0.98 \times 0.0314 \times 0.00785}{\sqrt{0.0314^2 - 0.00785^2}} \times \sqrt{2 \times 9.81 \times h}$$

$$= \frac{0.98 \times 0.00024649}{0.0304} \times 4.429\sqrt{h}$$

$$\therefore \quad \sqrt{h} = \frac{0.06 \times 0.0304}{0.98 \times 0.00024649 \times 4.429} = 1.705$$

$$\therefore \quad h = 1.705^2 = 2.908 \text{ m}$$

But for a vertical venturimeter, $h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right)$

$$\therefore \quad 2.908 = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) + z_1 - z_2$$

$$\frac{p_1 - p_2}{\rho g} = 2.908 + z_2 - z_1 = 2.908 + 0.3 \qquad (\because \ z_2 - z_1 = 0.3 \text{ m})$$

$$= 3.208 \text{ m of oil}$$

$$\therefore \quad p_1 - p_2 = \rho g \times 3.208$$

$$= 850 \times 9.81 \times 3.208 \text{ N/m}^2 = \frac{850 \times 9.81 \times 3.208}{10^4} \text{ N/cm}^2$$

$=$ **2.675 N/cm^2. Ans.**

(*ii*) *Difference in the levels of mercury columns* (*i.e.*, *x*)

The value of h is given by, $h = x\left[\dfrac{S_g}{S_o} - 1\right]$

$$\therefore \quad 2.908 = x\left[\frac{13.6}{0.85} - 1\right] = x\,[16 - 1] = 15\,x$$

$$\therefore \quad x = \frac{2.908}{15} = 0.1938 \text{ m} = \textbf{19.38 cm of oil. Ans.}$$

Problem 6.21 *In a 100 mm diameter horizontal pipe a venturimeter of 0.5 contraction ratio has been fixed. The head of water on the metre when there is no flow is 3 m (gauge). Find the rate of flow for which the throat pressure will be 2 metres of water absolute. The co-efficient of discharge is 0.97. Take atmospheric pressure head = 10.3 m of water.*

Solution. Given :

Dia. of pipe, $d_1 = 100 \text{ mm} = 10 \text{ cm}$

$\therefore$ Area, $a_1 = \dfrac{\pi}{4} d_1^{\,2} = \dfrac{\pi}{4}(10)^2 = 78.54 \text{ cm}^2$

Dia. at throat, $d_2 = 0.5 \times d_1 = 0.5 \times 10 = 5 \text{ cm}$

$\therefore$ Area, $a_2 = \dfrac{\pi}{4}(5)^2 = 19.635 \text{ cm}^2$

Head of water for no flow $= \dfrac{p_1}{\rho g} = 3 \text{ m (gauge)} = 3 + 10.3 = 13.3 \text{ m (abs.)}$

Throat pressure head $= \dfrac{p_2}{\rho g} = 2$ m of water absolute.

$\therefore$ Difference of pressure head, $h = \dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g} = 13.3 - 2.0 = 11.3 \text{ m} = 1130 \text{ cm}$

$\therefore$ Rate of flow, Q is given by $Q = C_d \dfrac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$

$$= 0.97 \times \frac{78.54 \times 19.635}{\sqrt{(78.54)^2 - (19.635)^2}} \times \sqrt{2 \times 981 \times 1130}$$

$$= \frac{2227318.17}{76} = 29306.8 \text{ cm}^3/\text{s} = \textbf{29.306 litres/s. Ans.}$$

6.7.2 Orifice Meter or Orifice Plate. It is a device used for measuring the rate of flow of a fluid through a pipe. It is a cheaper device as compared to venturimeter. It also works on the same principle as that of venturimeter. It consists of a flat circular plate which has a circular sharp edged hole called orifice, which is concentric with the pipe. The orifice diameter is kept generally 0.5 times the diameter of the pipe, though it may vary from 0.4 to 0.8 times the pipe diameter.

A differential manometer is connected at section (1), which is at a distance of about 1.5 to 2.0 times the pipe diameter upstream from the orifice plate, and at section (2), which is at a distance of about half the diameter of the orifice on the downstream side from the orifice plate.

Let p_1 = pressure at section (1),
v_1 = velocity at section (1),
a_1 = area of pipe at section (1), and

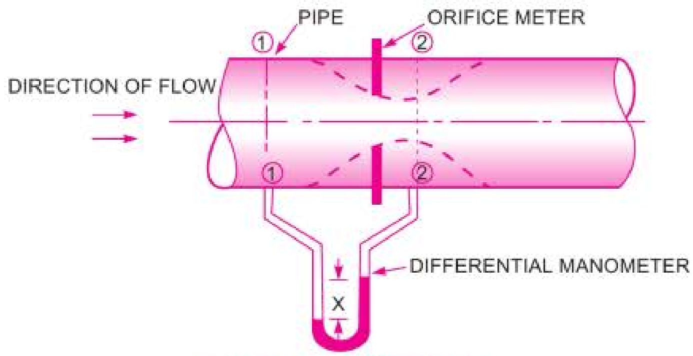


Fig. 6.12. *Orifice meter.*

p_2, v_2, a_2 are corresponding values at section (2). Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

or
$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But
$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = h = \text{Differential head}$$

$\therefore$
$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \quad \text{or} \quad 2gh = {v_2}^2 - {v_1}^2$$

or
$$v_2 = \sqrt{2gh + v_1^2} \qquad \ldots(i)$$

Now section (2) is at the vena-contracta and a_2 represents the area at the vena-contracta. If a_0 is the area of orifice then, we have

$$C_c = \frac{a_2}{a_0}$$

where C_c = Co-efficient of contraction

$\therefore$
$$a_2 = a_0 \times C_c \qquad \ldots(ii)$$

By continuity equation, we have

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2}{a_1} v_2 = \frac{a_0\, C_c}{a_1} v_2 \qquad \ldots(iii)$$

Substituting the value of v_1 in equation (*i*), we get

$$v_2 = \sqrt{2gh + \frac{a_0^2 C_c^2 v_2^2}{a_1^2}}$$

or $$v_2^2 = 2gh + \left(\frac{a_0}{a_1}\right)^2 C_c^2 v_2^2 \text{ or } v_2^2 \left[1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2\right] = 2gh$$

$\therefore$ $$v_2 = \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$\therefore$ The discharge $Q = v_2 \times a_2 = v_2 \times a_0\, C_c$ $\{\because a_2 = a_0 C_c \text{ from } (ii)\}$

$$= \frac{a_0 C_c \sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}} \qquad ...(iv)$$

The above expression is simplified by using

$$C_d = C_c \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$\therefore$ $$C_c = C_d \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}}$$

Substituting this value of C_c in equation (*iv*), we get

$$Q = a_0 \times C_d \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}} \times \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$$= \frac{C_d a_0 \sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}} = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}. \qquad ...(6.13)$$

where C_d = Co-efficient of discharge for orifice meter.

The co-efficient of discharge for orifice meter is much smaller than that for a venturimeter.

Problem 6.22 *An orifice meter with orifice diameter 10 cm is inserted in a pipe of 20 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter gives readings of 19.62 N/cm² and 9.81 N/cm² respectively. Co-efficient of discharge for the orifice meter is given as 0.6. Find the discharge of water through pipe.*

Solution. Given :

Dia. of orifice, $d_0 = 10$ cm

$\therefore$ Area, $a_0 = \dfrac{\pi}{4}(10)^2 = 78.54\text{ cm}^2$

Dia. of pipe, $d_1 = 20$ cm

$\therefore$ Area, $a_1 = \dfrac{\pi}{4}(20)^2 = 314.16\text{ cm}^2$

$p_1 = 19.62\text{ N/cm}^2 = 19.62 \times 10^4\text{ N/m}^2$

$$\therefore \quad \frac{p_1}{\rho g} = \frac{19.62 \times 10^4}{1000 \times 9.81} = 20 \text{ m of water}$$

Similarly
$$\frac{p_2}{\rho g} = \frac{9.81 \times 10^4}{1000 \times 9.81} = 10 \text{ m of water}$$

$$\therefore \quad h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 20.0 - 10.0 = 10 \text{ m of water} = 1000 \text{ cm of water}$$

$C_d = 0.6$

The discharge, Q is given by equation (6.13)

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.6 \times \frac{78.54 \times 314.16}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 1000}$$

$$= \frac{20736838.09}{304} = 68213.28\text{ cm}^3\text{/s} = \mathbf{68.21\ litres/s.}\ \textbf{Ans.}$$

Problem 6.23 *An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure difference measured by a mercury oil differential manometer on the two sides of the orifice meter gives a reading of 50 cm of mercury. Find the rate of flow of oil of sp. gr. 0.9 when the co-efficient of discharge of the orifice meter = 0.64.*

Solution. Given :

Dia. of orifice, $d_0 = 15$ cm

$\therefore$ Area, $a_0 = \dfrac{\pi}{4}(15)^2 = 176.7\text{ cm}^2$

Dia. of pipe, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \dfrac{\pi}{4}(30)^2 = 706.85\text{ cm}^2$

Sp. gr. of oil, $S_o = 0.9$

Reading of diff. manometer, $x = 50$ cm of mercury

$\therefore$ Differential head, $h = x\left[\dfrac{S_g}{S_o} - 1\right] = 50\left[\dfrac{13.6}{0.9} - 1\right]$ cm of oil

$$= 50 \times 14.11 = 705.5 \text{ cm of oil}$$

$$C_d = 0.64$$

∴ The rate of the flow, Q is given by equation (6.13)

$$Q = C_d \cdot \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.64 \times \frac{176.7 \times 706.85}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 705.5}$$

$$= \frac{94046317.78}{684.4} = 137414.25 \text{ cm}^3\text{/s} = \mathbf{137.414\ litres/s.}$$ **Ans.**

6.7.3 Pitot-tube. It is a device used for measuring the velocity of flow at any point in a pipe or a channel. It is based on the principle that if the velocity of flow at a point becomes zero, the pressure there is increased due to the conversion of the kinetic energy into pressure energy. In its simplest form, the pitot-tube consists of a glass tube, bent at right angles as shown in Fig. 6.13.

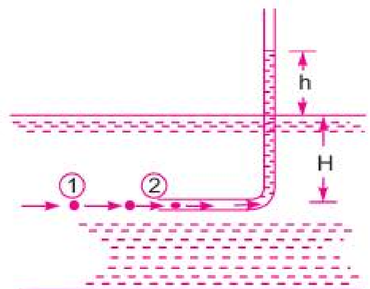


Fig. 6.13 *Pitot-tube.*

The lower end, which is bent through 90° is directed in the up-stream direction as shown in Fig. 6.13. The liquid rises up in the tube due to the conversion of kinetic energy into pressure energy. The velocity is determined by measuring the rise of liquid in the tube.

Consider two points (1) and (2) at the same level in such a way that point (2) is just as the inlet of the pitot-tube and point (1) is far away from the tube.

Let p_1 = intensity of pressure at point (1)

v_1 = velocity of flow at (1)

p_2 = pressure at point (2)

v_2 = velocity at point (2), which is zero

H = depth of tube in the liquid

h = rise of liquid in the tube above the free surface.

Applying Bernoulli's equation at points (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $z_1 = z_2$ as points (1) and (2) are on the same line and $v_2 = 0$.

$$\frac{p_1}{\rho g} = \text{pressure head at (1)} = H$$

$$\frac{p_2}{\rho g} = \text{pressure head at (2)} = (h + H)$$

Substituting these values, we get

$$\therefore \quad H + \frac{v_1^2}{2g} = (h + H) \quad \therefore \quad h = \frac{v_1^2}{2g} \quad \text{or} \quad v_1 = \sqrt{2gh}$$

This is theoretical velocity. Actual velocity is given by

$$(v_1)_{\text{act}} = C_v \sqrt{2gh}$$

where C_v = Co-efficient of pitot-tube

$\therefore$ Velocity at any point $\qquad v = C_v \sqrt{2gh}$ ...(6.14)

Velocity of flow in a pipe by pitot-tube. For finding the velocity at any point in a pipe by pitot-tube, the following arrangements are adopted :

1. Pitot-tube along with a vertical piezometer tube as shown in Fig. 6.14.
2. Pitot-tube connected with piezometer tube as shown in Fig. 6.15.
3. Pitot-tube and vertical piezometer tube connected with a differential U-tube manometer as shown in Fig. 6.16.

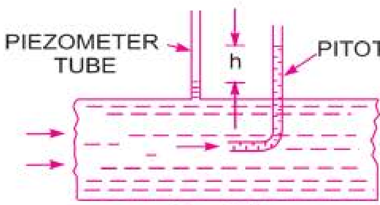


Fig. 6.14

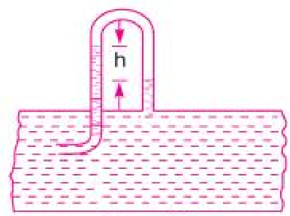


Fig. 6.15

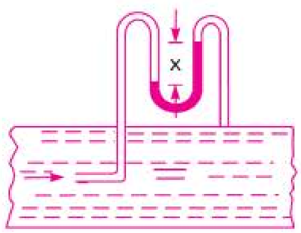


Fig. 6.16

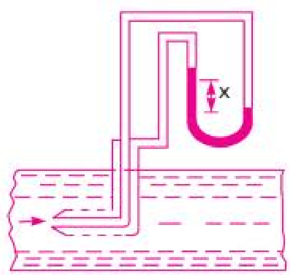


Fig. 6.17

4. Pitot-static tube, which consists of two circular concentric tubes one inside the other with some annular space in between as shown in Fig. 6.17. The outlet of these two tubes are connected to the differential manometer where the difference of pressure head 'h' is measured by knowing the difference of the levels of the manometer liquid say x. Then $h = x\left[\dfrac{S_g}{S_o} - 1\right]$.

Problem 6.24 *A pitot-static tube placed in the centre of a 300 mm pipe line has one orifice pointing upstream and other perpendicular to it. The mean velocity in the pipe is 0.80 of the central velocity. Find the discharge through the pipe if the pressure difference between the two orifices is 60 mm of water. Take the co-efficient of pitot tube as C_v = 0.98.*

Solution. Given :

Dia. of pipe, $\qquad d$ = 300 mm = 0.30 m

Diff. of pressure head, $\qquad h$ = 60 mm of water = .06 m of water

$C_v = 0.98$

Mean velocity, $\qquad \overline{V} = 0.80 \times$ Central velocity

Central velocity is given by equation (6.14)

$$= C_v \sqrt{2gh} = 0.98 \times \sqrt{2 \times 9.81 \times .06} = 1.063 \text{ m/s}$$

$\therefore$ $\overline{V} = 0.80 \times 1.063 = 0.8504$ m/s

Discharge, $Q =$ Area of pipe $\times \overline{V}$

$$= \frac{\pi}{4} d^2 \times \overline{V} = \frac{\pi}{4} (.30)^2 \times 0.8504 = \mathbf{0.06\ m^3/s.}\ \textbf{Ans.}$$

Problem 6.25 *Find the velocity of the flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 100 mm. Take co-efficient of pitot-tube 0.98 and sp. gr. of oil = 0.8.*

Solution. Given :

Diff. of mercury level, $x = 100$ mm $= 0.1$ m

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_g = 13.6$

$C_v = 0.98$

Diff. of pressure head, $h = x\left[\frac{S_g}{S_o} - 1\right] = .1\left[\frac{13.6}{0.8} - 1\right] = 1.6$ m of oil

$\therefore$ Velocity of flow $= C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1.6} = \mathbf{5.49\ m/s.}$ **Ans.**

Problem 6.26 *A pitot-static tube is used to measure the velocity of water in a pipe. The stagnation pressure head is 6 m and static pressure head is 5 m. Calculate the velocity of flow assuming the co-efficient of tube equal to 0.98.*

Solution. Given :

Stagnation pressure head, $h_s = 6$ m

Static pressure head, $h_t = 5$ m

$\therefore$ $h = 6 - 5 = 1$ m

Velocity of flow, $V = C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1} = \mathbf{4.34\ m/s.}$ **Ans.**

Problem 6.27 *A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.*

Solution. Given :

Diff. of mercury level, $x = 170$ mm $= 0.17$ m

Sp. gr. of mercury, $S_g = 13.6$

Sp. gr. of sea-water, $S_o = 1.026$

$$\therefore \quad h = x\left[\frac{S_g}{S_o} - 1\right] = 0.17\left[\frac{13.6}{1.026} - 1\right] = 2.0834 \text{ m}$$

$$\therefore \quad V = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2.0834} = 6.393 \text{ m/s}$$

$$= \frac{6.393 \times 60 \times 60}{1000} \text{ km/hr} = \mathbf{23.01\ km/hr.}\ \textbf{Ans.}$$

Problem 6.28 *A pitot-tube is inserted in a pipe of 300 mm diameter. The static pressure in pipe is 100 mm of mercury (vacuum). The stagnation pressure at the centre of the pipe, recorded by the* *pitot-tube is 0.981 N/cm². Calculate the rate of flow of water through pipe, if the mean velocity of flow is 0.85 times the central velocity. Take $C_v = 0.98$.*

Solution. Given :

Dia. of pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

$\therefore$ Area, $a = \dfrac{\pi}{4} d^2 = \dfrac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$

Static pressure head $= 100$ mm of mercury (vacuum)

$$= -\frac{100}{1000} \times 13.6 = -1.36 \text{ m of water}$$

Stagnation pressure $= .981 \text{ N/cm}^2 = .981 \times 10^4 \text{ N/m}^2$

$\therefore$ Stagnation pressure head $= \dfrac{.981 \times 10^4}{\rho g} = \dfrac{.981 \times 10^4}{1000 \times 9.81} = 1 \text{ m}$

$\therefore$ h = Stagnation pressure head – Static pressure head

$= 1.0 - (-1.36) = 1.0 + 1.36 = 2.36$ m of water

$\therefore$ Velocity at centre $= C_v \sqrt{2gh}$

$$= 0.98 \times \sqrt{2 \times 9.81 \times 2.36} = 6.668 \text{ m/s}$$

Mean velocity, $\overline{V} = 0.85 \times 6.668 = 5.6678$ m/s

$\therefore$ Rate of flow of water $= \overline{V} \times$ area of pipe

$= 5.6678 \times 0.07068 \text{ m}^3/\text{s} =$ **0.4006 m^3/s. Ans.**

6.8 THE MOMENTUM EQUATION

It is based on the law of conservation of momentum or on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in that direction. The force acting on a fluid mass 'm' is given by the Newton's second law of motion,

$$F = m \times a$$

where a is the acceleration acting in the same direction as force F.

But $a = \dfrac{dv}{dt}$

$\therefore$ $F = m \dfrac{dv}{dt}$

$= \dfrac{d(mv)}{dt}$ {m is constant and can be taken inside the differential}

$$\therefore \quad F = \frac{d(mv)}{dt} \qquad \text{...(6.15)}$$

Equation (6.15) is known as the momentum principle.

Equation (6.15) can be written as $F.dt = d(mv)$ ...(6.16)

which is known as the *impulse-momentum equation* and states that the impulse of a force F acting on a fluid of mass m in a short interval of time dt is equal to the change of momentum $d(mv)$ in the direction of force.

Force exerted by a flowing fluid on a pipe bend

The impulse-momentum equation (6.16) is used to determine the resultant force exerted by a flowing fluid on a pipe bend.

Consider two sections (1) and (2), as shown in Fig. 6.18.

Let v_1 = velocity of flow at section (1),

p_1 = pressure intensity at section (1),

A_1 = area of cross-section of pipe at section (1) and

v_2, p_2, A_2 = corresponding values of velocity, pressure and area at section (2).

Let F_x and F_y be the components of the forces exerted by the flowing fluid on the bend in x-and y-directions respectively. Then the force exerted by the bend on the fluid in the directions of x and y will be equal to F_x and F_y but in the opposite directions. Hence component of the force exerted by bend on the fluid in the x-direction $= -F_x$ and in the direction of $y = -F_y$. The other external forces acting on the fluid are p_1A_1 and p_2A_2 on the sections (1) and (2) respectively. Then momentum equation in x-direction is given by

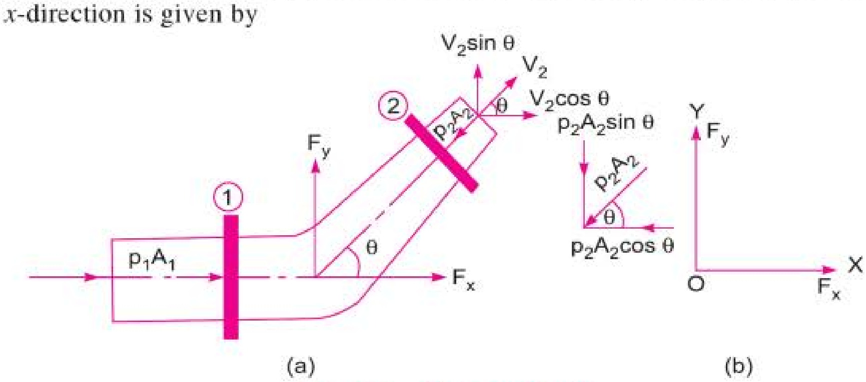


Fig. 6.18 *Forces on bend.*

Net force acting on fluid in the direction of x = Rate of change of momentum in x-direction

$\therefore \quad p_1A_1 - p_2A_2 \cos\theta - F_x$ = (Mass per sec) (change of velocity)

$= \rho Q$ (Final velocity in the direction of x – Initial velocity in the direction of x)

$$= \rho Q\,(V_2 \cos\theta - V_1) \quad \text{...(6.17)}$$

$$\therefore \quad F_x = \rho Q\,(V_1 - V_2 \cos\theta) + p_1A_1 - p_2A_2 \cos\theta \quad \text{...(6.18)}$$

Similarly the momentum equation in y-direction gives

$$0 - p_2A_2 \sin\theta - F_y = \rho Q\,(V_2 \sin\theta - 0) \quad \text{...(6.19)}$$

$$\therefore \quad F_y = \rho Q\,(-V_2 \sin\theta) - p_2A_2 \sin\theta \quad \text{...(6.20)}$$

Now the resultant force (F_R) acting on the bend

$$= \sqrt{F_x^2 + F_y^2} \quad \text{...(6.21)}$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan\theta = \frac{F_y}{F_x} \quad \text{...(6.22)}$$

Problem 6.29 *A 45° reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 600 mm and 300 mm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet to bend is 8.829 N/cm² and rate of flow of water is 600 litres/s.*

Solution. Given :

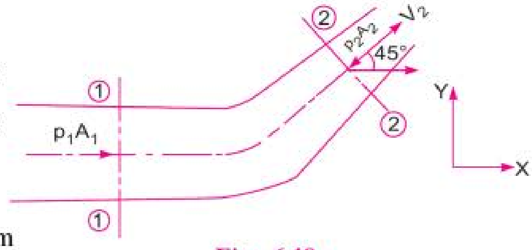


Fig. 6.19

Angle of bend, $\theta = 45°$

Dia. at inlet, $D_1 = 600$ mm $= 0.6$ m

$\therefore$ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.6)^2$

$= 0.2827 \text{ m}^2$

Dia. at outlet, $D_2 = 300$ mm $= 0.30$ m

$\therefore$ Area, $A_2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$

Pressure at inlet, $p_1 = 8.829 \text{ N/cm}^2 = 8.829 \times 10^4 \text{ N/m}^2$

$Q = 600 \text{ lit/s} = 0.6 \text{ m}^3\text{/s}$

$$V_1 = \frac{Q}{A_1} = \frac{0.6}{.2827} = 2.122 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.6}{.07068} = 8.488 \text{ m/s.}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

But $z_1 = z_2$

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad \text{or} \quad \frac{8.829 \times 10^4}{1000 \times 9.81} + \frac{2.122^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{8.488^2}{2 \times 9.81}$$

$$9 + .2295 = p_2/\rho g + 3.672$$

$$\therefore \quad \frac{p_2}{\rho g} = 9.2295 - 3.672 = 5.5575 \text{ m of water}$$

$$\therefore \quad p_2 = 5.5575 \times 1000 \times 9.81 \text{ N/m}^2 = 5.45 \times 10^4 \text{ N/m}^2$$

Forces on the bend in x- and y-directions are given by equations (6.18) and (6.20) as

$$F_x = \rho Q [V_1 - V_2 \cos \theta] + p_1 A_1 - p_2 A_2 \cos \theta$$

$$= 1000 \times 0.6 [2.122 - 8.488 \cos 45°]$$

$$+ 8.829 \times 10^4 \times .2827 - 5.45 \times 10^4 \times .07068 \times \cos 45°$$

$$= -2327.9 + 24959.6 - 2720.3 = 24959.6 - 5048.2$$

$$= 19911.4 \text{ N}$$

and

$$F_y = \rho Q [-V_2 \sin \theta] - p_2 A_2 \sin \theta$$

$$= 1000 \times 0.6 [-8.488 \sin 45°] - 5.45 \times 10^4 \times .07068 \times \sin 45°$$

$$= -3601.1 - 2721.1 = -6322.2 \text{ N}$$

–ve sign means F_y is acting in the downward direction

$\therefore$ Resultant force, $F_R = \sqrt{F_x^2 + F_y^2}$

$$= \sqrt{(19911.4)^2 + (-6322.2)^2}$$

$$= \mathbf{20890.9\ N.}\ \textbf{Ans.}$$

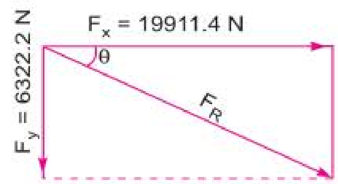


Fig. 6.20

The angle made by resultant force with x-axis is given by equation (6.22) or

$$\tan\theta = \frac{F_y}{F_x} = \frac{6322.2}{19911.4} = 0.3175$$

$$\therefore \qquad \theta = \tan^{-1} .3175 = \mathbf{17^\circ\ 36'}.\ \textbf{Ans.}$$

Problem 6.30 *250 litres/s of water is flowing in a pipe having a diameter of 300 mm. If the pipe is bent by 135° (that is change from initial to final direction is 135°), find the magnitude and direction of the resultant force on the bend. The pressure of water flowing is 39.24 N/cm².*

Solution. Given :

Pressure, $p_1 = p_2 = 39.24\ \text{N/cm}^2 = 39.24 \times 10^4\ \text{N/m}^2$

Discharge, $Q = 250\ \text{litres/s} = 0.25\ \text{m}^3\text{/s}$

Dia. of bend at inlet and outlet, $D_1 = D_2 = 300\ \text{mm} = 0.3\ \text{m}$

$\therefore$ Area, $A_1 = A_2 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068\ \text{m}^2$

Velocity of water at sections (1) and (2), $V = V_1 = V_2 = \frac{Q}{\text{Area}} = \frac{0.25}{.07068} = 3.537\ \text{m/s}.$

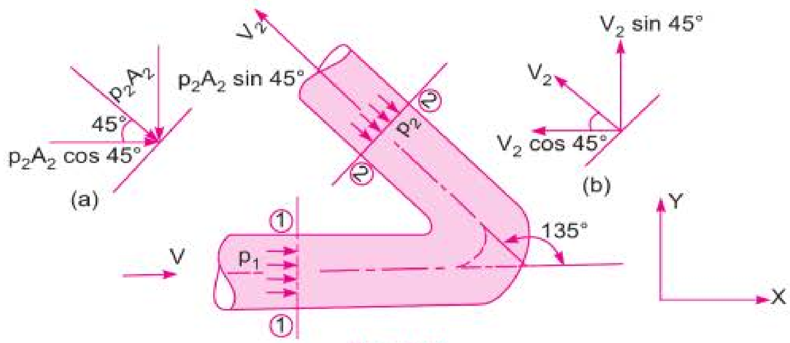


Fig. 6.21

Force along x-axis

$$= F_x = \rho Q[V_{1x} - V_{2x}] + p_{1x}A_1 + p_{2x}A_2$$

where, V_{1x} = initial velocity in the direction of $x = 3.537$ m/s

V_{2x} = final velocity in the direction of $x = -V_2 \cos 45^\circ = -3.537 \times .7071$

p_{1x} = pressure at section (1) in x-direction

$= 39.24\ \text{N/cm}^2 = 39.24 \times 10^4\ \text{N/m}^2$

p_{2x} = pressure at section (2) in x-direction

$= p_2 \cos 45^\circ = 39.24 \times 10^4 \times .7071$

$$\therefore \qquad F_x = 1000 \times .25[3.537 - (-3.537 \times .7071)] + 39.24 \times 10^4 \times .07068 + 39.24 \times 10^4 \times .07068 \times .7071$$

$$= 1000 \times .25[3.537 + 3.537 \times .7071] + 39.24 \times 10^4 \times .07068\ [1 + .7071]$$

$= 1509.4 + 47346 = 48855.4$ N

Force along y-axis

$= F_y = \rho Q[V_{1y} - V_{2y}] + (p_1A_1)_y + (p_2A_2)_y$

where V_{1y} = initial velocity in y-direction = 0

V_{2y} = final velocity in y-direction = $- V_2 \sin 45° = 3.537 \times .7071$

$(p_1A_1)_y$ = pressure force in y-direction = 0

$(p_2A_2)_y$ = pressure force at (2) in y-direction

$= - p_2A_2 \sin 45° = - 39.24 \times 10^4 \times .07068 \times .7071$

$\therefore$ $F_y = 1000 \times .25[0 - 3.537 \times .7071] + 0 + (- 39.24 \times 10^4 \times .07068 \times .7071)$

$= - 625.2 - 19611.1 = - 20236.3$ N

–ve sign means F_y is acting in the downward direction

$\therefore$ Resultant force, $F_R = \sqrt{F_x^2 + F_y^2}$

$= \sqrt{48855.4^2 + 20236.3^2}$

= 52880.6 N. Ans.

The direction of the resultant force F_R, with the x-axis is given as

$$\tan \theta = \frac{F_y}{F_x} = \frac{20236.3}{48855.4} = 0.4142$$

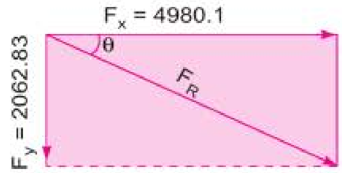


Fig. 6.22

$\therefore$ θ = **22° 30′. Ans.**

Problem 6.31 *A 300 mm diameter pipe carries water under a head of 20 metres with a velocity of 3.5 m/s. If the axis of the pipe turns through 45°, find the magnitude and direction of the resultant force at the bend.*

Solution. Given :

Dia. of bend, $D = D_1 = D_2 = 300$ mm $= 0.30$ m

$\therefore$ Area, $A = A_1 = A_2 = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times .3^2 = 0.07068$ m^2

Velocity, $V = V_1 = V_2 = 3.5$ m/s

$\theta = 45°$

Discharge, $Q = A \times V = 0.07068 \times 3.5 = 0.2475$ m^3/s

Pressure head = 20 m of water or $\frac{p}{\rho g}$ = 20 m of water

$\therefore$ $p = 20 \times \rho g = 20 \times 1000 \times 9.81$ N/m^2 = 196200 N/m^2

$\therefore$ Pressure intensity, $p = p_1 = p_2 = 196200$ N/m^2

Now $V_{1x} = 3.5$ m/s, $V_{2x} = V_2 \cos 45° = 3.5 \times .7071$

$V_{1y} = 0$, $V_{2y} = V_2 \sin 45° = 3.5 \times .7071$

$(p_1A_1)_x = p_1A_1 = 196200 \times .07068$, $(p_1A_1)_y = 0$

$(p_2A_2)_x = - p_2A_2 \cos 45°$, $(p_2A_2)_y = - p_2A_2 \sin 45°$

Force along x-axis, $F_x = \rho Q[V_{1x} - V_{2x}] + (p_1A_1)_x + (p_2A_2)_x$

$= 1000 \times .2475[3.5 - 3.5 \times .7071] + 196200 \times .07068 - p_2A_2 \times \cos 45°$

$$= 253.68 + 196200 \times .07068 - 196200 \times .07068 \times 0.7071$$
$$= 253.68 + 13871.34 - 9808.04 = 4316.98 \text{ N}$$

Force along y-axis, $F_y = \rho Q\,[V_1y - V_2y] + (p_1A_1)_y + (p_2A_2)_y$
$$= 1000 \times .2475[0 - 3.5 \times .7071] + 0 + [- p_2A_2 \sin 45°]$$
$$= - 612.44 - 196200 \times .07068 \times .7071$$
$$= - 612.44 - 9808 = - 10420.44 \text{ N}$$

$\therefore$ Resultant force $F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(4316.98)^2 + (10420.44)^2} =$ **11279 N. Ans.**

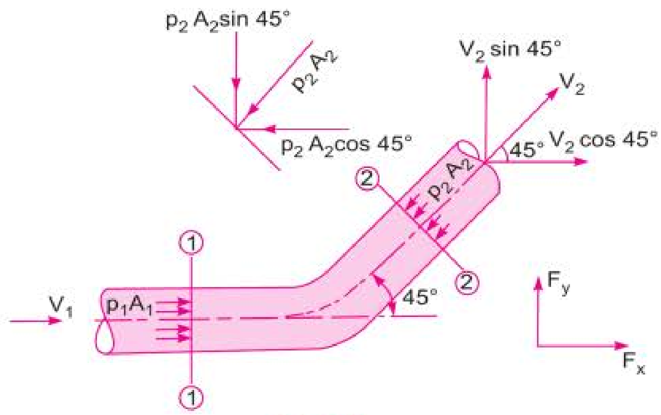


Fig. 6.23

The angle made by F_R with x-axis

$$\tan\theta = \frac{F_y}{F_x} = \frac{10420.44}{4316.98} = 2.411$$

$\therefore$ $\theta = \tan^{-1} 2.411 =$ **67° 28′. Ans.**

Problem 6.32 *In a 45° bend a rectangular air duct of 1 m² cross-sectional area is gradually reduced to 0.5 m² area. Find the magnitude and direction of the force required to hold the duct in position if the velocity of flow at the 1 m² section is 10 m/s, and pressure is 2.943 N/cm². Take density of air as 1.16 kg/m³.*

Solution. Given :

Area at section (1), $A_1 = 1 \text{ m}^2$

Area at section (2), $A_2 = 0.5 \text{ m}^2$

Velocity at section (1), $V_1 = 10$ m/s

Pressure at section (1), $p_1 = 2.943 \text{ N/cm}^2 = 2.943 \times 10^4 \text{ N/m}^2 = 29430 \text{ N/m}^2$

Density of air, $\rho = 1.16 \text{ kg/m}^3$

Applying continuity equation at sections (1) and (2)

$$A_1V_1 = A_2V_2$$

$$\therefore \quad V_2 = \frac{A_1V_1}{A_2} = \frac{1}{0.5} \times 10 = 20 \text{ m/s}$$

Discharge $\qquad Q = A_1V_1 = 1 \times 10 = 10 \text{ m}^3/\text{s}$

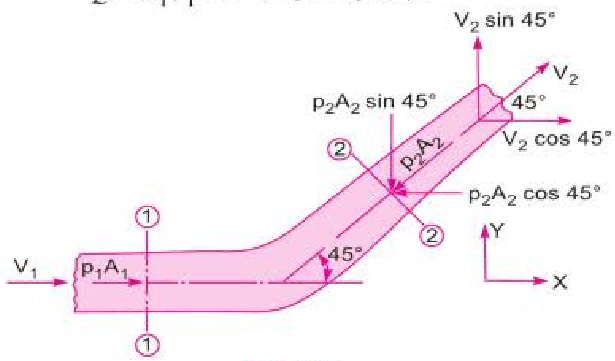


Fig. 6.24

Applying Bernoulli's equation at sections (1) and (2)

$$\therefore \qquad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \qquad \{\because \quad Z_1 = Z_2\}$$

or
$$\frac{2.943 \times 10^4}{1.16 \times 9.81} + \frac{10^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{20^2}{2 \times 9.81}$$

$$\therefore \qquad \frac{p_2}{\rho g} = \frac{2.943 \times 10^4}{1.16 \times 9.81} + \frac{10^2}{2 \times 9.81} - \frac{20^2}{2 \times 9.81}$$

$$= 2586.2 + 5.0968 - 20.387 = 2570.90 \text{ m}$$

$$\therefore \qquad p_2 = 2570.90 \times 1.16 \times 9.81 = 29255.8 \text{ N}$$

Force along x-axis, $\qquad F_x = \rho Q\, [V_{1x} - V_{2x}] + (p_1A_1)_x + (p_2A_2)_x$

where $\quad A_{1x} = 10$ m/s, $V_{2x} = V_2 \cos 45° = 20 \times .7071$,

$\quad (p_1A_1)_x = p_1A_1 = 29430 \times 1 = 29430$ N

and $\quad (p_2A_2)_x = -\,p_2A_2 \cos 45° = -\,29255.8 \times 0.5 \times .7071$

$$\therefore \qquad F_x = 1.16 \times 10[10 - 20 \times .7071] + 29430 \times 1 - 29255.8 \times .5 \times .7071$$

$$= -\,48.04 + 29430 - 10343.37 = 0 - 19038.59 \text{ N}$$

Similarly *force along y-axis,* $F_y = \rho Q[V_{1y} - V_{2y}] + (p_1A_1)_y + (p_2A_2)_y$

where $\quad V_{1y} = 0,\ V_{2y} = V_2 \sin 45° = 20 \times .7071 = 14.142$

$\quad (p_1A_1)_y = 0$ and $(p_2A_2)_y = -\,p_2A_2 \sin 45° = -\,29255.8 \times .5 \times .7071 = -\,10343.37$

$$F_y = 1.16 \times 10\ [0 - 14.142] + 0 - 10343.37$$

$$= -\,164.05 - 10343.37 = -\,10507.42 \text{ N}$$

$\therefore$ Resultant force, $\qquad F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(19038.6)^2 + (10507.42)^2} =$ **21746.6 N. Ans.**

The direction of F_R with x-axis is given as

$$\tan \theta = \frac{F_y}{F_x} = \frac{10507.42}{19038.6} = 0.5519$$

$\therefore \qquad \theta = \tan^{-1} .5519 =$ **28° 53′. Ans.**

F_R is the force exerted on bend. Hence the force required to hold the duct in position is equal to 21746.6 N but it is acting in the opposite direction of F_R. **Ans.**

Problem 6.33 *A pipe of 300 mm diameter conveying 0.30 m³/s of water has a right angled bend in a horizontal plane. Find the resultant force exerted on the bend if the pressure at inlet and outlet of the bend are 24.525 N/cm² and 23.544 N/cm².*

Solution. Given :

Dia. of bend, $D = 300 \text{ mm} = 0.3 \text{ m}$

$\therefore$ Area, $A = A_1 = A_2 = \dfrac{\pi}{4}(.3)^2 = 0.07068 \text{ m}^2$

$\therefore$ Discharge, $Q = 0.30 \text{ m}^3/\text{s}$

$\therefore$ Velocity, $V = V_1 = V_2 = \dfrac{Q}{A} = \dfrac{0.30}{.07068} = 4.244 \text{ m/s}$

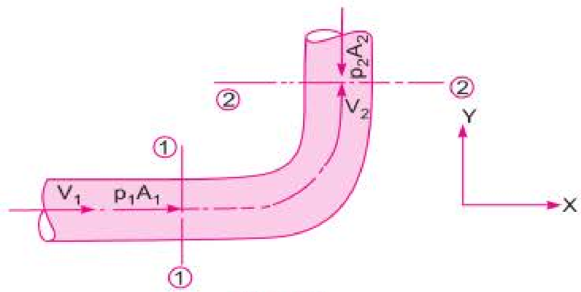


Fig. 6.25

Angle of bend, $\theta = 90°$

$$p_1 = 24.525 \text{ N/cm}^2 = 24.525 \times 10^4 \text{ N/m}^2 = 245250 \text{ N/m}^2$$

$$p_2 = 23.544 \text{ N/cm}^2 = 23.544 \times 10^4 \text{ N/m}^2 = 235440 \text{ N/m}^2$$

Force on bend along x-axis $F_x = \rho Q\,[V_{1x} - V_{2x}] + (p_1A_1)_x + (p_2A_2)_x$

where $\rho = 1000$, $V_{1x} = V_1 = 4.244$ m/s, $V_{2x} = 0$

$(p_1A_1)_x = p_1A_1 = 245250 \times .07068$

$(p_2A_2)_x = 0$

$$\therefore \quad F_x = 1000 \times 0.30\,[4.244 - 0] + 245250 \times .07068 + 0$$

$$= 1273.2 + 17334.3 = 18607.5 \text{ N}$$

Force on bend along y-axis, $F_y = \rho Q\,[V_{1y} - V_{2y}] + (p_1A_1)_y + (p_2A_2)_y$

where $V_{1y} = 0$, $V_{2y} = V_2 = 4.244$ m/s

$(p_1A_1)_y = 0$, $(p_2A_2)_y = -\,p_2A_2 = -\,235440 \times .07068 = -\,16640.9$

$$\therefore \quad F_y = 1000 \times 0.30[0 - 4.244] + 0 - 16640.9$$

$$= -\,1273.2 - 16640.9 = -\,17914.1 \text{ N}$$

$\therefore$ Resultant force, $F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(18607.5)^2 + (17914.1)^2} = -\,25829.3 \text{ N}$

and $\tan\theta = \dfrac{F_y}{F_x} = \dfrac{17914.1}{18607.5} = 0.9627$

$\therefore$ $\theta = $ **43° 54′. Ans.**

Problem 6.34 *A nozzle of diameter 20 mm is fitted to a pipe of diameter 40 mm. Find the force exerted by the nozzle on the water which is flowing through the pipe at the rate of 1.2 m^3/minute.*

Solution. Given :

Dia. of pipe, $D_1 = 40 \text{ mm} = 40 \times 10^{-3} \text{ m} = .04 \text{ m}$

$\therefore$ Area, $A_1 = \dfrac{\pi}{4} D_1^2 = \dfrac{\pi}{4}(.04)^2 = 0.001256 \text{ m}^2$

Dia. of nozzle, $D_2 = 20 \text{ mm} = 0.02 \text{ m}$

$\therefore$ Area, $A_2 = \dfrac{\pi}{4}(.02)^2 = .000314 \text{ m}^2$

Discharge, $Q = 1.2 \text{ m}^3/\text{minute} = \dfrac{1.2}{60} \text{ m}^3/\text{s} = 0.02 \text{ m}^3/\text{s}$

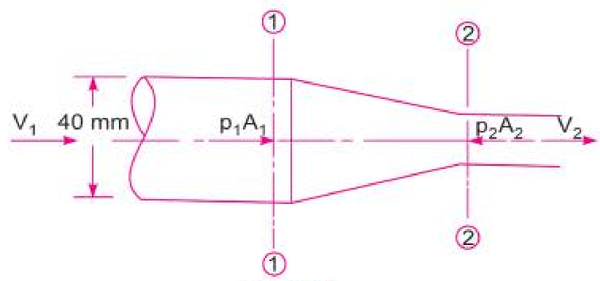


Fig. 6.26

Applying continuity equation at sections (1) and (2),

$$A_1V_1 = A_2V_2 = Q$$

$\therefore$
$$V_1 = \frac{Q}{A_1} = \frac{0.2}{.001256} = 15.92 \text{ m/s}$$

and
$$V_2 = \frac{Q}{A_2} = \frac{0.2}{.000314} = 63.69 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Now
$$z_1 = z_2, \ \frac{p_2}{\rho g} = \text{atmospheric pressure} = 0$$

$\therefore$
$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{V_2^2}{2g}$$

$\therefore$
$$\frac{p_1}{\rho g} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} = \frac{(63.69^2)}{2 \times 9.81} - \frac{(15.92^2)}{2 \times 9.81} = 206.749 - 12.917$$
$$= 193.83 \text{ m of water}$$

$\therefore$
$$p_1 = 193.83 \times 1000 \times 9.81 \ \frac{\text{N}}{\text{m}^2} = 1901472 \ \frac{\text{N}}{\text{m}^2}$$

Let the force exerted by the nozzle on water = F_x

Net force in the direction of x = rate of change of momentum in the direction of x

$$\therefore \qquad p_1A_1 - p_2A_2 + F_x = \rho Q(V_2 - V_1)$$

where p_2 = atmospheric pressure = 0 and ρ = 1000

$\therefore$ $1901472 \times .001256 - 0 + F_x = 1000 \times 0.02(63.69 - 15.92)$ or $2388.24 + F_x = 916.15$

$$\therefore \qquad F_x = -2388.24 + 916.15 = \mathbf{-1472.09.}\ \textbf{Ans.}$$

–ve sign indicates that the force exerted by the nozzle on water is acting from right to left.

Problem 6.35 *The diameter of a pipe gradually reduces from 1 m to 0.7 m as shown in Fig. 6.27. The pressure intensity at the centre-line of 1 m section 7.848 kN/m² and rate of flow of water through the pipe is 600 litres/s. Find the intensity of pressure at the centre-line of 0.7 m section. Also determine the force exerted by flowing water on transition of the pipe.*

Solution. Given :

Dia. of pipe at section 1, $D_1 = 1$ m

$\therefore$ Area, $A_1 = \dfrac{\pi}{4}(1)^2 = 0.7854\ \text{m}^2$

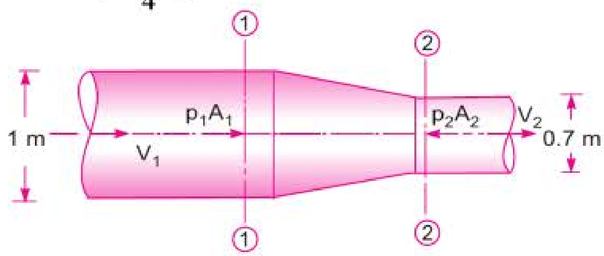


Fig. 6.27

Dia. of pipe at section 2, $D_2 = 0.7$ m

$\therefore$ Area, $A_2 = \dfrac{\pi}{4}(0.7)^2 = 0.3848\ \text{m}^2$

Pressure at section 1, $p_1 = 7.848\ \text{kN/m}^2 = 7848\ \text{N/m}^2$

Discharge, $Q = 600\ \text{litres/s} = \dfrac{600}{1000} = 0.6\ \text{m}^3/\text{s}$

Applying continuity equation,

$$A_1V_1 = A_2V_2 = Q$$

$$\therefore \qquad V_1 = \frac{Q}{A_2} = \frac{0.6}{0.7854} = 0.764\ \text{m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.6}{.3854} = 1.55\ \text{m/s}$$

Applying Bernoulli's equation at sections (1) and (2),

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \qquad \{\because \text{ pipe is horizontal, } \therefore\ z_1 = z_2\}$$

or
$$\frac{7848}{1000 \times 9.81} + \frac{(.764)^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{(1.55)^2}{2 \times 9.81}$$

$$\therefore \quad \frac{p_2}{\rho g} = 0.8 + \frac{(.764)^2}{2 \times 9.81} - \frac{(1.55)^2}{2 \times 9.81}$$

$$= 0.8 + 0.0297 - 0.122 = 0.7077 \text{ m of water}$$

$$\therefore \quad p_2 = 0.7077 \times 9.81 \times 1000$$

$$= \mathbf{6942.54 \ N/m^2 \ or \ 6.942 \ kN/m^2}. \textbf{ Ans.}$$

Let F_x = the force exerted by pipe transition on the flowing water in the direction of flow

Then net force in the direction of flow = rate of change of momentum in the direction of flow

or $\quad p_1A_1 - p_2A_2 + F_x = \rho(V_2 - V_1)$

$\therefore \quad 7848 \times .7854 - 6942.54 \times .3848 + F_x = 1000 \times 0.6[1.55 - .764]$

or $\quad 6163.8 - 2671.5 + F_x = 471.56$

$\therefore \quad F_x = 471.56 - 6163.8 + 2671.5 = -3020.74$ N

$\therefore$ The force exerted by water on pipe transition

$$= -F_x = -(-3020.74) = \mathbf{3020.74 \ N.} \textbf{ Ans.}$$

▶ 6.9 MOMENT OF MOMENTUM EQUATION

Moment of momentum equation is derived from moment of momentum principle which states that the resulting torque acting on a rotating fluid is equal to the rate of change of moment of momentum.

Let $\quad V_1$ = velocity of fluid at section 1,

r_1 = radius of curvature at section 1,

Q = rate of flow of fluid,

ρ = density of fluid,

and $\quad V_2$ and r_2 = velocity and radius of curvature at section 2

Momentum of fluid at section 1 = mass × velocity = $\rho Q \times V_1/s$

$\therefore$ Moment of momentum per second at section 1,

$$= \rho Q \times V_1 \times r_1$$

Similarly moment of momentum per second of fluid at section 2

$$= \rho Q \times V_2 \times r_2$$

$\therefore$ Rate of change of moment of momentum

$$= \rho Q V_2 r_2 - \rho Q V_1 r_1 = \rho Q[V_2 r_2 - V_1 r_1]$$

According to moment of momentum principle

Resultant torque = rate of change of moment of momentum

or
$$T = \rho Q[V_2 r_2 - V_1 r_1] \qquad \text{...(6.23)}$$

Equation (6.23) is known as moment of momentum equation. This equation is applied :

1. For analysis of flow problems in turbines and centrifugal pumps.
2. For finding torque exerted by water on sprinkler.

Problem 6.36 *A lawn sprinkler with two nozzles of diameter 4 mm each is connected across a tap of water as shown in Fig. 6.28. The nozzles are at a distance of 30 cm and 20 cm from the centre of the tap. The rate of flow of water through tap is 120 cm^3/s. The nozzles discharge water in the downward direction. Determine the angular speed at which the sprinkler will rotate free.*

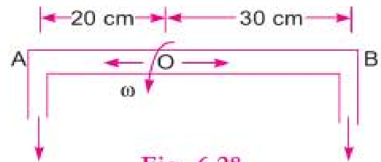


Fig. 6.28

Solution. Given :

Dia. of nozzles A and B,

$$D = D_A = D_B = 4 \text{ mm} = .004 \text{ m}$$

$\therefore$ Area,
$$A = \frac{\pi}{4}(.004)^2 = .00001256 \text{ m}^2$$

Discharge
$$Q = 120 \text{ cm}^3/\text{s}$$

Assuming the discharge to be equally divided between the two nozzles, we have

$$Q_A = Q_B = \frac{Q}{2} = \frac{120}{2} = 60 \text{ cm}^3/\text{s} = 60 \times 10^{-6} \text{ m}^3/\text{s}$$

$\therefore$ Velocity of water at the outlet of each nozzle,

$$V_A = V_B = \frac{Q_A}{A} = \frac{60 \times 10^{-6}}{.00001256} = 4.777 \text{ m/s.}$$

The jet of water coming out from nozzles A and B is having velocity 4.777 m/s. These jets of water will exert force in the opposite direction, *i.e.*, force exerted by the jets will be in the upward direction. The torque exerted will also be in the opposite direction. Hence torque at B will be in the anti-clockwise direction and at A in the clockwise direction. But torque at B is more than the torque at A and hence sprinkle, if free, will rotate in the anti-clockwise direction as shown in Fig. 6.28.

Let ω = angular velocity of the sprinkler.

Then absolute velocity of water at A,

$$V_1 = V_A + \omega \times r_A$$

where r_A = distance of nozzle A from the centre of tap

$= 20 \text{ cm} = 0.2 \text{ m}$ $\qquad \{\omega \times r_A$ = tangential velocity due to rotation$\}$

$$V_1 = (4.777 + \omega \times 0.2) \text{ m/s}$$

Here $\omega \times r_A$ is added to V_A as V_A and tangential velocity due to rotation $(\omega \times r_A)$ are in the same direction as shown in Fig. 6.28.

Similarly, absolute velocity of water at B,

$$V_2 = V_B - \text{tangential velocity due to rotation}$$
$$= 4.777 - \omega \times r_B \qquad \{\text{where } r_B = 30 \text{ cm} = 0.3 \text{ m}\}$$
$$= (4.777 - \omega \times 0.3)$$

Now applying equation (6.23), we get

$$T = \rho Q[V_2 r_2 - V_1 r_1] \qquad \text{Here } r_2 = r_B,\ r_1 = r_A$$
$$= \rho Q_A[V_2 r_B - V_1 r_A] \qquad Q = Q_A = Q_B$$
$$= 1000 \times 60 \times 10^{-6}\,[(4.777 \times 0.3\,\omega) \times .3 - (4.777 + 0.2\,\omega) \times .2]$$

The moment of momentum of the fluid entering sprinkler is given zero and also there is no external torque applied on the sprinkler. Hence resultant external torque is zero, *i.e.*, $T = 0$

$$\therefore \quad 1000 \times 60 \times 10^{-6}\,[(4.777 - 0.3\,\omega) \times .3 - (4.777 + 0.2\,\omega) \times .2] = 0$$

or
$$(4.777 - 0.3\,\omega) \times 0.3 - (4.777 + 0.2\,\omega) \times .2 = 0$$

or
$$4.777 \times .3 - .09\,\omega - 4.777 \times .2 - .04\,\omega = 0$$

or
$$0.1 \times 4.777 = (.09 + .04)\omega = .13\,\omega$$

$$\therefore \quad \omega = \frac{.4777}{0.13} = \mathbf{3.6746\ rad/s.}\ \textbf{Ans.}$$

Problem 6.37 *A lawn sprinkler shown in Fig. 6.29 has 0.8 cm diameter nozzle at the end of a rotating arm and discharges water at the rate of 10 m/s velocity. Determine the torque required to hold the rotating arm stationary. Also determine the constant speed of rotation of the arm, if free to rotate.*

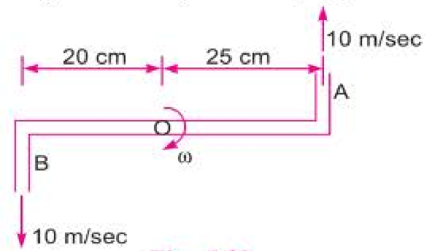


Fig. 6.29

Solution. Dia. of each nozzle $= 0.8 \text{ cm} = .008 \text{ m}$

$\therefore$ Area of each nozzle $= \frac{\pi}{4}(.008)^2 = .00005026 \text{ m}^2$

Velocity of flow at each nozzle = 10 m/s.

$\therefore$ Discharge through each nozzle,

$$Q = \text{Area} \times \text{Velocity}$$
$$= .00005026 \times 10 = .0005026 \text{ m}^3/\text{s}$$

Torque exerted by water coming through nozzle A on the sprinkler = moment of momentum of water through A

$$= r_A \times \rho \times Q \times V_A = 0.25 \times 1000 \times .0005026 \times 10 \text{ clockwise}$$

Torque exerted by water coming through nozzle B on the sprinkler

$$= r_B \times \rho \times Q \times V_B = 0.20 \times 1000 \times .0005026 \times 10 \text{ clockwise}$$

$\therefore$ Total torque exerted by water on sprinkler

$$= .25 \times 1000 \times .0005026 \times 10 + .20 \times 1000 \times .0005026 \times 10$$
$$= 1.2565 + 1.0052 = 2.26 \text{ Nm}$$

$\therefore$ Torque required to hold the rotating arm stationary = Torque exerted by water on sprinkler

$$= \mathbf{2.26 \text{ Nm.}} \textbf{ Ans.}$$

Speed of rotation of arm, if free to rotate

Let ω = speed of rotation of the sprinkler

The absolute velocity of flow of water at the nozzles A and B are

$$V_1 = 10.0 - 0.25 \times \omega \text{ and } V_2 = 10.0 - 0.20 \times \omega$$

Torque exerted by water coming out at A, on sprinkler

$$= r_A \times \rho \times Q \times V_1 = 0.25 \times 1000 \times .0005026 \times (10 - 0.25\ \omega)$$
$$= 0.12565\ (10 - 0.25\ \omega)$$

Torque exerted by water coming out at B, on sprinkler

$$= r_B \times \rho \times Q \times V_2 = 0.20 \times 1000 \times .0005026 \times (10.0 - 0.2\ \omega)$$
$$= 0.10052\ (10.0 - 0.2\ \omega)$$

$\therefore$ Total torque exerted by water $= 0.12565\ (10.0 - 0.25\ \omega) + 0.10052\ (10.0 - 0.2\ \omega)$

Since moment of momentum of the flow entering is zero and no external torque is applied on sprinkler, so the resultant torque on the sprinkler must be zero.

$$\therefore\ 0.12565\ (10.0 - 0.25\ \omega) + 0.10052(10.0 - 0.2\ \omega) = 0$$
$$1.2565 - 0.0314\ \omega + 1.0052 - 0.0201\ \omega = 0$$
$$1.2565 + 1.0052 = \omega\ (0.0314 + 0.0201)$$
$$2.2617 = 0.0515\ \omega$$

$$\therefore \qquad \omega = \frac{2.2617}{0.0515} = \mathbf{43.9 \text{ rad/s.}} \textbf{ Ans.}$$

and $$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 43.9}{2\pi} = \textbf{419.2 r.p.m.} \text{ Ans.}$$

▶ 6.10 FREE LIQUID JETS

Free liquid jet is defined as the jet of water coming out from the nozzle in atmosphere. The path travelled by the free jet is parabolic.

Consider a jet coming from the nozzle as shown in Fig. 6.30. Let the jet at A, makes an angle θ with the horizontal direction. If U is the velocity of jet of water, then the horizontal component and vertical component of this velocity at A are $U \cos \theta$ and $U \sin \theta$.

Consider another point $P(x, y)$ on the centre line of the jet. The co-ordinates of P from A are x and y. Let the velocity of jet at P in the x- and y-directions are u and v. Let a liquid particle takes time 't' to reach from A to P. Then the horizontal and vertical distances travelled by the liquid particle in time 't' are :

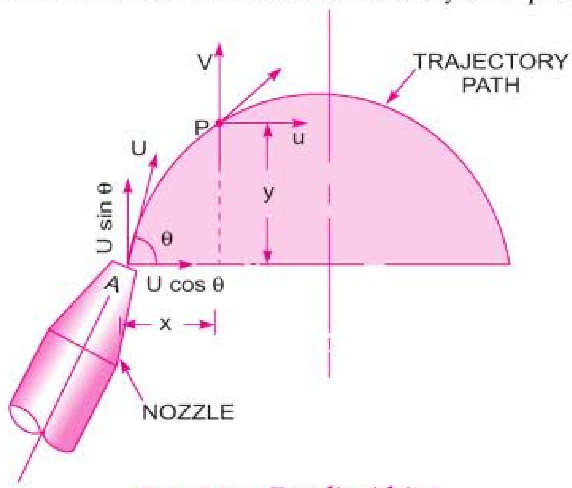


Fig. 6.30 *Free liquid jet.*

$$x = \text{velocity component in } x\text{-direction} \times t$$
$$= U \cos \theta \times t \qquad ...(i)$$

and $$y = (\text{vertical component in } y\text{-direction} \times \text{time} - \frac{1}{2} gt^2)$$
$$= U \sin \theta \times t - \frac{1}{2} gt^2 \qquad ...(ii)$$

{$\because$ Horizontal component of velocity is constant while the vertical distance is affected by gravity}

From equation (i), the value of t is given as $t = \dfrac{x}{U \cos \theta}$

Substituting this value in equation (ii)

$$y = U \sin \theta \times \frac{x}{U \cos \theta} - \frac{1}{2} \times g \times \left(\frac{x}{U \cos \theta}\right)^2 = x \frac{\sin \theta}{\cos \theta} - \frac{gx^2}{2U^2 \cos^2 \theta}$$
$$= x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta \qquad \left\{\because \quad \frac{1}{\cos^2 \theta} = \sec^2 \theta\right\} \quad ...(6.24)$$

Equation (6.24) gives the variation of y with the square of x. Hence this is the equation of a parabola. Thus the path travelled by the free jet in atmosphere is parabolic.

(*i*) **Maximum height attained by the jet.** Using the relation $V_2^2 - V_1^2 = -2gS$, we get in this case $V_1 = 0$ at the highest point

$$V_1 = \text{Initial vertical component}$$
$$= U \sin \theta$$

–ve sign on right hand side is taken as g is acting in the downward direction but particles is moving up.

$$\therefore \quad 0 - (U \sin \theta)^2 = -2g \times S$$

where S is the maximum vertical height attained by the particle.

or $$-U^2 \sin^2\theta = -2gS$$

$$\therefore \quad S = \frac{U^2 \sin^2 \theta}{2g} \qquad ...(6.25)$$

(*ii*) **Time of flight.** It is the time taken by the fluid particle in reaching from A to B as shown in Fig. 6.30. Let T is the time of flight.

Using equation (*ii*), we have $y = U \sin \theta \times t - \frac{1}{2} gt^2$

when the particle reaches at B, $y = 0$ and $t = T$

$\therefore$ Above equation becomes as $0 = U \sin \theta \times T - \frac{1}{2} g \times T^2$

or $$0 = U \sin \theta - \frac{1}{2} gT \qquad \{\text{Cancelling } T\}$$

or $$T = \frac{2U \sin \theta}{g} \qquad ...(6.26)$$

(*iii*) **Time to reach highest point.** The time to reach highest point is half the time of flight. Let T^* is the time to reach highest point, then

$$T^* = \frac{T}{2} = \frac{2U \sin \theta}{g \times 2} = \frac{U \sin \theta}{g} \qquad ...(6.27)$$

(*iv*) **Horizontal range of the jet.** The total horizontal distance travelled by the fluid particle is called horizontal range of the jet, *i.e.*, the horizontal distance AB in Fig. 6.30 is called horizontal range of the jet. Let this range is denoted by x^*.

Then x^* = velocity component in x-direction × time taken by the particle to reach from A to B

$= U \cos \theta \times$ Time of flight

$$= U \cos \theta \times \frac{2U \sin \theta}{g} \qquad \left\{\because T = \frac{2U \sin \theta}{g}\right\}$$

$$= \frac{U^2}{g} 2 \cos \theta \sin \theta = \frac{U^2}{g} \sin 2\theta \qquad ...(6.28)$$

(*v*) **Value of θ for maximum range.** The range x^* will be maximum for a given velocity of projection (U), when $\sin 2\theta$ is maximum

or when $$\sin 2\theta = 1 \text{ or } \sin 2\theta = \sin 90° = 1$$

$$\therefore \quad 2\theta = 90° \text{ or } \theta = 45°$$

Then maximum range, $$x^*_{max} = \frac{U^2}{g} \sin^2 \theta = \frac{U^2}{g} \qquad \{\because \sin 90° = 1\} \quad ...(6.29)$$

Problem 6.38 *A vertical wall is of 8 m in height. A jet of water is coming out from a nozzle with a velocity of 20 m/s. The nozzle is situated at a distance of 20 m from the vertical wall. Find the angle of projection of the nozzle to the horizontal so that the jet of water just clears the top of the wall.*

Solution. Given :

Height of wall $= 8$ m

Velocity of jet, $U = 20$ m/s

Distance of jet from wall, $x = 20$ m

Let the required angle $= \theta$

Using equation (6.24), we have

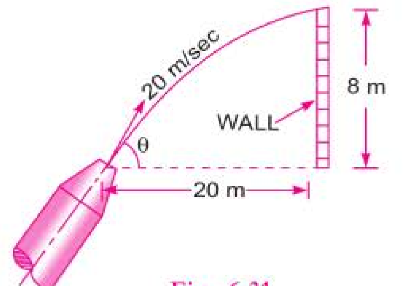


Fig. 6.31

$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

where $y = 8$ m, $x = 20$ m, $U = 20$ m/s

$$8 = 20 \tan \theta - \frac{9.81 \times 20^2}{2 \times 20^2} \sec^2 \theta$$

$$= 20 \tan \theta - 4.905 \sec^2 \theta$$

$$= 20 \tan \theta - 4.905 [1 + \tan^2 \theta] \qquad \{\because \ \sec^2 \theta = 1 + \tan^2 \theta\}$$

$$= 20 \tan \theta - 4.905 - 4.905 \tan^2 \theta$$

or $\quad 4.905 \tan^2 \theta - 20 \tan \theta + 8 + 4.905 = 0$

or $\quad 4.905 \tan^2 \theta - 20 \tan \theta + 12.905 = 0$

$$\therefore \quad \tan \theta = \frac{20 \pm \sqrt{20^2 - 4 \times 12.905 \times 4.905}}{2 \times 4.905} = \frac{20 \pm \sqrt{400 - 253.19}}{9.81}$$

$$= \frac{20 \pm \sqrt{146.81}}{9.81} = \frac{20 \pm 12.116}{9.81} = \frac{32.116}{9.81} \text{ or } \frac{7.889}{9.81}$$

$$\therefore \quad = 3.273 \text{ or } 0.8036$$

$$\therefore \quad \theta = \mathbf{73^\circ\ 0.8' \text{ or } 38^\circ\ 37'}. \text{ Ans.}$$

Problem 6.39 *A fire-brigade man is holding a fire stream nozzle of 50 mm diameter as shown in Fig. 6.32. The jet issues out with a velocity of 13 m/s and strikes the window. Find the angle or angles of inclination with which the jet issues from the nozzle. What will be the amount of water falling on the window ?*

Solution. Given :

Dia. of nozzle, $d = 50$ mm $= .05$ m

$\therefore$ Area, $A = \frac{\pi}{4}(.05)^2 = 0.001963 \text{ m}^2$

Velocity of jet, $U = 13$ m/s.

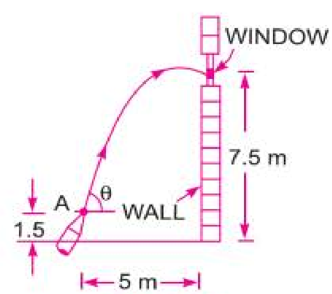


Fig. 6.32

The jet is coming out from nozzle at A. It strikes the window and let the angle made by the jet at A with horizontal is equal to θ.

The co-ordinates of window, with respect to origin at A.

$$x = 5 \text{ m}, y = 7.5 - 1.5 = 6.0 \text{ m}$$

The equation of the jet is given by (6.24) as

$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

or $$6.0 = 5 \times \tan \theta - \frac{9.81 \times 5}{2 \times 13^2} [1 + \tan^2 \theta] \qquad \{\because \quad \sec^2 \theta = 1 + \tan^2 \theta\}$$

or $$6.0 = 5 \tan \theta - .7256 (1 + \tan^2 \theta)$$

$$= 5 \tan \theta - .7256 - .7256 \tan^2 \theta$$

or $$0.7256 \tan^2 \theta - 5 \tan \theta + 6 + .7256 = 0$$

or $$0.7256 \tan^2 \theta - 5 \tan \theta + 6.7256 = 0$$

This is a quadratic equation in tan θ. Hence solution is

$$\tan \theta = \frac{5 \pm \sqrt{5^2 - 4 \times .7256 \times 6.7256}}{2 \times .7256}$$

$$= \frac{5 \pm \sqrt{25 - 19.52}}{1.4512} = \frac{5 + 2.341}{1.4512} = 5.058 \text{ or } 1.8322$$

∴ $$\theta = \tan^{-1} 5.058 \text{ or } \tan^{-1} 1.8322 = \mathbf{78.8° \text{ or } 61.37°}. \text{ Ans.}$$

Amount of water falling on window = Discharge from nozzle

= Area of nozzle × Velocity of jet at nozzle

= $0.001963 \times U = 0.001963 \times 13.0 =$ **0.0255 m³/s. Ans.**

Problem 6.40 *A nozzle is situated at a distance of 1 m above the ground level and is inclined at an angle of 45° to the horizontal. The diameter of the nozzle is 50 mm and the jet of water from the nozzle strikes the ground at a horizontal distance of 4 m. Find the rate of flow of water.*

Solution. Given :

Distance of nozzle above ground = 1 m

Angle of inclination, $\theta = 45°$

Dia. of nozzle, $d = 50 \text{ mm} = .05 \text{ m}$

∴ Area, $A = \frac{\pi}{4}(.05)^2 = .001963 \text{ m}^2$

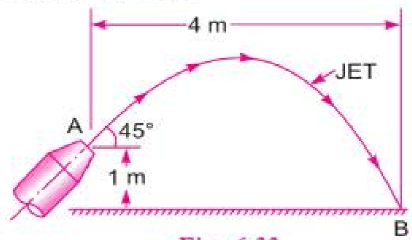


Fig. 6.33

The horizontal distance $x = 4$ m

The co-ordinates of the point B, which is on the centre-line of the jet of water and is situated on the ground, with respect to A (origin) are

$x = 4$ m and $y = -1.0$ m {From A, point B is vertically down by 1 m}

The equation of the jet is given by (6.24) as $y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$

Substituting the known values as

$$-1.0 = 4 \tan 45° - \frac{9.81 \times 4^2}{2U^2} \times \sec^2 45°$$

$$= 4 - \frac{78.48}{U^2} \times \left(\sqrt{2}\right)^2 \qquad \left\{\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}\right\}$$

$$-1.0 = 4 - \frac{78.48 \times 2}{U^2} \quad \text{or} \quad \frac{78.48 \times 2}{U^2} = +4.0 + 1.0 = 5.0$$

$$\therefore \qquad U^2 = \frac{78.48 \times 2.0}{5.0} = 31.39$$

$$\therefore \qquad U = \sqrt{31.39} = 5.60 \text{ m/s}$$

Now the rate of flow of fluid = Area × Velocity of jet

$= A \times U = .001963 \times 5.6$ m^3/sec

$= 0.01099 \approx$ **.011 m^3/s. Ans.**

Problem 6.41 *A window, in a vertical wall, is at a distance of 30 m above the ground level. A jet of water, issuing from a nozzle of diameter 50 mm is to strike the window. The rate of flow of water through the nozzle is 3.5 m^3/minute and nozzle is situated at a distance of 1 m above ground level. Find the greatest horizontal distance from the wall of the nozzle so that jet of water strikes the window.*

Solution. Given :

Distance of window from ground level = 30 m

Dia. of nozzle, $d = 50$ mm $= 0.05$ m

$\therefore$ Area $A = \frac{\pi}{4}(.05)^2 = 0.001963$ m^2

The discharge, $Q = 3.5$ m^3/minute

$= \frac{3.5}{60} = 0.0583$ m^3/s

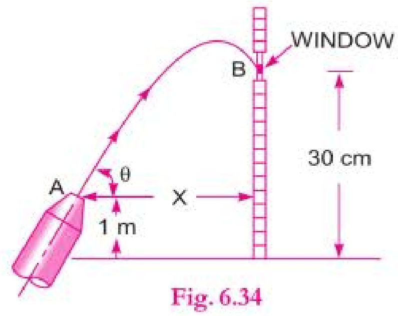


Fig. 6.34

Distance of nozzle from ground = 1 m.

Let the greatest horizontal distance of the nozzle from the wall = x and let angle of inclination = θ. If the jet reaches the window, then the point B on the window is on the centre-line of the jet. The co-ordinates of B with respect to A are

$$x = x,\ y = 30 - 1.0 = 29 \text{ m}$$

The velocity of jet, $U = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{A} = \frac{.0583}{.001963} = 29.69$ m/sec

Using the equation (6.34), which is the equation of jet,

$$y = x \tan\theta - \frac{gx^2}{2U^2}\sec^2\theta$$

or

$$29.0 = x\tan\theta - \frac{9.81x^2}{2 \times (29.69)^2}\sec^2\theta$$

$$= x\tan\theta - 0.0055\sec^2\theta \times x^2$$

$$= x\tan\theta - \frac{.0055\,x^2}{\cos^2\theta}$$

$$x\tan\theta - .0055\,x^2/\cos^2\theta - 29 = 0 \qquad \ldots(i)$$

The maximum value of x with respect to θ is obtained, by differentiating the above equation w.r.t. θ and substituting the value of $\frac{dx}{d\theta} = 0$. Hence differentiating the equation (i) w.r.t. θ, we have

$$\left[x\sec^2\theta + \tan\theta \times \frac{dx}{d\theta}\right] - 0.0055\left[x^2 \times \left(\frac{(-2)}{\cos^3\theta}\right)(-\sin\theta) + \frac{1}{\cos^2\theta} \times 2x\frac{dx}{d\theta}\right]$$

$$\left\{\because \frac{d}{d\theta}(x\tan\theta) = x\sec^2\theta + \tan\theta\frac{dx}{d\theta} \text{ and } \frac{d}{d\theta}\left(\frac{x^2}{\cos^2\theta}\right) = x^2\frac{d}{d\theta}\left(\frac{1}{\cos^2\theta}\right) + \frac{1}{\cos^2\theta}\frac{d}{d\theta}\left(x^2\right)\right\}$$

$$\therefore \qquad x\sec^2\theta + \tan\theta\frac{dx}{d\theta} - .0055\left[\frac{2x^2\sin\theta}{\cos^3\theta} + \frac{2x}{\cos^2\theta}\frac{dx}{d\theta}\right] = 0$$

For maximum value of x, w.r.t. θ, we have $\frac{dx}{d\theta} = 0$

Substituting this value in the above equation, we have

$$x\sec^2\theta - .0055\left[\frac{2x^2\sin\theta}{\cos^3\theta}\right] = 0$$

or $$\frac{x}{\cos^2\theta} - \frac{.0055\times 2x^2\sin\theta}{\cos^3\theta} = 0 \quad \text{or} \quad x - .011\times x^2\frac{\sin\theta}{\cos\theta} = 0$$

or $$x - .011\,x^2\tan\theta = 0 \quad \text{or} \quad 1 - .011\,x\tan\theta = 0$$

or $$x\tan\theta = \frac{1}{.011} = 90.9 \qquad ...(ii)$$

or $$x = \frac{90.9}{\tan\theta} \qquad ...(iii)$$

Substituting this value of x in equation (i), we get

$$\frac{90.9}{\tan\theta}\times\tan\theta - .0055\times\frac{(90.9)^2}{\tan^2\theta}\times\frac{1}{\cos^2\theta} - 29 = 0$$

$$90.9 - \frac{45.445}{\sin^2\theta} - 29 = 0 \quad \text{or} \quad 61.9 - \frac{45.445}{\sin^2\theta} = 0$$

or $$61.9 = \frac{45.445}{\sin^2\theta} \quad \text{or} \quad \sin^2\theta = \frac{45.445}{61.90} = 0.7341$$

$$\therefore \qquad \sin\theta = \sqrt{0.7341} = 0.8568$$

$$\therefore \qquad \theta = \tan^{-1} .8568 = 58°\ 57.8'$$

Substituting this value of θ in equation (iii), we get

$$x = \frac{90.9}{\tan\theta} = \frac{90.9}{\tan 58°\ 57.8'} = \frac{90.9}{\tan 58.95} = \frac{90.9}{1.66} = 54.759 \text{ m}$$

= 54.76 m. Ans.

HIGHLIGHTS

1. The study of fluid motion with the forces causing flow is called dynamics of fluid flow, which is analysed by the Newton's second law of motion.
2. Bernoulli's equation is obtained by integrating the Euler's equation of motion. Bernoulli's equation states "For a steady, ideal flow of an incompressible fluid, the total energy which consists of pressure energy, kinetic energy and datum energy, at any point of the fluid is constant". Mathematically,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

where $\frac{p_1}{\rho g}$ = pressure energy per unit weight = pressure head

$\frac{v_1^2}{2g}$ = kinetic energy per unit weight = kinetic head

z_1 = datum energy per unit weight = datum head.

3. Bernoulli's equation for real fluids

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

where h_L = loss of energy between sections 1 and 2.

4. The discharge, Q, through a venturimeter or an orifice meter is given by

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

where a_1 = area at the inlet of venturimeter,
a_2 = area at the throat of venturimeter,
C_d = co-efficient of venturimeter,
h = difference of pressure head in terms of fluid head flowing through venturimeter.

5. The value of h is given by differential U-tube manometer

$$h = x\left[\frac{S_h}{S_o} - 1\right]$$ ... (when differential manometer contains heavier liquid)

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$ ...(when differential manometer contains lighter liquid)

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right]$$ ...(for inclined venturimeter in which differential manometer contains heavier liquid)

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right]$$ ...(for inclined venturimeter in which differential manometer contains lighter liquid)

where x = difference in the readings of differential manometer,
S_h = sp. gr. of heavier liquid
S_o = sp. gr. of fluid flowing through venturimeter
S_l = sp. gr. of lighter liquid.

6. Pitot-tube is used to find the velocity of a flowing fluid at any point in a pipe or a channel. The velocity is given by the relation

$$V = C_v \sqrt{2gh}$$

where C_v = co-efficient of Pitot-tube

h = rise of liquid in the tube above free surface of liquid

$$= x\left[\frac{S_g}{S_o} - 1\right] \text{ (for pipes or channels).}$$

7. The momentum equation states that the net force acting on a fluid mass is equal to the change in momentum per second in that direction. This is given as $F = \frac{d}{dt}(mv)$

The impulse-momentum equation is given by $F \,.\, dt = d(mv)$.

8. The force exerted by a fluid on a pipe bend in the directions of x and y are given by

$$F_x = \frac{\text{mass}}{\text{sec}}(\text{Initial velocity in the direction of } x - \text{Final velocity in } x\text{-direction})$$

$$+ \text{ Initial pressure force in } x\text{-direction} + \text{Final pressure force in } x\text{-direction}$$

$$= \rho Q[V_{1x} - V_{2x}] + (p_1A_1)_x + (p_2A_2)_x$$

and $$F_y = \rho Q[V_{1y} - V_{2y}] + (p_1A_1)_y + (p_2A_2)_y$$

Resultant force, $$F_R = \sqrt{F_x^2 + F_y^2}$$

and the direction of the resultant with horizontal is $\tan\theta = \frac{F_y}{F_x}$.

9. The force exerted by the nozzle on the water is given by $F_x = \rho Q[V_{2x} - V_{1x}]$ and force exerted by the water on the nozzle is $= -F_x = \rho Q[V_{1x} - V_{2x}]$.

10. Moment of momentum equation states that the resultant torque acting on a rotating fluid is equal to the rate of change of moment of momentum. Mathematically, it is given by $T = \rho Q[V_2r_2 - V_1r_1]$.

11. Free liquid jet is the jet of water issuing from a nozzle in atmosphere. The path travelled by the free jet is parabolic. The equation of the jet is given by

$$y = x\tan\theta - \frac{gx^2}{2U^2}\sec^2\theta$$

where x, y = co-ordinates of any point on jet w.r.t. to the nozzle

U = velocity of jet of water issuing from nozzle

θ = inclination of jet issuing from nozzle with horizontal.

12. (*i*) Maximum height attained by jet $= \frac{U^2\sin^2\theta}{2g}$

(*ii*) Time of flight, $T = \frac{2U\sin\theta}{g}$

(*iii*) Time to reach highest point, $T^* = \frac{T}{2} = \frac{U\sin\theta}{2g}$

(*iv*) Horizontal range of the jet, $x^* = \frac{U^2}{g}\sin 2\theta$

(*v*) Value of θ for maximum range, $\theta = 45°$

(*vi*) Maximum range, $x^*_{max} = U^2/g$.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Name the different forces present in a fluid flow. For the Euler's equation of motion, which forces are taken into consideration.
2. What is Euler's equation of motion ? How will you obtain Bernoulli's equation from it ?
3. Derive Bernoulli's equation for the flow of an incompressible frictionless fluid from consideration of momentum.
4. State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's theorem from first principle and state the assumptions made for such a derivation.
5. What is a venturimeter ? Derive an expression for the discharge through a venturimeter.
6. Explain the principle of venturimeter with a neat sketch. Derive the expression for the rate of flow of fluid through it.
7. Discuss the relative merits and demerits of venturimeter with respect to orifice-meter. *(Delhi University, Dec. 2002)*
8. Define an orifice-meter. Prove that the discharge through an orifice-meter is given by the relation

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

where a_1 = area of pipe in which orifice-meter is fitted
a_0 = area of orifice *(Technical University of M.P., S 2002)*

9. What is a pitot-tube ? How will you determine the velocity at any point with the help of pitot-tube ? *(Delhi University, Dec. 2002)*
10. What is the difference between pitot-tube and pitot-static tube ?
11. State the momentum equation. How will you apply momentum equation for determining the force exerted by a flowing liquid on a pipe bend ?
12. What is the difference between momentum equation and impulse momentum equation.
13. Define moment of momentum equation. Where this equation is used.
14. What is a free jet of liquid ? Derive an expression for the path travelled by free jet issuing from a nozzle.
15. Prove that the equation of the free jet of liquid is given by the expression,

$$y = x \tan\theta - \frac{gx^2}{2U^2}\sec^2\theta$$

where x, y = co-ordinates of a point on the jet
U = velocity of issuing jet
θ = inclination of the jet with horizontal.

16. Which of the following statement is correct in case of pipe flow :
    (*a*) flow takes place from higher pressure to lower pressure ;
    (*b*) flow takes place from higher velocity to lower velocity ;
    (*c*) flow takes place from higher elevation to lower elevation ;
    (*d*) flow takes place from higher energy to lower energy.
17. Derive Euler's equation of motion along a stream line for an ideal fluid stating clearly the assumptions. Explain how this is integrated to get Bernoulli's equation along a stream-line.
18. State Bernoulli's theorem. Mention the assumptions made. How is it modified while applying in practice? List out its engineering applications.
19. Define continuity equation and Bernoulli's equation.

20. What are the different forms of energy in a flowing fluid ? Represent schematically the Bernoulli's equation for flow through a tapering pipe and show the position of total energy line and the datum line.
21. Write Euler's equation of motion along a stream line and integrate it to obtain Bernoulli's equation. State all assumptions made.
22. Describe with the help of sketch the construction, operation and use of Pitot-static tube.
23. Starting with Euler's equation of motion along a stream line, obtain Bernoulli's equation by its integration. List all the assumptions made.
24. State the different devices that one can use to measure the discharge through a pipe and also through an open channel. Describe one of such devices with a neat sketch and explain how one can obtain the actual discharge with its help?
25. Derive Bernoulli's equation from fundamentals.

(B) NUMERICAL PROBLEMS

1. Water is flowing through a pipe of 100 mm diameter under a pressure of 19.62 N/cm^2 (gauge) and with mean velocity of 3.0 m/s. Find the total head of the water at a cross-section, which is 8 m above the datum line. [**Ans.** 28.458 m]
2. A pipe, through which water is flowing is having diameters 40 cm and 20 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 5.0 m/s. Find the velocity head at the sections 1 and 2 and also rate of discharge. [**Ans.** 1.274 m ; 20.387 m ; 0.628 m^3/s]
3. The water is flowing through a pipe having diameters 20 cm and 15 cm at sections 1 and 2 respectively. The rate of flow through pipe is 40 litres/s. The section 1 is 6 m above datum line and section 2 is 3 m above the datum. If the pressure at section 1 is 29.43 N/cm^2, find the intensity of pressure at section 2. [**Ans.** 32.19 N/cm^2]
4. Water is flowing through a pipe having diameters 30 cm and 15 cm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 29.43 N/cm^2 and the pressure at the upper end is 14.715 N/cm^2. Determine the difference in datum head if the rate of flow through pipe is 50 lit/s. [**Ans.** 14.618 m]
5. The water is flowing through a taper pipe of length 50 m having diameters 40 cm at the upper end and 20 cm at the lower end, at the rate of 60 litres/s. The pipe has a slope of 1 in 40. Find the pressure at the lower end if the pressure at the higher level is 24.525 N/cm^2. [**Ans.** 25.58 N/cm^2]
6. A pipe of diameter 30 cm carries water at a velocity of 20 m/sec. The pressures at the points A and B are given as 34.335 N/cm^2 and 29.43 N/cm^2 respectively, while the datum head at A and B are 25 m and 28 m. Find the loss of head between A and B. [**Ans.** 2 m]
7. A conical tube of length 3.0 m is fixed vertically with its smaller end upwards. The velocity of flow at the smaller end is 4 m/s while at the lower end it is 2 m/s. The pressure head at the smaller end is 2.0 m of liquid. The loss of head in the tube is $0.95\,(v_1 - v_2)^2/2g$, where v_1 is the velocity at the smaller end and v_2 at the lower end respectively. Determine the pressure head at the lower end. Flow takes place in downward direction. [**Ans.** 5.56 m of fluid]
8. A pipe line carrying oil of specific gravity 0.8, changes in diameter from 300 mm at a position A to 500 mm diameter to a position B which is 5 m at a higher level. If the pressures at A and B are 19.62 N/cm^2 and 14.91 N/cm^2 respectively, and the discharge is 150 litres/s, determine the loss of head and direction of flow. [**Ans.** 1.45 m, Flow takes place from A to B]
9. A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to inlet and throat is 10 cm of mercury. Determine the rate of flow. Take $C_d = 0.98$. [**Ans.** 88.92 litres/s]

10. An oil of sp. gr. 0.9 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 20 cm. Calculate the discharge of oil through the horizontal venturimeter. Take $C_d = 0.98$. [**Ans.** 59.15 litres/s]

11. A horizontal venturimeter with inlet diameter 30 cm and throat diameter 15 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 50 litres/s, find the reading of the oil-mercury differential manometer. Take $C_d = 0.98$. [**Ans.** 2.489 cm]

12. A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of water. The pressure at inlet is 14.715 N/cm^2 and vacuum pressure at the throat is 40 cm of mercury. Find the discharge of water through venturimeter. [**Ans.** 162.539 lit./s]

13. A 30 cm × 15 cm venturimeter is inserted in a vertical pipe carrying water, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 30 cm. Find the discharge. Take $C_d = 0.98$. [**Ans.** 154.02 lit/s]

14. If in the problem 13, instead of water, oil of sp. gr. 0.8 is flowing through the venturimeter, determine the rate of flow of oil in litres/s. [**Ans.** 173.56 lit/s]

15. The water is flowing through a pipe of diameter 30 cm. The pipe is inclined and a venturimeter is inserted in the pipe. The diameter of venturimeter at throat is 15 cm. The difference of pressure between the inlet and throat of the venturimeter is measured by a liquid of sp. gr. 0.8 in an inverted *U*-tube which gives a reading of 40 cm. The loss of head between the inlet and throat is 0.3 times the kinetic head of the pipe. Find the discharge. [**Ans.** 22.64 lit./s]

16. A 20 × 10 cm venturimeter is provided in a vertical pipe line carrying oil of sp. gr. 0.8, the flow being upwards. The difference in elevation of the throat section and entrance section of the venturimeter is 50 cm. The differential *U*-tube mercury manometer shows a gauge deflection of 40 cm. Calculate : (*i*) the discharge of oil, and (*ii*) the pressure difference between the entrance section and the throat section. Take $C_d = 0.98$ and sp. gr. of mercury as 13.6. [**Ans.** (*i*) 89.132 lit/s, (*ii*) 5.415 N/cm^2]

17. In a 200 mm diameter horizontal pipe a venturimeter of 0.5 contraction ratio has been fixed. The head of water on the venturimeter when there is no flow is 4 m (gauge). Find the rate of flow for which the throat pressure will be 4 metres of water absolute. Take $C_d = 0.97$ and atmospheric pressure head = 10.3 m of water. [**Ans.** 111.92 lit/s]

18. An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter give readings of 14.715 N/cm^2 and 9.81 N/cm^2 respectively. Find the rate of flow of water through the pipe in litres/s. Take $C_d = 0.6$. [**Ans.** 108.434 lit/s]

19. If in problem 18, instead of water, oil of sp. gr. 0.8 is flowing through the orifice meter in which the pressure difference is measured by a mercury oil differential manometer on the two sides of the orifice meter, find the rate of flow of oil when the reading of manometer is 40 cm. [**Ans.** 122.68 lit/s]

20. The pressure difference measured by the two tappings of a pitot-static tube, one tapping pointing upstream and other perpendicular to the flow, placed in the centre of a pipe line of diameter 40 cm is 10 cm of water. The mean velocity in the pipe is 0.75 times the central velocity. Find the discharge through the pipe. Take co-efficient of pitot-tube as 0.98. [**Ans.** 0.1293 m^3/s]

21. Find the velocity of flow of an oil through a pipe, when the difference of mercury level in a differential *U*-tube manometer connected to the two tappings of the pitot-tube is 15 cm. Take sp. gr. of oil = 0.8 and co-efficient of pitot-tube as 0.98. [**Ans.** 6.72 m/s]

22. A sub-marine moves horizontally in sea and has its axis 20 m below the surface of water. A pitot-static tube placed in front of sub-marine and along its axis, is connected to the two limbs of a *U*-tube containing mercury. The difference of mercury level is found to be 20 cm. Find the speed of sub-marine. Take sp. gr. of mercury 13.6 and of sea-water 1.026. [**Ans.** 24.958 km/hr.]

23. A 45° reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 40 cm and 20 cm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet of bend is 21.58 N/cm^2. The rate of flow of water is 500 litres/s. [**Ans.** 22696.5 N ; 20° 3.5′]

24. The discharge of water through a pipe of diameter 40 cm is 400 litres/s. If the pipe is bend by 135°, find the magnitude and direction of the resultant force on the bend. The pressure of flowing water is 29.43 N/cm^2. [**Ans.** 7063.2 N, $\theta = 22° 29.9'$ with x-axis clockwise]

25. A 30 cm diameter pipe carries water under a head of 15 metres with a velocity of 4 m/s. If the axis of the pipe turns through 45°, find the magnitude and direction of the resultant force at the bend. [**Ans.** 8717.5 N, $\theta = 67° 30'$]

26. A pipe of 20 cm diameter conveying 0.20 m^3/sec of water has a right angled bend in a horizontal plane. Find the resultant force exerted on the bend if the pressure at inlet and outlet of the bend are 22.563 N/cm^2 and 21.582 N/cm^2 respectively. [**Ans.** 11604.7 N, $\theta = 43° 54.2'$]

27. A nozzle of diameter 30 mm is fitted to a pipe of 60 mm diameter. Find the force exerted by the nozzle on the water which is flowing through the pipe at the rate of 4.0 m^3/minute. [**Ans.** 7057.7 N]

28. A lawn sprinkler with two nozzles of diameters 3 mm each is connected across a tap of water. The nozzles are at a distance of 40 cm and 30 cm from the centre of the tap. The rate of water through tap is 100 cm^3/s. The nozzle discharges water in the downward directions. Determine the angular speed at which the sprinkler will rotate free. [**Ans.** 2.83 rad/s]

29. A lawn sprinkler has two nozzles of diameters 8 mm each at the end of a rotating arm and the velocity of flow of water from each nozzle is 12 m/s. One nozzle discharges water in the downward direction, while the other nozzle discharges water vertically up. The nozzles are at a distance of 40 cm from the centre of the rotating arm. Determine the torque required to hold the rotating arm stationary. Also determine the constant speed of rotation of arm, if it is free to rotate. [**Ans.** 5.78 Nm, 30 rad/s]

30. A vertical wall is of 10 m in height. A jet of water is issuing from a nozzle with a velocity of 25 m/s. The nozzle is situated at a horizontal distance of 20 m from the vertical wall. Find the angle of projection of the nozzle to the horizontal so that the jet of water just clears the tap of the wall. [**Ans.** 79° 55′ or 36° 41′]

31. A fire-brigade man is holding a fire stream nozzle of 50 mm diameter at a distance of 1 m above the ground and 6 m from a vertical wall. The jet is coming out with a velocity of 15 m/s. This jet is to strike a window, situated at a distance of 10 m above ground in the vertical wall. Find the angle or angles of inclination with the horizontal made by the jet, coming out from the nozzle. What will be the amount of water falling on the window ? [**Ans.** 79° 16.7′ or 67° 3.7′ ; 0.0294 m^3/s]

32. A window, in a vertical wall, is at a distance of 12 m above the ground level. A jet of water, issuing from a nozzle of diameter 50 mm, is to strike the window. The rate of flow of water through the nozzle is 40 litres/sec. The nozzle is situated at a distance of 1 m above ground level. Find the greatest horizontal distance from the wall of the nozzle so that jet of water strikes the window. [**Ans.** 29.38 m]

33. Explain in brief the working of a pitot-tube. Calculate the velocity of flow of water in a pipe of diameter 300 mm at a point, where the stagnation pressure head is 5 m and static pressure head is 4 m. Given the co-efficient of pitot-tube = 0.97. [**Ans.** 4.3 m/sec]

34. Find the rate of flow of water through a venturimeter fitted in a pipeline of diameter 30 cm. The ratio of diameter of throat and inlet of the venturimeter is *. The pressure at the inlet of the venturimeter is 13.734 N/cm^2 (gauge) and vacuum in the throat is 37.5 cm of mercury. The co-efficient of venturimeter is given as 0.98. [**Ans.** 0.15 m^3/s]

35. A 30 cm × 15 cm venturimeter is inserted in a vertical pipe carrying an oil of sp. gr. 0.8, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 30 cm. The difference in the elevation of the throat section and inlet section is 50 cm. Find the rate of flow of oil.

36. A venturimeter is used for measurement of discharge of water in horizontal pipe line. If the ratio of upstream pipe diameter to that of throat is 2 : 1, upstream diameter is 300 mm, the difference in pressure between the throat and upstream is equal to 3 m head of water and loss of head through meter is one-eighth of the throat velocity head, calculate the discharge in the pipe. [**Ans.** 0.107 m^3/s]

37. A liquid of specific gravity 0.8 is flowing upwards at the rate of 0.08 m^3/s, through a vertical venturimeter with an inlet diameter of 200 mm and throat diameter of 100 mm. The C_d = 0.98 and the vertical distance between pressure tappings is 300 mm. Find :

(*i*) the difference in readings of the two pressure gauges, which are connected to the two pressure tappings, and

(*ii*) the difference in the level of the mercury columns of the differential manometer which is connected to the tappings, in place of pressure gauges. [**Ans.** (*i*) 42.928 kN/m^2, (*ii*) 32.3 cm]

[**Hint.** $Q = 0.08$ m^3/s, $d_1 = 200$ mm $= 0.2$ m, $d_2 = 100$ mm $= 0.1$ m,

$$C_d = 0.98,\ z_2 - z_1 = 300 \text{ mm} = 0.3 \text{ m},\ a_1 = \frac{\pi}{4}(.2^2) = 0.0314 \text{ m}^2$$

$$a_2 = \frac{\pi}{4}(.1^2) = 0.007854 \text{ m}^2. \text{ Using } Q = C_d\,\frac{a_1 \times a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

Find 'h'. This value of $h = 5.17$ m.

Now use $h = \left(\dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g}\right) + (z_1 - z_2)$, where $\rho = 800$ kg/m^3. Find $(p_1 - p_2)$.

Now use the formula $h = x\left[\dfrac{S_g}{S_f} - 1\right]$,

where $h = 5.17$ m, $S_g = 13.6$ and $S_f = 0.8$. Find the value of x which will be 32.3 cm.]

38. A venturimeter is installed in a 300 mm diameter horizontal pipe line. The throat pipe rates is 1/3. Water flows through the installation. The pressure in the pipe line is 13.783 N/cm^2 (gauge) and vacuum in the throat is 37.5 cm of mercury. Neglecting head loss in the venturimeter, determine the rate of flow in the pipe line. [**Ans.** 0.153 m^3/sec]

[**Hint.** $d_1 = 300$ mm $= 0.3$ m, $d_2 = \dfrac{1}{3} \times 300 = 100$ mm $= 0.1$ m, $p_1 = 13.783$ N/cm^2 $= 13.783 \times 10^4$ N/m^2.

Hence $p_1/\rho \times g = 13.783 \times 10^4/1000 \times 9.81$
$= 14.05$ m, $p_2/\rho g = -37.5$ cm of Hg $= -0.375 \times 13.6$ m of water
$= -5.1$ m of water. Hence $h = 14.05 - (-5.1) = 19.15$ m of water.

Value of $C_d = 1.0$. Now use the formula $Q = C_d\,\dfrac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$]

39. The maximum flow through a 300 mm diameter horizontal main pipe line is 18200 litre/minute. A venturimeter is introduced at a point of the pipe line where the pressure head is 4.6 m of water. Find the smallest dia. of throat so that the pressure at the throat is never negative. Assume co-efficient of meter as unity. [**Ans.** $d_2 = 192.4$ mm]

[**Hint.** $d_1 = 300$ mm $= 0.3$ m, $Q = 18200$ litres/minute $= 18200/60 = 303.33$ litres/s $= 0.3033$ m^3/s, $p_1/\rho g$ $= 4.6$ m, $p_2/\rho g = 0$. Hence $h = 4.6$ m, $C_d = 1$. $d_2 =$ dia. at throat. Use formula $Q = C_d\dfrac{a_1 \times a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$ and

find the value of a_2. Then $a_2 = \dfrac{\pi}{4}\, d_2^2$ and find d_2.]

40. The following are the data given of a change in diameter effected in laying a water supply pipe. The change in diameter is gradual from 20 cm at A to 50 cm at B. Pressures at A and B are 7.848 N/cm^2 and 5.886 N/cm^2 respectively with the end B being 3 m higher than A. If the flow in the pipe line is 200 litre/s, find : (*i*) direction of flow, (*ii*) the head lost in friction between A and B.

[**Ans.** (*i*) From A to B, (*ii*) 1.015 m]

[**Hint.** $D_A = 20$ cm $= 0.2$ m, $D_B = 50$ cm $= 0.5$ m, $p_A = 7.848$ N/cm^2 $= 7.848 \times 10^4$ N/m^2
$p_B = 5.886$ N/cm^2 $= 5.886 \times 10^4$ N/m^2, $Z_A = 0$, $Z_B = 3$ m, $Q = 0.2$ m^3/s

$V_A = 0.2/\frac{\pi}{4}(.2^2) = 6.369$ m/s, $V_B = 0.2/\frac{\pi}{4}(.5^2) = 1.018$ m/s

$$E_A = (p_A/\rho \times g) + \frac{V_A^2}{2g} + Z_A = (7.848 \times 10^4/1000 \times 9.81) + (6.369^2/2 \times 9.81) + 0 = 10.067 \text{ m}$$

$$E_B = (p_B/\rho \times g) + \frac{V_B^2}{2g} + Z_B = (5.886 \times 10^4/1000 \times 9.81) + (1.018^2/2 \times 9.81) + 3 = 9.052 \text{ m}]$$

41. A venturimeter of inlet diameter 300 mm and throat diameter 150 mm is fixed in a vertical pipe line. A liquid of sp. gr. 0.8 is flowing upward through the pipe line. A differential manometer containing mercury gives a reading of 100 mm when connected at inlet and throat. The vertical difference between inlet and throat is 500 mm. If $C_d = 0.98$, then find : (*i*) rate of flow of liquid in litre per second and (*ii*) difference of pressure between inlet and throat in N/m^2. [**Ans.** (*i*) 100 litre/s, (*ii*) 15980 N/m^2]

42. A venturimeter with a throat diameter of 7.5 cm is installed in a 15 cm diameter pipe. The pressure at the entrance to the meter is 70 kPa (gauge) and it is desired that the pressure at any point should not fall below 2.5 m of absolute water. Determine the maximum flow rate of water through the meter. Take $C_d = 0.97$ and atmospheric pressure as 100 kPa. (*J.N.T.U., Hyderabad S 2002*)

[**Hint.** The pressure at the throat will be minimum. Hence $\frac{p_2}{\rho g} = 2.5$ m (abs.)

Given : $d_1 = 15$ cm $\therefore A_1 = \frac{\pi}{4}(15^2) = 176.7$ cm^2

$d_2 = 7.5$ cm $\therefore A_2 = \frac{\pi}{4}(7.5^2) = 44.175$ cm^2

$p_1 = 70$ kPa $= 70 \times 10^3$ N/m^2 (gauge), $p_{atm} = 100$ kPa $= 100 \times 10^3$ N/m^2

$\therefore$ p_1 (abs.) $= 70 \times 10^3 + 100 \times 10^3 = 170 \times 10^3$ N/m^2 (abs.)

$$\therefore \quad \frac{p_1}{\rho g} = \frac{170 \times 10^3}{1000 \times 9.81} = 17.33 \text{ m of water (abs.)}$$

$$\therefore \quad h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 17.33 - 2.5 = 14.83 \text{ m of water} = 1483 \text{ cm of water}$$

Now
$$Q = \frac{C_d A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh} = \frac{0.97 \times 176.7 \times 44.175 \times \sqrt{2 \times 981 \times 1483}}{\sqrt{176.7^2 - 44 \cdot 175^2}} = 75488 \text{ cm}^3/\text{s}$$

$= \mathbf{75.488}$ **litre/s.**]

43. Find the discharge of water flowing through a pipe 20 cm diameter placed in an inclined position, where a venturimeter is inserted, having a throat diameter of 10 cm. The difference of pressure between the main and throat is measured by a liquid of specific gravity 0.4 in an inverted *U*-tube, which gives a reading of 30 cm. The loss of head between the main and throat is 0.2 times the kinetic head of pipe.

(*Delhi University, Dec. 2002*)

[**Hint.** Given : $d_1 = 20$ cm $\therefore a_1 = \frac{\pi}{4}(20^2) = 100\,\pi$ cm^2 ; $d_2 = 10$ cm $\therefore a_2 = \frac{\pi}{4}(10^2) = 25\,\pi$ cm^2.

$$x = 30 \text{ cm}, h = x\left(1 - \frac{S_l}{S_o}\right) = 30\left(1 - \frac{0.4}{1.0}\right) = 18 \text{ cm} = 0.18 \text{ m}$$

But h is also
$$= \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) \therefore \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 18 \text{ cm} = 0.18 \text{ m}$$

$$h_L = 0.2 \times \frac{V_1^2}{2g}$$

From Bernoulli's equation, $\dfrac{p_1}{\rho g} + \dfrac{V_1^2}{2g} + z_1 = \dfrac{p_2}{\rho g} + \dfrac{V_2^2}{2g} + z_2 + h_L$

or $\left(\dfrac{p_1}{\rho g} + z_1\right) - \left(\dfrac{p_2}{\rho g} + z_2\right) + \dfrac{V_1^2}{2g} - \dfrac{V_2^2}{2g} = h_L$

or $0.18 + \dfrac{V_1^2}{2g} - \dfrac{V_2^2}{2g} = \dfrac{0.2\,V_1^2}{2g}$ $\left(\because \left(\dfrac{p_1}{\rho g} + z_1\right) - \left(\dfrac{p_2}{\rho g} + z_2\right) = 0.18 \text{ m and } h_L = \dfrac{0.2\,V_1^2}{2g}\right)$

or $0.18 + \dfrac{V_1^2}{2g} - \dfrac{V_2^2}{2g} - \dfrac{0.2\,V_1^2}{2g} = 0$ or $0.18 + \dfrac{0.8\,V_1^2}{2g} - \dfrac{V_2^2}{2g} = 0$

From continuity equation, $a_1 V_1 = a_2 V_2$ or $V_2 = \dfrac{a_1 V_1}{a_2} = \dfrac{\frac{\pi}{4}(20^2)\,V_1}{\frac{\pi}{4}(10^2)} = 4V_1$

Now $0.18 + \dfrac{0.8\,V_1^2}{2g} - \dfrac{V_2^2}{2g} = 0$ or $0.18 + \dfrac{0.8\,V_1^2}{2g} - \dfrac{(4V_1)^2}{2g} = 0$

or $0.18 + \dfrac{0.8\,V_1^2}{2g} - \dfrac{16V_1^2}{2g} = 0$ or $0.18 = \dfrac{16V_1^2}{2g} - \dfrac{0.8\,V_1^2}{2g} = \dfrac{15.2V_1^2}{2g}$

$\therefore$ $V_1 = \sqrt{\dfrac{0.18 \times 2 \times 9.81}{15.2}} = 0.48 \text{ m/s} = 48 \text{ cm/s}$

$\therefore$ $Q = A_1 V_1 = \dfrac{\pi}{4}(20^2) \times 48 = 15140 \text{ cm}^3\text{/s} =$ **15.14 litre/s.**]

CHAPTER 7

ORIFICES AND MOUTHPIECES

▶ 7.1 INTRODUCTION

Orifice is a small opening of any cross-section (such as circular, triangular, rectangular etc.) on the side or at the bottom of a tank, through which a fluid is flowing. A mouthpiece is a short length of a pipe which is two to three times its diameter in length, fitted in a tank or vessel containing the fluid. Orifices as well as mouthpieces are used for measuring the rate of flow of fluid.

▶ 7.2 CLASSIFICATIONS OF ORIFICES

The orifices are classified on the basis of their size, shape, nature of discharge and shape of the upstream edge. The following are the important classifications :

1. The orifices are classified as **small orifice** or **large orifice** depending upon the size of orifice and head of liquid from the centre of the orifice. If the head of liquid from the centre of orifice is more than five times the depth of orifice, the orifice is called small orifice. And if the head of liquids is less than five times the depth of orifice, it is known as large orifice.

2. The orifices are classified as (*i*) Circular orifice, (*ii*) Triangular orifice, (*iii*) Rectangular orifice and (*iv*) Square orifice depending upon their cross-sectional areas.

3. The orifices are classified as (*i*) Sharp-edged orifice and (*ii*) Bell-mouthed orifice depending upon the shape of upstream edge of the orifices.

4. The orifices are classified as (*i*) Free discharging orifices and (*ii*) Drowned or sub-merged orifices depending upon the nature of discharge.

The sub-merged orifices are further classified as (*a*) Fully sub-merged orifices and (*b*) Partially sub-merged orifices.

▶ 7.3 FLOW THROUGH AN ORIFICE

Consider a tank fitted with a circular orifice in one of its sides as shown in Fig. 7.1. Let H be the head of the liquid above the centre of the orifice. The liquid flowing through the orifice forms a jet of liquid whose area of cross-section is less than that of orifice. The area of jet of fluid goes on decreasing and at a section C-C, the area is minimum. This section is approximately at a distance of half of diameter of the orifice. At this section, the streamlines are straight and parallel to each other and perpendicular to the plane of the orifice. This section is called **Vena-contracta**. Beyond this section, the jet diverges and is attracted in the downward direction by the gravity.

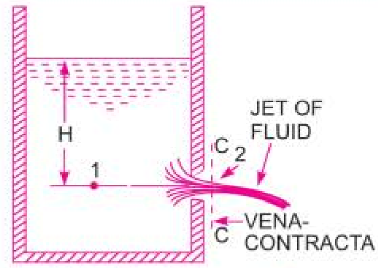


Fig. 7.1 *Tank with an orifice.*

Consider two points 1 and 2 as shown in Fig. 7.1. Point 1 is inside the tank and point 2 at the vena-contracta. Let the flow is steady and at a constant head H. Applying Bernoulli's equation at points 1 and 2.

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $$z_1 = z_2$$

$\therefore$ $$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

Now $$\frac{p_1}{\rho g} = H$$

$$\frac{p_2}{\rho g} = 0 \text{ (atmospheric pressure)}$$

v_1 is very small in comparison to v_2 as area of tank is very large as compared to the area of the jet of liquid.

$\therefore$ $$H + 0 = 0 + \frac{v_2^2}{2g}$$

$\therefore$ $$v_2 = \sqrt{2gH} \quad \text{...(7.1)}$$

This is theoretical velocity. Actual velocity will be less than this value.

▶ 7.4 HYDRAULIC CO-EFFICIENTS

The hydraulic co-efficients are

1. Co-efficient of velocity, C_v
2. Co-efficient of contraction, C_c
3. Co-efficient of discharge, C_d.

7.4.1 Co-efficient of Velocity (C_v). It is defined as the ratio between the actual velocity of a jet of liquid at vena-contracta and the theoretical velocity of jet. It is denoted by C_v and mathematically, C_v is given as

$$C_v = \frac{\text{Actual velocity of jet at vena-contracta}}{\text{Theoretical velocity}}$$

$$= \frac{V}{\sqrt{2gH}}, \text{ where } V = \text{actual velocity}, \sqrt{2gH} = \text{Theoretical velocity} \quad \text{...(7.2)}$$

The value of C_v varies from 0.95 to 0.99 for different orifices, depending on the shape, size of the orifice and on the head under which flow takes place. Generally the value of $C_v = 0.98$ is taken for sharp-edged orifices.

7.4.2 Co-efficient of Contraction (C_c). It is defined as the ratio of the area of the jet at vena-contracta to the area of the orifice. It is denoted by C_c.

Let a = area of orifice and

a_c = area of jet at vena-contracta.

Then
$$C_c = \frac{\text{area of jet at vena-contracta}}{\text{area of orifice}}$$

$$= \frac{a_c}{a} \qquad \text{...(7.3)}$$

The value of C_c varies from 0.61 to 0.69 depending on shape and size of the orifice and head of liquid under which flow takes place. In general, the value of C_c may be taken as 0.64.

7.4.3 Co-efficient of Discharge (C_d). It is defined as the ratio of the actual discharge from an orifice to the theoretical discharge from the orifice. It is denoted by C_d. If Q is actual discharge and Q_{th} is the theoretical discharge then mathematically, C_d is given as

$$C_d = \frac{Q}{Q_{th}} = \frac{\text{Actual velocity} \times \text{Actual area}}{\text{Theoretical velocity} \times \text{Theoretical area}}$$

$$= \frac{\text{Actual velocity}}{\text{Theoretical velocity}} \times \frac{\text{Actual area}}{\text{Theoretical area}}$$

$$\therefore \qquad C_d = C_v \times C_c \qquad \text{...(7.4)}$$

The value of C_d varies from 0.61 to 0.65. For general purpose the value of C_d is taken as 0.62.

Problem 7.1 *The head of water over an orifice of diameter 40 mm is 10 m. Find the actual discharge and actual velocity of the jet at vena-contracta. Take $C_d = 0.6$ and $C_v = 0.98$.*

Solution. Given :

Head, $H = 10$ cm

Dia. of orifice, $d = 40$ mm $= 0.04$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.04)^2 = .001256 \text{ m}^2$

$C_d = 0.6$

$C_v = 0.98$

(*i*) $$\frac{\text{Actual discharge}}{\text{Theoretical discharge}} = 0.6$$

But Theoretical discharge = V_{th} × Area of orifice

V_{th} = Theoretical velocity, where $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 10} = 14$ m/s

$\therefore$ Theoretical discharge $= 14 \times .001256 = 0.01758 \ \frac{\text{m}^2}{\text{s}}$

$\therefore$ Actual discharge $= 0.6 \times$ Theoretical discharge

$= 0.6 \times .01758 =$ **0.01054 m^3/s.** **Ans.**

(*ii*) $$\frac{\text{Actual velocity}}{\text{Theoretical velocity}} = C_v = 0.98$$

$\therefore$ Actual velocity = 0.98 × Theoretical velocity
= 0.98 × 14 = **13.72 m/s. Ans.**

Problem 7.2 *The head of water over the centre of an orifice of diameter 20 mm is 1 m. The actual discharge through the orifice is 0.85 litre/s. Find the co-efficient of discharge.*

Solution. Given :

Dia. of orifice, $d = 20 \text{ mm} = 0.02 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4}(0.02)^2 = 0.000314 \text{ m}^2$

Head, $H = 1 \text{ m}$

Actual discharge, $Q = 0.85 \text{ litre/s} = 0.00085 \text{ m}^3/\text{s}$

Theoretical velocity, $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 1} = 4.429 \text{ m/s}$

$\therefore$ Theoretical discharge, $Q_{th} = V_{th} \times \text{Area of orifice}$
$= 4.429 \times 0.000314 = 0.00139 \text{ m}^3/\text{s}$

$\therefore$ Co-efficient of discharge $= \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{0.00085}{0.00139} = \mathbf{0.61.}$ **Ans.**

▶ 7.5 EXPERIMENTAL DETERMINATION OF HYDRAULIC CO-EFFICIENTS

7.5.1 Determination of Co-efficient of Discharge (C_d). The water is allowed to flow through an orifice fitted to a tank under a constant head, H as shown in Fig. 7.2. The water is collected in a measuring tank for a known time, t. The height of water in the measuring tank is noted down. Then actual discharge through orifice,

$$Q = \frac{\text{Area of measuring tank } \times \text{Height of water in measuring tank}}{\text{Time } (t)}$$

and theoretical discharge = area of orifice × $\sqrt{2gH}$

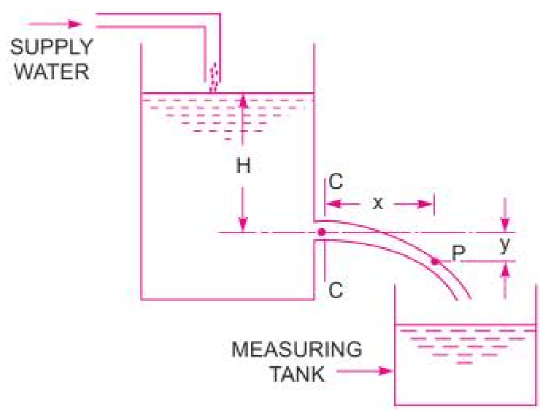


Fig. 7.2 *Value of C_d.*

$$\therefore \quad C_d = \frac{Q}{a \times \sqrt{2gH}} \quad \ldots(7.5)$$

7.5.2 Determination of Co-efficient of Velocity (C_v). Let C-C represents the vena-contracta of a jet of water coming out from an orifice under constant head H as shown in Fig. 7.2. Consider a liquid particle which is at vena-contracta at any time and takes the position at P along the jet in time 't'.

Let x = horizontal distance travelled by the particle in time 't'
y = vertical distance between P and C-C
V = actual velocity of jet at vena-contracta.

Then horizontal distance, $x = V \times t$ ...(*i*)

and vertical distance, $y = \frac{1}{2} gt^2$ ...(*ii*)

From equation (*i*), $t = \frac{x}{V}$

Substituting this value of 't' in (*ii*), we get

$$y = \frac{1}{2} g \times \frac{x^2}{V^2}$$

$$V^2 = \frac{gx^2}{2y}$$

$$\therefore \quad V = \sqrt{\frac{gx^2}{2y}}$$

But theoretical velocity,

$$V_{th} = \sqrt{2gH}$$

$$\therefore \text{ Co-efficient of velocity, } C_v = \frac{V}{V_{th}} = \sqrt{\frac{gx^2}{2y}} \times \frac{1}{\sqrt{2gH}} = \sqrt{\frac{x^2}{4yH}}$$

$$= \frac{x}{\sqrt{4yH}}. \qquad \text{...(7.6)}$$

7.5.3 Determination of Co-efficient of Contraction (C_c). The co-efficient of contraction is determined from the equation (7.4) as

$$C_d = C_v \times C_c$$

$$\therefore \quad C_c = \frac{C_d}{C_v} \qquad \text{...(7.7)}$$

Problem 7.3 *A jet of water, issuing from a sharp-edged vertical orifice under a constant head of 10.0 cm, at a certain point, has the horizontal and vertical co-ordinates measured from the vena-contracta as 20.0 cm and 10.5 cm respectively. Find the value of C_v. Also find the value of C_c if C_d = 0.60.*

Solution. Given :

Head, $H = 10.0$ cm
Horizontal distance, $x = 20.0$ cm
Vertical distance, $y = 10.5$ cm
$C_d = 0.6$

The value of C_v is given by equation (7.6) as

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{20.0}{\sqrt{4 \times 10.5 \times 10.0}} = \frac{20}{20.493} = 0.9759 = \mathbf{0.976.\ Ans.}$$

The value of C_c is given by equation (7.7) as

$$C_c = \frac{C_d}{C_v} = \frac{0.6}{0.976} = 0.6147 = \mathbf{0.615.\ Ans.}$$

Problem 7.4 *The head of water over an orifice of diameter 100 mm is 10 m. The water coming out from orifice is collected in a circular tank of diameter 1.5 m. The rise of water level in this tank is 1.0 m in 25 seconds. Also the co-ordinates of a point on the jet, measured from vena-contracta are 4.3 m horizontal and 0.5 m vertical. Find the co-efficients, C_d, C_v and C_c.*

Solution. Given :

Head, $H = 10$ m

Dia. of orifice, $d = 100$ mm $= 0.1$ m

$\therefore$ Area of orifice, $a = \frac{\pi}{4}(.1)^2 = 0.007853\ \text{m}^2$

Dia. of measuring tank, $D = 1.5$ m

$\therefore$ Area, $A = \frac{\pi}{4}(1.5)^2 = 1.767\ \text{m}^2$

Rise of water, $h = 1$ m

Time, $t = 25$ seconds

Horizontal distance, $x = 4.3$ m

Vertical distance, $y = 0.5$ m

Now theoretical velocity, $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 10} = 14.0$ m/s

$\therefore$ Theoretical discharge, $Q_{th} = V_{th} \times$ Area of orifice $= 14.0 \times 0.007854 = 0.1099\ \text{m}^3/\text{s}$

Actual discharge, $$Q = \frac{A \times h}{t} = \frac{1.767 \times 1.0}{25} = 0.07068$$

$\therefore$ $$C_d = \frac{Q}{Q_{th}} = \frac{0.07068}{0.1099} = \mathbf{0.643.\ Ans.}$$

The value of C_v is given by equation (7.6) as

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{4.3}{\sqrt{4 \times 0.5 \times 10}} = \frac{4.3}{4.472} = \mathbf{0.96.\ Ans.}$$

C_c is given by equation (7.7) as $C_c = \dfrac{C_d}{C_v} = \dfrac{0.643}{0.96} = \mathbf{0.669.\ Ans.}$

Problem 7.5 *Water discharge at the rate of 98.2 litres/s through a 120 mm diameter vertical sharp-edged orifice placed under a constant head of 10 metres. A point, on the jet, measured from the vena-contracta of the jet has co-ordinates 4.5 metres horizontal and 0.54 metres vertical. Find the co-efficient C_v, C_c and C_d of the orifice.*

Solution. Given :

Discharge, $Q = 98.2$ lit/s $= 0.0982$ m^3/s

Dia. of orifice, $d = 120$ mm $= 0.12$ m

$\therefore$ Area of orifice, $a = \frac{\pi}{4}(0.12)^2 = 0.01131 \text{ m}^2$

Head, $H = 10$ m

Horizontal distance of a point on the jet from vena-contracta, $x = 4.5$ m and vertical distance, $y = 0.54$ m

Now theoretical velocity, $V_{th} = \sqrt{2g \times H} = \sqrt{2 \times 9.81 \times 10} = 14.0$ m/s

Theoretical discharge, $Q_{th} = V_{th} \times$ Area of orifice
$= 14.0 \times 0.01131 = 0.1583$ m^3/s

The value of C_d is given by, $C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{Q}{Q_{th}} = \frac{0.0982}{0.1583} = \mathbf{0.62.}$ **Ans.**

The value of C_c is given by equation (7.6),

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{4.5}{\sqrt{4 \times 0.54 \times 10}} = \mathbf{0.968.}\ \textbf{Ans.}$$

The value of C_c is given by equation (7.7) as

$$C_c = \frac{C_d}{C_v} = \frac{0.62}{0.968} = \mathbf{0.64.}\ \textbf{Ans.}$$

Problem 7.6 *A 25 mm diameter nozzle discharges 0.76 m^3 of water per minute when the head is 60 m. The diameter of the jet is 22.5 mm. Determine : (i) the values of co-efficients* C_c, C_v *and* C_d *and (ii) the loss of head due to fluid resistance.*

Solution. Given :

Dia. of nozzle, $D = 25$ mm $= 0.025$ m

Actual discharge, $Q_{act} = 0.76 \text{ m}^3/\text{minute} = \frac{0.76}{60} = 0.01267 \text{ m}^3/\text{s}$

Head, $H = 60$ m

Dia. of jet, $d = 22.5$ mm $= 0.0225$ m.

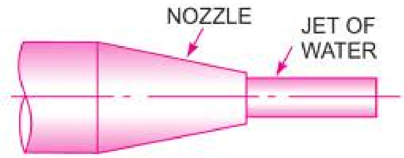


Fig. 7.3

(*i*) *Values of co-efficients :*

Co-efficient of contraction (C_c) is given by,

$$C_c = \frac{\text{Area of jet}}{\text{Area of nozzle}}$$

$$= \frac{\frac{\pi}{4} d^2}{\frac{\pi}{4} D^2} = \frac{d^2}{D^2} = \frac{0.0225^2}{0.025^2} = \mathbf{0.81.}\ \textbf{Ans.}$$

Co-efficient of discharge (C_d) is given by,

$$C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}}$$

$$= \frac{0.01267}{\text{Theoretical velocity} \times \text{Area of nozzle}}$$

$$= \frac{0.01267}{\sqrt{2gH} \times \frac{\pi}{4} D^2} = \frac{0.01267}{\sqrt{2 \times 9.81 \times 60} \times \frac{\pi}{4} (0.025)^2}$$

$= \mathbf{0.752.}$ **Ans.**

Co-efficient of velocity (C_v) is given by,

$$C_v = \frac{C_d}{C_c} = \frac{0.752}{0.81} = \mathbf{0.928.} \textbf{ Ans.}$$

(*ii*) *Loss of head due to fluid resistance* :

Applying Bernoulli's equation at the outlet of nozzle and to the jet of water, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{Loss of head}$$

But $\quad \frac{p_1}{\rho g} = \frac{p_2}{\rho g} =$ Atmospheric pressure head

$$z_1 = z_2,\ V_1 = \sqrt{2gH},\ V_2 = \text{Actual velocity of jet} = C_v \sqrt{2gH}$$

$$\therefore \quad \frac{\left(\sqrt{2gH}\right)^2}{2g} = \frac{\left(C_v\sqrt{2gH}\right)^2}{2g} + \text{Loss of head}$$

or $\quad H = C_v^2 \times H + \text{Loss of head}$

$\therefore$ Loss of head $\quad = H - C_v^2 \times H = H(1 - C_v^2)$

$= 60(1 - 0.928^2) = 60 \times 0.1388 = \mathbf{8.328\ m.}$ **Ans.**

Problem 7.7 *A pipe, 100 mm in diameter, has a nozzle attached to it at the discharge end, the diameter of the nozzle is 50 mm. The rate of discharge of water through the nozzle is 20 litres/s and the pressure at the base of the nozzle is 5.886 N/cm². Calculate the co-efficient of discharge. Assume that the base of the nozzle and outlet of the nozzle are at the same elevation.*

Solution. Given :

Dia. of pipe, $\quad D = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore \quad A_1 = \frac{\pi}{4}(.1)^2 = .007854 \text{ m}^2$

Dia. of nozzle, $\quad d = 50 \text{ mm} = 0.05 \text{ m}$

$\therefore \quad A_2 = \frac{\pi}{4}(.05)^2 = .001963 \text{ m}^2$

Actual discharge, $\quad Q = 20 \text{ lit/s} = 0.02 \text{ m}^3/\text{s}$

Pressure at the base, $\quad p_1 = 5.886 \text{ N/cm}^2 = 5.886 \times 10^4 \frac{\text{N}}{\text{m}^2}$

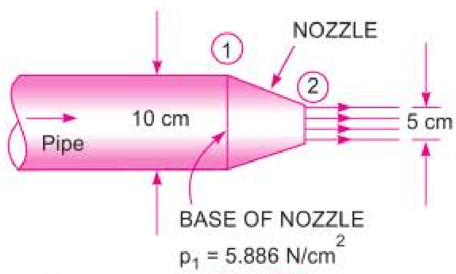


Fig. 7.4

From continuity equation, $A_1V_1 = A_2V_2$

or $\quad .007854\ V_1 = .001963\ V_2$

$$\therefore \qquad V_1 = \frac{.001963 V_2}{.007854} = \frac{V_2}{4}$$

where V_1 and V_2 are theoretical velocity at sections (1) and (2).

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

But $\qquad z_1 = z_2$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g}$$

or $$\frac{5.886 \times 10^4}{1000 \times 9.81} + \frac{\left(\frac{V_2}{4}\right)^2}{2g} = 0 + \frac{V_2^2}{2g} \qquad \left\{\because \frac{p_2}{\rho g} = \text{Atmospheric pressure} = 0\right\}$$

$$6.0 + \frac{V_2^2}{2g \times 16} = \frac{V_2^2}{2g}$$

or $$\frac{V_2^2}{2g}\left[1 - \frac{1}{16}\right] = 6.0 \quad \text{or} \quad \frac{V_2^2}{2g}\left[\frac{15}{16}\right] = 6.0$$

$$\therefore \qquad V_2 = \sqrt{6.0 \times 2 \times 9.81 \times \frac{16}{15}} = 11.205 \text{ m/sec}$$

$\therefore$ Theoretical discharge $= V_2 \times A_2 = 11.205 \times .001963 = 0.022 \text{ m}^3\text{/s}$

$$\therefore \qquad C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{0.02}{0.022} = \mathbf{0.909.} \textbf{ Ans.}$$

Problem 7.8 *A tank has two identical orifices on one of its vertical sides. The upper orifice is 3 m below the water surface and lower one is 5 m below the water surface. If the value of C_v for each orifice is 0.96, find the point of intersection of the two jets.*

Solution. Given :

Height of water from orifice (1), $H_1 = 3$ m

From orifice (2), $H_2 = 5$ m

C_v for both $= 0.96$

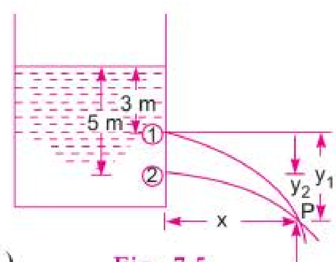


Fig. 7.5

Let P is the point of intersection of the two jets coming from orifices (1) and (2), such that

x = horizontal distance of P

y_1 = vertical distance of P from orifice (1)

y_2 = vertical distance of P from orifice (2)

Then $\qquad y_1 = y_2 + (5 - 3) = y_2 + 2$ m

The value of C_v is given by equation (7.6) as

For orifice (1), $$C_{v_1} = \frac{x}{\sqrt{4y_1 H_1}} = \frac{x}{\sqrt{4y_1 \times 3.0}} \quad ...(i)$$

For orifice (2), $$C_{v_2} = \frac{x}{\sqrt{4y_2 H_2}} = \frac{x}{\sqrt{4 \times y_2 \times 5.0}} \quad ...(ii)$$

As both the orifices are identical

$\therefore$ $$C_{v_1} = C_{v_2}$$

or $$\frac{x}{\sqrt{4y_1 \times 3.0}} = \frac{x}{\sqrt{4y_2 \times 5.0}} \text{ or } 3y_1 = 5y_2$$

But $$y_1 = y_2 + 2.0$$

$\therefore$ $$3(y_2 + 2.0) = 5y_2$$

$\therefore$ $$2y_2 = 6.0$$

$\therefore$ $$y_2 = 3.0$$

From (ii), $$C_{v_2} = \frac{x}{\sqrt{4y_2 \times H_2}}$$

or $$0.96 = \frac{x}{\sqrt{4 \times 3.0 \times 5.0}}$$

$\therefore$ $$x = 0.96 \times \sqrt{4 \times 3.0 \times 5.0} = \mathbf{7.436 \text{ m.}}$$ **Ans.**

Problem 7.9 *A closed vessel contains water upto a height of 1.5 m and over the water surface there is air having pressure 7.848 N/cm² (0.8 kgf/cm²) above atmospheric pressure. At the bottom of the vessel there is an orifice of diameter 100 mm. Find the rate of flow of water from orifice. Take $C_d = 0.6$.*

Solution. Given :

Dia. of orifice, $d = 100 \text{ mm} = 0.1 \text{ m}$

$C_d = 0.6$

Height of water, $H = 1.5 \text{ m}$

Air pressure, $p = 7.848 \text{ N/cm}^2 = 7.848 \times 10^4 \text{ N/m}^2$

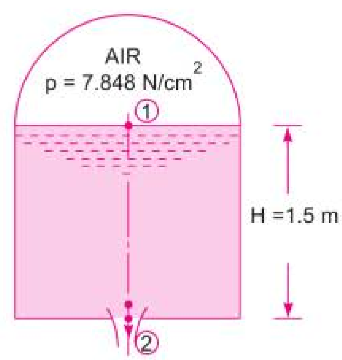


Fig. 7.6

Applying Bernoulli's equation at sections (1) (water surface) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Taking datum line passing through (2) which is very close to the bottom surface of the tank. Then $z_2 = 0$, $z_1 = 1.5$ m

Also $$\frac{p_2}{\rho g} = 0 \text{ (atmospheric pressure)}$$

and $$\frac{p_1}{\rho g} = \frac{7.848 \times 10^4}{1000 \times 9.81} = 8 \text{ m of water}$$

$\therefore$ $$8 + 0 + 1.5 = 0 + \frac{V_2^2}{2g} + 0 \quad \{V_1 \text{ is negligible}\}$$

$\therefore$ $$9.5 = \frac{V_2^2}{2g}$$

$\therefore \quad V_2 = \sqrt{2 \times 9.81 \times 9.5} = 13.652$ m/s

$\therefore$ Rate of flow of water $= C_d \times a_2 \times V_2$

$$= 0.6 \times \frac{\pi}{4} (.1)^2 \times 13.652 \text{ m}^3\text{/s} = \mathbf{0.0643\ m^3/s.}\ \textbf{Ans.}$$

Problem 7.10 *A closed tank partially filled with water upto a height of 0.9 m having an orifice of diameter 15 mm at the bottom of the tank. The air is pumped into the upper part of the tank. Determine the pressure required for a discharge of 1.5 litres/s through the orifice. Take $C_d = 0.62$.*

Solution. Given :

Height of water above orifice, $H = 0.9$ m

Dia. of orifice, $d = 15$ mm $= 0.015$ m

$\therefore$ Area, $a = \frac{\pi}{4}\left[d^2\right] = \frac{\pi}{4} (.015)^2 = 0.0001767 \text{ m}^2$

Discharge, $Q = 1.5$ litres/s $= .0015 \text{ m}^3\text{/s}$

$C_d = 0.62$

Let p is intensity of pressure required above water surface in N/cm^2.

Then pressure head of air$= \frac{p}{\rho g} = \frac{p \times 10^4}{1000 \times 9.81} = \frac{10p}{9.81}$ m of water.

If V_2 is the velocity at outlet of orifice, then

$$V_2 = \sqrt{2g\left(H + \frac{p}{\rho g}\right)} = \sqrt{2 \times 9.81\left(0.9 + \frac{10p}{9.81}\right)}$$

$\therefore$ Discharge $Q = C_d \times a \times \sqrt{2g\,(H + p/\rho g)}$

$$.0015 = 0.6 \times .0001767 \times \sqrt{2 \times 9.81\,(0.9 + p\,/\,\rho g)}$$

$$\therefore \quad \sqrt{2 \times 9.81\left(0.9 + \frac{10p}{9.81}\right)} = \frac{.0015}{0.6 \times .0001767} = 14.148$$

or

$$2 \times 9.81\left(0.9 + \frac{10p}{9.81}\right) = 14.148 \times 14.148$$

$$\therefore \quad \frac{10p}{9.81} = \frac{14.148 \times 14.148}{2 \times 9.81} - 0.9 = 10.202 - 0.9 = 9.302$$

$$\therefore \quad p = \frac{9.302 \times 9.81}{10} = \mathbf{9.125\ N/cm^2.}\ \textbf{Ans.}$$

▶ 7.6 FLOW THROUGH LARGE ORIFICES

If the head of liquid is less than 5 times the depth of the orifice, the orifice is called large orifice. In case of small orifice, the velocity in the entire cross-section of the jet is considered to be constant and discharge can be calculated by $Q = C_d \times a \times \sqrt{2gh}$. But in case of a large orifice, the velocity is not constant over the entire cross-section of the jet and hence Q cannot be calculated by $Q = C_d \times a \times \sqrt{2gh}$.

7.6.1 Discharge Through Large Rectangular Orifice. Consider a large rectangular orifice in one side of the tank discharging freely into atmosphere under a constant head, H as shown in Fig. 7.7.

Let H_1 = height of liquid above top edge of orifice

H_2 = height of liquid above bottom edge of orifice

b = breadth of orifice

d = depth of orifice = $H_2 - H_1$

C_d = co-efficient of discharge.

Consider an elementary horizontal strip of depth 'dh' at a depth of 'h' below the free surface of the liquid in the tank as shown in Fig. 7.7 (b).

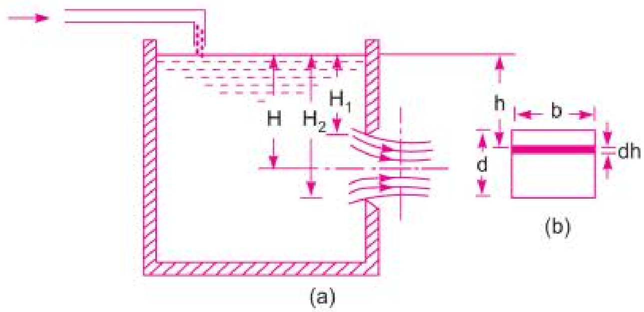


Fig. 7.7 *Large rectangular orifice.*

$\therefore$ Area of strip = $b \times dh$

and theoretical velocity of water through strip = $\sqrt{2gh}$.

$\therefore$ Discharge through elementary strip is given

$$dQ = C_d \times \text{Area of strip} \times \text{Velocity}$$
$$= C_d \times b \times dh \times \sqrt{2gh} = C_d\, b \times \sqrt{2gh}\ dh$$

By integrating the above equation between the limits H_1 and H_2, the total discharge through the whole orifice is obtained

$$\therefore \quad Q = \int_{H_1}^{H_2} C_d \times b \times \sqrt{2gh}\ dh$$
$$= C_d \times b \times \sqrt{2g}\int_{H_1}^{H_2} \sqrt{h}\ dh = C_d \times b \times \sqrt{2g}\left[\frac{h^{3/2}}{3/2}\right]_{H_1}^{H_2}$$
$$= \frac{2}{3} C_d \times b\ \sqrt{2g}\left[H_2^{3/2} - H_1^{3/2}\right]. \quad \text{...(7.8)}$$

Problem 7.11 *Find the discharge through a rectangular orifice 2.0 m wide and 1.5 m deep fitted to a water tank. The water level in the tank is 3.0 m above the top edge of the orifice. Take C_d = 0.62.*

Solution. Given :

Width of orifice, b = 2.0 m

Depth of orifice, d = 1.5 m

Height of water above top edge of the orifice, H_1 = 3 m

Height of water above bottom edge of the orifice,

$$H_2 = H_1 + d = 3 + 1.5 = 4.5 \text{ m}$$

$$C_d = 0.62$$

Discharge Q is given by equation (7.8) as

$$Q = \frac{2}{3} C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.62 \times 2.0 \times \sqrt{2+9.81}\,[4.5^{1.5} - 3^{1.5}] \text{ m}^3\text{/s}$$

$$= 3.66[9.545 - 5.196] \text{ m}^3\text{/s} = \mathbf{15.917\ m^3/s.}\ \textbf{Ans.}$$

Problem 7.12 *A rectangular orifice, 1.5 m wide and 1.0 m deep is discharging water from a tank. If the water level in the tank is 3.0 m above the top edge of the orifice, find the discharge through the orifice. Take the co-efficient of discharging for the orifice = 0.6.*

Solution. Given :

Width of orifice, $b = 1.5$ m

Depth of orifice, $d = 1.0$ m

$$H_1 = 3.0 \text{ m}$$

$$H_2 = H_1 + d = 3.0 + 1.0 = 4.0 \text{ m}$$

$$C_d = 0.6$$

Discharge, Q is given by the equation (7.8) as

$$Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.6 \times 1.5 \times \sqrt{2+9.81}\,[4.0^{1.5} - 3.0^{1.5}] \text{ m}^3\text{/s}$$

$$= 2.657\ [8.0 - 5.196] \text{ m}^3\text{/s} = \mathbf{7.45\ m^3/s.}\ \textbf{Ans.}$$

Problem 7.13 *A rectangular orifice 0.9 m wide and 1.2 m deep is discharging water from a vessel. The top edge of the orifice is 0.6 m below the water surface in the vessel. Calculate the discharge through the orifice if* C_d *= 0.6 and percentage error if the orifice is treated as a small orifice.*

Solution. Given :

Width of orifice, $b = 0.9$ m

Depth of orifice, $d = 1.2$ m

$$H_2 = 0.6 \text{ m}$$

$$H_2 = H_1 + d = 0.6 + 1.2 = 1.8 \text{ m}$$

$$C_d = 0.6$$

Discharge Q is given as $Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g} \times [H_2^{3/2} - H_1^{3/2}]$

$$= \frac{2}{3} \times 0.6 \times 2.9 \times \sqrt{2 \times 9.81}\ [1.8^{3/2} - 0.6^{3/2}] \text{ m}^3\text{/s}$$

$$= 1.5946\ [2.4149 - .4647] = \mathbf{3.1097\ m^3/s.}\ \textbf{Ans.}$$

Discharging for a small orifice

$$Q_1 = C_d \times a \times \sqrt{2gh}$$

where $h = H_1 + \frac{d}{2} = 0.6 + \frac{1.2}{2} = 1.2$ m and $a = b \times d = 0.9 \times 1.2$

$$Q_1 = 0.6 \times .9 \times 1.2 \times \sqrt{2 \times 9.81 \times 1.2} = 3.1442 \text{ m}^3/\text{s}$$

$$\% \text{ error} = \frac{Q_1 - Q}{Q} = \frac{3.1442 - 3.1097}{3.1097} = \mathbf{0.01109} \text{ or } \mathbf{1.109\%}. \textbf{ Ans.}$$

7.7 DISCHARGE THROUGH FULLY SUB-MERGED ORIFICE

Fully sub-merged orifice is one which has its whole of the outlet side sub-merged under liquid so that it discharges a jet of liquid into the liquid of the same kind. It is also called totally drowned orifice. Fig. 7.8 shows the fully sub-merged orifice. Consider two points (1) and (2), point 1 being in the reservoir on the upstream side of the orifice and point 2 being at the vena-contracta as shown in Fig. 7.8.

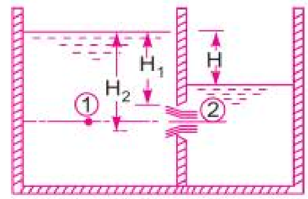


Fig. 7.8 Fully sub-merged orifice.

Let H_1 = Height of water above the top of the orifice on the upstream side,

H_2 = Height of water above the bottom of the orifice,

H = Difference in water level,

b = Width of orifice,

C_d = Co-efficient of discharge.

Height of water above the centre of orifice on upstream side

$$= H_1 + \frac{H_2 - H_1}{2} = \frac{H_1 + H_2}{2} \qquad ...(1)$$

Height of water above the centre of orifice on downstream side

$$= \frac{H_1 + H_2}{2} - H \qquad ...(2)$$

Applying Bernoulli's equation at (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \qquad [\because z_1 = z_2]$$

Now $\dfrac{p_1}{\rho g} = \dfrac{H_1 + H_2}{2}, \dfrac{p_2}{\rho g} = \dfrac{H_1 + H_2}{2} - H$ and V_1 is negligible

$$\therefore \quad \frac{H_1 + H_2}{2} + 0 = \frac{H_1 + H_2}{2} - H + \frac{V_2^2}{2g}$$

$$\therefore \quad \frac{V_2^2}{2g} = H$$

$$\therefore \quad V_2 = \sqrt{2gH}$$

Area of orifice $= b \times (H_2 - H_1)$

$\therefore$ Discharge through orifice $= C_d \times \text{Area} \times \text{Velocity}$

$$= C_d \times b\,(H_2 - H_1) \times \sqrt{2gH}$$

$$\therefore \quad Q = C_d \times b\,(H_2 - H_1) \times \sqrt{2gH}. \qquad ...(7.9)$$

Problem 7.14 *Find the discharge through a fully sub-merged orifice of width 2 m if the difference of water levels on both sides of the orifice be 50 cm. The height of water from top and bottom of the orifice are 2.5 m and 2.75 m respectively. Take $C_d = 0.6$.*

Solution. Given :

Width of orifice, $b = 2$ m

Difference of water level, $H = 50$ cm $= 0.5$ m

Height of water from top of orifice, $H_1 = 2.5$ m

Height of water from bottom of orifice, $H_2 = 2.5$ m

$C_d = 0.6$

Discharge through fully sub-merged orifice is given by equation (7.9)

or $$Q = C_d \times b \times (H_2 - H_1) \times \sqrt{2gH}$$

$$= 0.6 \times 2.0 \times (2.75 - 2.5) \times \sqrt{2 \times 9.81 \times 0.5} \text{ m}^3\text{/s}$$

$$= \mathbf{0.9396\ m^3/s.}\ \textbf{Ans.}$$

Problem 7.15 *Find the discharge through a totally drowned orifice 2.0 m wide and 1 m deep, if the difference of water levels on both the sides of the orifice be 3 m. Take $C_d = 0.62$.*

Solution. Given :

Width of orifice, $b = 2.0$ m

Depth of orifice, $d = 1$ m.

Difference of water level on both the sides

$H = 3$ m

$C_d = 0.62$

Discharge through orifice is $Q = C_d \times \text{Area} \times \sqrt{2gH}$

$$= 0.62 \times b \times d \times \sqrt{2gH}$$

$$= 0.62 \times 2.0 \times 1.0 \times \sqrt{2 \times 9.81 \times 3} \text{ m}^3\text{/s} = \mathbf{9.513\ m^3/s.}\ \textbf{Ans.}$$

▶ 7.8 DISCHARGE THROUGH PARTIALLY SUB-MERGED ORIFICE

Partially sub-merged orifice is one which has its outlet side partially sub-merged under liquid as shown in Fig. 7.9. It is also known as partially drowned orifice. Thus the partially sub-merged orifice has two portions. The upper portion behaves as an orifice discharging free while the lower portion behaves as a sub-merged orifice. Only a large orifice can behave as a partially sub-merged orifice. The total discharge Q through partially sub-merged orifice is equal to the discharges through free and the sub-merged portions.

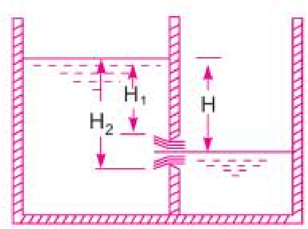


Fig. 7.9 *Partially sub-merged orifice.*

Discharge through the sub-merged portion is given by equation (7.9)

$$Q_1 = C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

Discharge through the free portion is given by equation (7.8) as

$$Q_2 = \frac{2}{3} C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H_1^{3/2}]$$

$\therefore$ Total discharge

$$Q = Q_1 + Q_2$$

$$= C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

$$+ \frac{2}{3} C_d \times b \times \sqrt{2g}\ [H_2^{3/2} - H^{3/2}]. \quad ...(7.10)$$

Problem 7.16 *A rectangular orifice of 2 m width and 1.2 m deep is fitted in one side of a large tank. The water level on one side of the orifice is 3 m above the top edge of the orifice, while on the other side of the orifice, the water level is 0.5 m below its top edge. Calculate the discharge through the orifice if $C_d = 0.64$.*

Solution. Given : Width of orifice, $b = 2$ m

Depth of orifice, $d = 1.2$ m

Height of water from top edge of orifice, $H_1 = 3$ m

Difference of water level on both sides, $H = 3 + 0.5 = 3.5$ m

Height of water from the bottom edge of orifice, $H_2 = H_1 + d = 3 + 1.2 = 4.2$ m

The orifice is partially sub-merged. The discharge through sub-merged portion,

$$Q_1 = C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

$$= 0.64 \times 2.0 \times (4.2 - 3.5) \times \sqrt{2 \times 9.81 \times 3.5} = 7.4249 \text{ m}^3/\text{s}$$

The discharge through free portion is

$$Q_2 = \frac{2}{3} C_d \times b \times \sqrt{2g}\ [H^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.64 \times 2.0 \times \sqrt{2 \times 9.81}\ [3.5^{3/2} - 3.0^{3/2}]$$

$$= 3.779\ [6.5479 - 5.1961] = 5.108 \text{ m}^3/\text{s}$$

$\therefore$ Total discharge through the orifice is

$$Q = Q_1 + Q_2 = 7.4249 + 5.108 = \mathbf{12.5329\ m^3/s.}\ \textbf{Ans.}$$

▶ 7.9 TIME OF EMPTYING A TANK THROUGH AN ORIFICE AT ITS BOTTOM

Consider a tank containing some liquid upto a height of H_1. Let an orifice is fitted at the bottom of the tank. It is required to find the time for the liquid surface to fall from the height H_1 to a height H_2.

Let A = Area of the tank

a = Area of the orifice

H_1 = Initial height of the liquid

H_2 = Final height of the liquid

T = Time in seconds for the liquid to fall from H_1 to H_2.

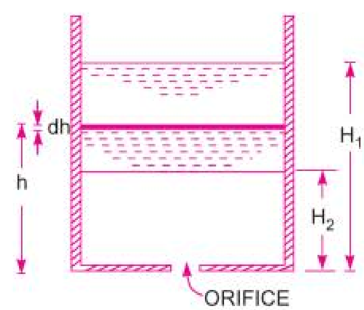


Fig. 7.9. (*a*)

Let at any time, the height of liquid from orifice is h and let the liquid surface fall by a small height dh in time dT. Then

Volume of liquid leaving the tank in time, $dT = A \times dh$

Also the theoretical velocity through orifice, $V = \sqrt{2gh}$

$\therefore$ Discharge through orifice/sec,

$$dQ = C_d \times \text{Area of orifice} \times \text{Theoretical velocity} = C_d \,.\, a \,.\, \sqrt{2gh}$$

$\therefore$ Discharge through orifice in time interval

$$dT = C_d \,.\, a \,.\, \sqrt{2gh} \,\,.\, dT$$

As the volume of liquid leaving the tank is equal to the volume of liquid flowing through orifice in time dT, we have

$$A(-\,dh) = C_d \,.\, a \,.\, \sqrt{2gh} \,\,.\, dT$$

– ve sign is inserted because with the increase of time, head on orifice decreases.

$$\therefore \qquad -\,Adh = C_d \,.\, a \,.\, \sqrt{2gh} \,\,.\, dT \text{ or } dT = \frac{-A\,dh}{C_d \,.\, a \,.\, \sqrt{2gh}} = \frac{-A(h)^{-1/2}}{C_d \,.\, a \,.\, \sqrt{2g}}dh$$

By integrating the above equation between the limits H_1 and H_2, the total time, T is obtained as

$$\int_0^T dT = \int_{H_1}^{H_2} \frac{-Ah^{-1/2}\,dh}{C_d \,.\, a \,.\, \sqrt{2g}} = \frac{-A}{C_d \,.\, a \,.\, \sqrt{2g}} \int_{H_1}^{H_2} h^{-1/2}\,dh$$

or

$$T = \frac{-A}{C_d \,.a \,.\, \sqrt{2g}} \left[\frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_{H_1}^{H_2} = \frac{-A}{C_d \,.a \,.\, \sqrt{2g}} \left[\frac{\sqrt{h}}{\frac{1}{2}}\right]_{H_1}^{H_2}$$

$$= \frac{-2A}{C_d \,.\, a \,.\, \sqrt{2g}}\left[\sqrt{H_2} - \sqrt{H_1}\right] = \frac{2A\left[\sqrt{H_1} - \sqrt{H_2}\right]}{C_d \,.\, a \,.\, \sqrt{2g}} \qquad \text{...(7.11)}$$

For emptying the tank completely, $H_2 = 0$ and hence

$$T = \frac{2A\sqrt{H_1}}{C_d \,.\, a \,.\sqrt{2g}}. \qquad \text{...(7.12)}$$

Problem 7.17 *A circular tank of diameter 4 m contains water upto a height of 5 m. The tank is provided with an orifice of diameter 0.5 m at the bottom. Find the time taken by water (i) to fall from 5 m to 2 m (ii) for completely emptying the tank. Take C_d = 0.6.*

Solution. Given :

Dia. of tank, $D = 4$ m

$\therefore$ Area, $A = \frac{\pi}{4}(4)^2 = 12.566 \text{ m}^2$

Dia. of orifice, $d = 0.5$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.5)^2 = 0.1963 \text{ m}^2$

Initial height of water, $H_1 = 5$ m

Final height of water, (*i*) $H_2 = 2$ m (*ii*) $H_2 = 0$

First Case. When $H_2 = 2$ m

Using equation (7.11), we have $T = \frac{2A}{C_d \,.\, a \,.\, \sqrt{2g}}\left[\sqrt{H_1} - \sqrt{H_2}\right]$

$$= \frac{2 \times 12.566}{0.6 \times .1963 \times \sqrt{2 \times 9.81}} \left[\sqrt{5} - \sqrt{2.0}\right] \text{ seconds}$$

$$= \frac{20.653}{0.5217} = \mathbf{39.58\ seconds.\ Ans.}$$

Second Case. When $H_2 = 0$

$$T = \frac{2A}{C_d \cdot a \cdot \sqrt{2g}} \sqrt{H_1} = \frac{2 \times 12.566 \times \sqrt{5}}{0.6 \times .1963 \times \sqrt{2 \times 9.81}}$$

$$= \mathbf{107.7\ seconds.\ Ans.}$$

Problem 7.18 *A circular tank of diameter 1.25 m contains water upto a height of 5 m. An orifice of 50 mm diameter is provided at its bottom. If $C_d = 0.62$, find the height of water above the orifice after 1.5 minutes.*

Solution. Given :

Dia. of tank, $D = 1.25$ m

∴ Area, $A = \frac{\pi}{4}(1.25)^2 = 1.227\ \text{m}^2$

Dia. of orifice, $d = 50$ mm $= .05$ m

∴ Area, $a = \frac{\pi}{4}(.05)^2 = .001963\ \text{m}^2$

$C_d = 0.62$

Initial height of water, $H_1 = 5$ m

Time in seconds, $T = 1.5 \times 60 = 90$ seconds

Let the height of water after 90 seconds $= H_2$

Using equation (7.11), we have $T = \dfrac{2A\left[\sqrt{H_1} - \sqrt{H_2}\right]}{C_d \cdot a \cdot \sqrt{2g}}$

or
$$90 = \frac{2 \times 1.227\left[\sqrt{5} - \sqrt{H_2}\right]}{0.62 \times 0.001963 \times \sqrt{2 \times 9.81}} = 455.215\left[2.236 - \sqrt{H_2}\right]$$

∴
$$\sqrt{H_2} = 2.236 - \frac{90}{455.215} = 236 - 0.1977 = 2.0383$$

∴
$$H_2 = 2.0383 \times 2.0383 = \mathbf{4.154\ m.\ Ans.}$$

▶ 7.10 TIME OF EMPTYING A HEMISPHERICAL TANK

Consider a hemispherical tank of radius R fitted with an orifice of area 'a' at its bottom as shown in Fig. 7.10. The tank contains some liquid whose initial height is H_1 and in time T, the height of liquid falls to H_2. It is required to find the time T.

Let at any instant of time, the head of liquid over the orifice is h and at this instant let x be the radius of the liquid surface. Then

Area of liquid surface, $A = \pi x^2$

and theoretical velocity of liquid $= \sqrt{2gh}$.

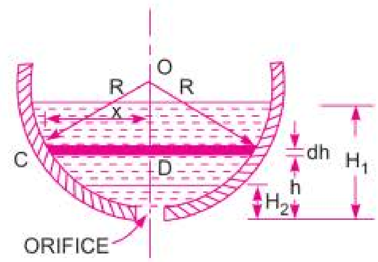


Fig. 7.10 *Hemispherical tank.*

Let the liquid level falls down by an amount of dh in time dT.

$\therefore$ Volume of liquid leaving tank in time $dT = A \times dh$

$$= \pi x^2 \times dh \qquad ...(i)$$

Also volume of liquid flowing through orifice

$$= C_d \times \text{area of orifice} \times \text{velocity} = C_d . a. \sqrt{2gh} \text{ second}$$

$\therefore$ Volume of liquid flowing through orifice in time dT

$$= C_d . a. \sqrt{2gh} \times dT \qquad ...(ii)$$

From equations (i) and (ii), we get

$$\pi x^2 (-dh) = C_d . a. \sqrt{2gh} \; . \; dT$$

$-$ve sign is introduced, because with the increase of T, h will decrease

$$\therefore \qquad -\pi x^2 \, dh = C_d . a. \sqrt{2gh} \; . \; dT \qquad ...(iii)$$

But from Fig. 7.10, for ΔOCD, we have $OC = R$

$$DO = R - h$$

$$\therefore \qquad CD = x = \sqrt{OC^2 - OD^2} = \sqrt{R^2 - (R - h)^2}$$

$$\therefore \qquad x^2 = R^2 - (R - h)^2 = R^2 - (R^2 + h^2 - 2Rh) = 2Rh - h^2$$

Substituting x^2 in equation (iii), we get

$$-\pi(2Rh - h^2)dh = C_d . a. \sqrt{2gh} \; . \; dT$$

or

$$dT = \frac{-\pi\left(2Rh - h^2\right)dh}{C_d \, . \, a \, . \sqrt{2gh}} = \frac{-\pi}{C_d \, . \, a \, . \sqrt{2g}}\left(2Rh - h^2\right)h^{-1/2} \; dh$$

$$= \frac{-\pi}{C_d \, . \, a \, . \sqrt{2g}} (2Rh^{1/2} - h^{3/2})dh$$

The total time T required to bring the liquid level from H_1 to H_2 is obtained by integrating the above equation between the limits H_1 and H_2.

$$\therefore \qquad T = \int_{H_1}^{H_2} \frac{-\pi}{C_d \, . \, a \, . \sqrt{2g}} (2Rh^{1/2} - h^{3/2})dh$$

$$= \frac{-\pi}{C_d \, . \, a \, . \sqrt{2g}} \int_{H_1}^{H_2} (2Rh^{1/2} - h^{3/2})dh$$

$$= \frac{-\pi}{C_d \times a \times \sqrt{2g}} \left[2R \frac{h^{1/2+1}}{\frac{1}{2}+1} - \frac{h^{3/2}+1}{\frac{3}{2}+1} \right]_{H_1}^{H_2}$$

$$= \frac{-\pi}{C_d \times a \times \sqrt{2g}} \left[2 \times \frac{2}{3} Rh^{3/2} - \frac{2}{5} h^{5/2} \right]_{H_1}^{H_2}$$

$$= \frac{-\pi}{C_d \times a \times \sqrt{2g}} \left[\frac{4}{3} R\left(H_2^{3/2} - H_1^{3/2}\right) - \frac{2}{5}\left(H_2^{5/2} - H_1^{5/2}\right) \right]$$

$$= \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[\frac{4}{3} R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right) \right] \quad ...(7.13)$$

For completely emptying the tank, $H_2 = 0$ and hence

$$T = \frac{\pi}{C_d \,.\, a \,.\, \sqrt{2g}} \left[\frac{4}{3} RH_1^{3/2} - \frac{2}{5} H_1^{5/2} \right]. \quad ...(7.14)$$

Problem 7.19 *A hemispherical tank of diameter 4 m contains water upto a height of 1.5 m. An orifice of diameter 50 mm is provided at the bottom. Find the time required by water (i) to fall from 1.5 m to 1.0 m (ii) for completely emptying the tank. Tank $C_d = 0.6$.*

Solution. Given :

Dia. of hemispherical tank, $D = 4$ m

$\therefore$ Radius, $R = 2.0$ m

Dia. of orifice, $d = 50$ mm $= 0.05$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.05)^2 = 0.001963 \text{ m}^2$

Initial height of water, $H_1 = 1.5$ m

$C_d = 0.6$

First Case. $H_2 = 1.0$

Time T is given by equation (7.13)

$$\therefore \quad T = \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[\frac{4}{3} R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right) \right]$$

$$= \frac{\pi}{0.6 \times .001963 \times \sqrt{2 \times 9.81}} \times \left[\frac{4}{3} \times 2.0\left(1.5^{3/2} - 1.0^{3/2}\right) - \frac{2}{5}\left(1.5^{5/2} - 1.0^{5/2}\right) \right]$$

$= 602.189$ [2.2323 – 0.7022] $= 921.4$ second

= **15 min 21.4 sec. Ans.**

Second Case. $H_2 = 0$ and hence time T is given by equation (7.14)

$$\therefore \quad T = \frac{\pi}{C_d \,.\, a \,.\, \sqrt{2g}} \left[\frac{4}{3} RH_1^{3/2} - \frac{2}{5} H_1^{5/2} \right]$$

$$= \frac{\pi}{0.6 \times .001963 \sqrt{2 \times 9.81}} \left[\frac{4}{3} \times 2.0 \times 1.5^{3/2} - \frac{2}{5} \times 1.5^{5/2} \right]$$

$= 602.189\ [4.8989 - 1.1022]$ sec $= 2286.33$ sec

$=$ **38 min 6.33 sec. Ans.**

Problem 7.20 *A hemispherical cistern of 6 m radius is full of water. It is fitted with a 75 mm diameter sharp edged orifice at the bottom. Calculate the time required to lower the level in the cistern by 2 metres. Assume co-efficient of discharge for the orifice is 0.6.*

Solution. Given :

Radius of hemispherical cistern, $R = 6$ m

Initial height of water, $H_1 = 6$ m

Dia. of orifice, $d = 75$ mm $= 0.075$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.075)^2 = .004418\ \text{m}^2$

Fall of height of water $= 2$ m

$\therefore$ Final height of water, $H_2 = 6 - 2 = 4$ m

$C_d = 0.6$

The time T is given by equation (7.31)

$$T = \frac{\pi}{C_d \times a \times \sqrt{2g}}\left[\frac{4}{3}R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right)\right]$$

$$= \frac{\pi}{0.6 \times .004418 \times \sqrt{2 \times 9.81}} \times \left[\frac{4}{3} \times 6\left(6.0^{3/2} - 4.0^{3/2}\right) - \frac{2}{5}\left(6.0^{5/2} - 4.0^{5/2}\right)\right]$$

$= 267.56\ [8(14.6969 - 8.0) - 0.4\ (88.18 - 32.0)]$

$= 267.56\ [53.575 - 22.472]$ sec

$= 8321.9$ sec $=$ **2hrs 18 min 42 sec. Ans.**

Problem 7.21 *A cylindrical tank is having a hemispherical base. The height of cylindrical portion is 5 m and diameter is 4 m. At the bottom of this tank an orifice of diameter 200 mm is fitted. Find the time required to completely emptying the tank. Take $C_d = 0.6$.*

Solution. Given :

Height of cylindrical portion (II) $= 5$ m

Dia. of tank $= 4.0$ m

$\therefore$ Area, $A = \frac{\pi}{4}(4)^2 = 12.566\ \text{m}^2$

Dia. of orifice, $d = 200$ mm $= 0.2$ m

$\therefore$ Area, $a = \frac{\pi}{4}(.2)^2 = 0.0314\ \text{m}^2$

$C_d = 0.6$

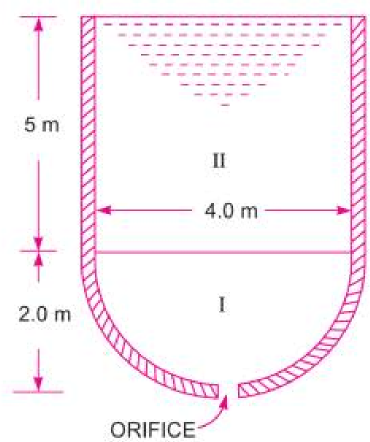


Fig. 7.11

The tank is splitted in two portions. First portion is a hemispherical tank and second portion is cylindrical tank.

Let T_1 = time for emptying hemispherical portion I.

T_2 = time for emptying cylindrical portion II.

Then total time $T = T_1 + T_2$.

For Portion I. $H_1 = 2.0$ m, $H_2 = 0$. Then T_1 is given by equation (7.14) as

$$T_1 = \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[\frac{4}{3} R H_1^{3/2} - \frac{2}{5} H_1^{5/2}\right]$$

$$= \frac{\pi}{0.6 \times .0314 \times \sqrt{2 \times 9.81}} \left[\frac{4}{3} \times 2.0 \times 2.0^{3/2} - \frac{2}{5} \times 2.0^{5/2}\right]$$

$$= 37.646\ [7.5424 - 2.262] \text{ sec} = 198.78 \text{ sec.}$$

For Portion II. $H_1 = 2.0 + 5.0 = 7.0$ m, $H_2 = 2.0$. Then T_2 is given by equation (7.11) as

$$T_2 = \frac{2A\left[\sqrt{H_1} - \sqrt{H_2}\right]}{C_d \times a \times \sqrt{2g}} = \frac{2 \times 12.566\left[\sqrt{7} - \sqrt{2.0}\right]}{0.6 \times .0314 \times \sqrt{2 \times 9.81}} \text{ sec} = 370.92 \text{ sec}$$

$\therefore$ Total time, $T = T_1 + T_2 = 198.78 + 370.92 = 569.7$ sec

= **9 min 29 sec. Ans.**

▶ 7.11 TIME OF EMPTYING A CIRCULAR HORIZONTAL TANK

Consider a circular horizontal tank of length L and radius R, containing liquid upto a height of H_1. Let an orifice of area 'a' is fitted at the bottom of the tank. Then the time required to bring the liquid level from H_1 to H_2 is obtained as :

Let at any time, the height of liquid over orifice is 'h' and in time dT, let the height falls by an height of 'dh'. Let at this time, the width of liquid surface = AC as shown in Fig. 7.12.

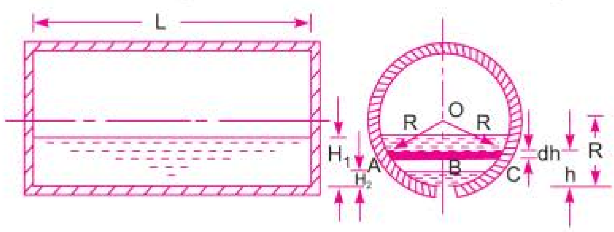


Fig. 7.12

$\therefore$ Surface area of liquid = $L \times AC$

But $$AC = 2 \times AB = 2\left[\sqrt{AO^2 - OB^2}\right] = 2\left[\sqrt{R^2 - (R-h)^2}\right]$$

$$= 2\sqrt{R^2 - \left(R^2 + h^2 - 2Rh\right)} = 2\sqrt{2Rh - h^2}$$

$\therefore$ Surface area, $A = L \times 2\sqrt{2Rh - h^2}$

$\therefore$ Volume of liquid leaving tank in time dT

$$= A \times dh = 2L\sqrt{2Rh - h^2} \times dh \qquad ...(i)$$

Also the volume of liquid flowing through orifice in time dT

$$= C_d \times \text{Area of orifice} \times \text{Velocity} \times dT$$

But the velocity of liquid at the time considered = $\sqrt{2gh}$

$\therefore$ Volume of liquid flowing through orifice in time dT

$$= C_d \times a \times \sqrt{2gh} \times dT \qquad ...(ii)$$

Equating (*i*) and (*ii*), we get

$$2L\sqrt{2Rh - h^2} \times (-dh) = C_d \times a \times \sqrt{2gh \times dT}$$

– ve sign is introduced as with the increase of T, the height h decreases,

$$\therefore \qquad dT = \frac{-2L\sqrt{2Rh - h^2}\,dh}{C_d \times a \times \sqrt{2gh}} = \frac{-2L\sqrt{(2R-h)}\,dh}{C_d \times a \times \sqrt{2g}}$$

[Taking $\sqrt{h}$ common]

$$\therefore \qquad \text{Total time, } T = \int_{H_1}^{H_2} \frac{-2L(2R-h)^{1/2}\,dh}{C_d \times a \times \sqrt{2g}}$$

$$= \frac{-2L}{C_d \times a \times \sqrt{2g}} \int_{H_1}^{H_2} [2R-h]^{1/2}\,dh$$

$$= \frac{-2L}{C_d \times a \times \sqrt{2g}} \left[\frac{(2R-h)^{1/2+1}}{\frac{1}{2}+1} \times (-1) \right]_{H_1}^{H_2}$$

$$= \frac{2L}{C_d \times a \times \sqrt{2g}} \times \frac{2}{3} \times \left[(2R-h)^{3/2} \right]_{H_1}^{H_2}$$

$$= \frac{4L}{3C_d \times a \times \sqrt{2g}} \left[(2R-H_2)^{3/2} - (2R-H_1)^{3/2} \right] \qquad ...(7.15)$$

For completely emptying the tank, $H_2 = 0$ and hence

$$T = \frac{4L}{3C_d \times a \times \sqrt{2g}} \left[(2R)^{3/2} - (2R-H_1)^{3/2} \right]. \qquad ...(7.16)$$

Problem 7.22 *An orifice of diameter 100 mm is fitted at the bottom of a boiler drum of length 5 m and of diameter 2 m. The drum is horizontal and half full of water. Find the time required to empty the boiler, given the value of $C_d = 0.6$.*

Solution. Given :

Dia. of orifice, $d = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4}(.1)^2 = .007854 \text{ m}^2$

Length, $L = 5 \text{ m}$

Dia. of drum, $D = 2 \text{ m}$

$\therefore$ Radius, $R = 1 \text{ m}$

Initial height of water, $H_1 = 1 \text{ m}$

Final height of water, $H_2 = 0$

$C_d = 0.6$

For completely emptying the tank, T is given by equation (7.16)

$$\therefore \quad T = \frac{4L}{3 \times C_d \times a \times \sqrt{2g}}\left[(2R)^{3/2} - (2R - H_1)^{3/2}\right]$$

$$= \frac{4 \times 5.0}{3 \times .06 \times .007854 \times \sqrt{2 \times 9.81}}\left[(2 \times 1)^{3/2} - (2 \times 1 - 1)^{3/2}\right]$$

$= 319.39\ [2.8284 - 1.0] = 583.98 \text{ sec} =$ **9 min 44 sec. Ans.**

Problem 7.23 *An orifice of diameter 150 mm is fitted at the bottom of a boiler drum of length 8 m and of diameter 3 metres. The drum is horizontal and contains water upto a height of 2.4 m. Find the time required to empty the boiler. Take $C_d = 0.6$.*

Solution. Given :

Dia. of orifice, $d = 150 \text{ mm} = 0.15 \text{ m}$

$\therefore$ Area, $a = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$

Length, $L = 8.0 \text{ m}$

Dia. of boiler, $D = 3.0 \text{ m}$

$\therefore$ Radius, $R = 1.5 \text{ m}$

Initial height of water, $H_1 = 2.4 \text{ m}$

Find height of water, $H_2 = 0$

$C_d = 0.6.$

For completely emptying the tank, T is given by equation (7.16) as

$$T = \frac{4L}{3C_d \times a \times \sqrt{2g}}\left[(2\text{R})^{3/2} - (2R - H_1)^{3/2}\right]$$

$$= \frac{4 \times 8.0}{3 \times .6 \times .01767 \times \sqrt{2 \times 9.81}}\left[(2 \times 1.5)^{3/2} - (2 \times 1.5 - 2.4)^{3/2}\right]$$

$= 227.14\ [5.196 - 0.4647] = 1074.66 \text{ sec}$

$=$ **17 min 54.66 sec. Ans.**

7.12 CLASSIFICATION OF MOUTHPIECES

1. The mouthpieces are classified as (*i*) External mouthpiece or (*ii*) Internal mouthpiece depending upon their position with respect to the tank or vessel to which they are fitted.

2. The mouthpiece are classified as (*i*) Cylindrical mouthpiece or (*ii*) Convergent mouthpiece or (*iii*) Convergent-divergent mouthpiece depending upon their shapes.

3. The mouthpieces are classified as (*i*) Mouthpieces running full or (*ii*) Mouthpieces running free, depending upon the nature of discharge at the outlet of the mouthpiece. This classification is only for internal mouthpieces which are known Borda's or Re-entrant mouthpieces. A mouthpiece is said to be running free if the jet of liquid after contraction does not touch the sides of the mouthpiece. But if the jet after contraction expands and fills the whole mouthpiece it is known as running full.

7.13 FLOW THROUGH AN EXTERNAL CYLINDRICAL MOUTHPIECE

A mouthpiece is a short length of a pipe which is two or three times its diameter in length. If this pipe is fitted externally to the orifice, the mouthpiece is called external cylindrical mouthpiece and the discharge through orifice increases.

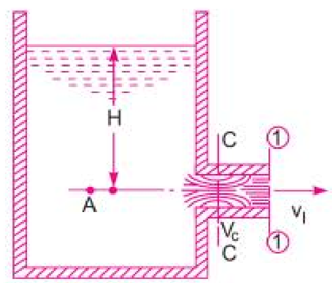


Fig. 7.13 *External cylindrical mouthpieces.*

Consider a tank having an external cylindrical mouthpiece of cross-sectional area a_1, attached to one of its sides as shown in Fig. 7.13. The jet of liquid entering the mouthpiece contracts to form a vena-contracta at a section *C-C*. Beyond this section, the jet again expands and fill the mouthpiece completely.

Let H = Height of liquid above the centre of mouthpiece
v_c = Velocity of liquid at *C-C* section
a_c = Area of flow at vena-contracta
v_1 = Velocity of liquid at outlet
a_1 = Area of mouthpiece at outlet
C_c = Co-efficient of contraction.

Applying continuity equation at *C-C* and (1)-(1), we get

$$a_c \times v_c = a_1 v_1$$

$$\therefore \qquad v_c = \frac{a_1 v_1}{a_c} = \frac{v_1}{a_c / a_1}$$

But $$\frac{a_c}{a_1} = C_c = \text{Co-efficient of contraction}$$

Taking $C_c = 0.62$, we get $\dfrac{a_c}{a_1} = 0.62$

$$\therefore \qquad v_c = \frac{v_1}{0.62}$$

The jet of liquid from section *C-C* suddenly enlarges at section (1)-(1). Due to sudden enlargement, there will be a loss of head, h_L* which is given as $h_L = \dfrac{(v_c - v_1)^2}{2g}$

* Please refer Art. 11.4.1 for loss of head due to sudden enlargement.

But $v_c = \dfrac{v_1}{0.62}$ hence $h_L = \dfrac{\left(\dfrac{v_1}{0.62} - v_1\right)^2}{2g} = \dfrac{v_1^2}{2g}\left[\dfrac{1}{0.62} - 1\right]^2 = \dfrac{0.375\, v_1^2}{2g}$

Applying Bernoulli's equation to point A and (1)-(1)

$$\frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 + h_L$$

where $z_A = z_1$, v_A is negligible,

$\dfrac{p_1}{\rho g}$ = atmospheric pressure = 0

$$\therefore \qquad H + 0 = 0 + \frac{v_1^2}{2g} + .375\,\frac{v_1^2}{2g}$$

$$\therefore \qquad H = 1.375\,\frac{v_1^2}{2g}$$

$$\therefore \qquad v_1 = \sqrt{\frac{2gH}{1.375}} = 0.855\,\sqrt{2gH}$$

Theoretical velocity of liquid at outlet is $v_{th} = \sqrt{2gH}$

$\therefore$ Co-efficient of velocity for mouthpiece

$$C_v = \frac{\text{Actual velocity}}{\text{Theoretical velocity}} = \frac{0.855\sqrt{2gH}}{\sqrt{2gH}} = 0.855.$$

C_c for mouthpiece = 1 as the area of jet of liquid at outlet is equal to the area of mouthpiece at outlet.

Thus $\qquad C_d = C_c \times C_v = 1.0 \times .855 = 0.855$

Thus the value of C_d for mouthpiece is more than the value of C_d for orifice, and so discharge through mouthpiece will be more.

Problem 7.24 *Find the discharge from a 100 mm diameter external mouthpiece, fitted to a side of a large vessel if the head over the mouthpiece is 4 metres.*

Solution. Given :

Dia. of mouthpiece = 100 m = 0.1 m

$\therefore$ Area, $\qquad a = \dfrac{\pi}{4}(0.1)^2 = 0.007854\ \text{m}^2$

Head, $\qquad H = 4.0$ m

C_d for mouthpiece $\qquad = 0.855$

$\therefore$ Discharge $\qquad = C_d \times \text{Area} \times \text{Velocity} = 0.855 \times a \times \sqrt{2gH}$

$\qquad = .855 \times .007854 \times \sqrt{2 \times 9.81 \times 4.0}$ = **.05948 m³/s. Ans.**

Problem 7.25 *An external cylindrical mouthpiece of diameter 150 mm is discharging water under a constant head of 6 m. Determine the discharge and absolute pressure head of water at vena-contracta. Take C_d = 0.855 and C_c for vena-contracta = 0.62. Atmospheric pressure head = 10.3 m of water.*

Solution. Given :

Dia. of mouthpiece, $d = 150 \text{ mm} = 0.15 \text{ cm}$

$\therefore$ Area, $a = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$

Head, $H = 6.0 \text{ m}$

$C_d = 0.855$

C_c at vena-contracta = 0.62

Atmospheric pressure head, $H_a = 10.3$ m

$\therefore$ Discharge $= C_d \times a \times \sqrt{2gH}$

$= 0.855 \times .01767 \times \sqrt{2 \times 9.81 \times 6.0}$ = **0.1639 m³/s. Ans.**

Pressure Head at Vena-contracta

Applying Bernoulli's equation at *A* and *C-C*, we get

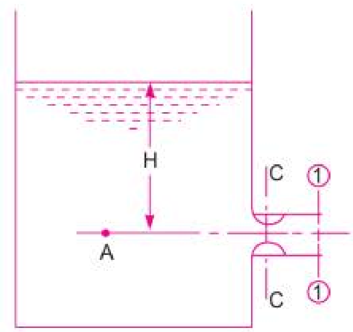


Fig. 7.14

$$\frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c$$

But $\frac{p_A}{\rho g} = H_a + H,\ v_A = 0,$

$z_A = z_c$

$$\therefore \qquad H_a + H + 0 = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} = H_c + \frac{v_c^2}{2g}$$

$$\therefore \qquad H_c = H_a + H - \frac{v_c^2}{2g}$$

But $v_c = \frac{v_1}{0.62}$

$$\therefore \qquad H_c = H_a + H - \left(\frac{v_1}{.62}\right)^2 \times \frac{1}{2g} = H_a + H - \frac{v_1^2}{2g} \times \frac{1}{(.62)^2}$$

But $\text{H} = 1.375 \ \frac{v_1^2}{2g}$

$$\therefore \qquad \frac{v_1^2}{2g} = \frac{H}{1.375} = 0.7272\ H$$

$$\therefore \qquad H_c = H_a + H - .7272\ H \times \frac{1}{(.62)^2}$$

$= H_a + H - 1.89\ H = H_a - .89\ H$

$= 10.3 - .89 \times 6.0$ $\{\because H_a = 10.3 \text{ and } H = 6.0\}$

$= 10.3 - 5.34$ = **4.96 m (Absolute). Ans.**

▶ 7.14 FLOW THROUGH A CONVERGENT-DIVERGENT MOUTHPIECE

If a mouthpiece converges upto vena-contracta and then diverges as shown in Fig. 7.15 then that type of mouthpiece is called Convergent-Divergent Mouthpiece. As in this mouthpiece there is no sudden enlargement of the jet, the loss of energy due to sudden enlargement is eliminated. The co-efficient of discharge for this mouthpiece is unity. Let H is the head of liquid over the mouthpiece.

Applying Bernoulli's equation to the free surface of water in tank and section C-C, we have

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c$$

Taking datum passing through the centre of orifice, we get

$$\frac{p}{\rho g} = H_a,\ v = 0,\ z = H,\ \frac{p_c}{\rho g} = H_c,\ z_c = 0$$

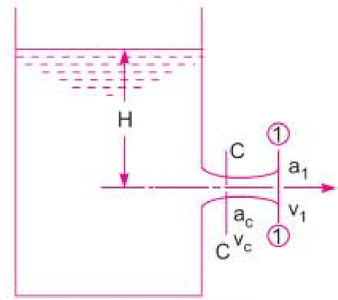


Fig. 7.15 *Convergent-divergent mouthpiece.*

$$\therefore \quad H_a + 0 + H = H_c + \frac{v_c^2}{2g} + 0 \qquad \ldots(i)$$

$$\therefore \quad \frac{v_c^2}{2g} = H_a + H - H_c \qquad \ldots(ii)$$

or

$$v_c = \sqrt{2g(H_a + H - H_c)}$$

Now applying Bernoulli's equation at sections C-C and (1)-(1)

$$\frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1$$

But

$$z_c = z_1 \text{ and } \frac{p_1}{\rho g} = H_a$$

$$\therefore \quad H_c + \frac{v_c^2}{2g} = H_a + \frac{v_1^2}{2g}$$

Also from (*i*), $\quad H_c + v_c^2/2g = H + H_a$

$$\therefore \quad H_a + v_1^2/2g = H + H_a$$

$$\therefore \quad v_1 = \sqrt{2gH} \qquad \ldots(iii)$$

Now by continuity equation, $a_c\, v_c = v_1 \times a_1$

$$\therefore \quad \frac{a_1}{a_c} = \frac{v_c}{v_1} = \frac{\sqrt{2g(H_a + H - H_c)}}{\sqrt{2gH}} = \sqrt{\frac{H_a}{H} + 1 - \frac{H_c}{H}}$$

$$= \sqrt{1 + \frac{H_a - H_c}{H}} \qquad \ldots(7.17)$$

The discharge, Q is given as $Q = a_c \times \sqrt{2gH}$ $\qquad \ldots(7.18)$

where a_c = area at vena-contracta.

Problem 7.26 *A convergent-divergent mouthpiece having throat diameter of 4.0 cm is discharging water under a constant head of 2.0 m, determine the maximum outer diameter for maximum discharge. Find maximum discharge also. Take H_a = 10.3 m of water and H_{sep} = 2.5 m of water (absolute).*

Solution. Given :

Dia. of throat, $d_c = 4.0$ cm

$\therefore$ Area, $a_c = \frac{\pi}{4}(4)^2 = 12.566 \text{ cm}^2$

Constant head, $H = 2.0$ m

Find max. dia. at outlet, d_1 and Q_{max}

$H_a = 10.3$ m

$H_{sep} = 2.5$ m (absolute)

The discharge, Q in convergent-divergent mouthpiece depends on the area at throat.

$$\therefore \quad Q_{max} = a_c \times \sqrt{2gH} = 12.566 \times \sqrt{2 \times 9.81 \times 200} = \mathbf{7871.5\ cm^3/s.}\ \textbf{Ans.}$$

Now ratio of areas at outlet and throat is given by equation (7.17) as

$$\frac{a_1}{a_c} = \sqrt{1 + \frac{H_a - H_c}{H}} = \sqrt{1 + \frac{10.3 - 2.5}{2.0}} \qquad \{\because H_c = H_{sep} = 2.5\}$$

$$= 2.2135$$

$$\frac{\pi}{4} d_1^2 \Big/ \frac{\pi}{4} d_c^2 = 2.2135 \text{ or } \left(\frac{d_1}{d_c}\right)^2 = 2.2135$$

$$\therefore \quad \frac{d_1}{d_c} = \sqrt{2.2135} = 1.4877$$

$$\therefore \quad d_1 = 1.4877 \times d_c = 1.4877 \times 4.0 = \mathbf{5.95\ cm.}\ \textbf{Ans.}$$

Problem 7.27 *The throat and exit diameters of convergent-divergent mouthpiece are 5 cm and 10 cm respectively. It is fitted to the vertical side of a tank, containing water. Find the maximum head of a water for steady flow. The maximum vacuum pressure is 8 m of water and take atmospheric pressure = 10.3 m water.*

Solution. Given :

Dia. at throat, $d_c = 5$ cm

Dia. at exit, $d_1 = 10$ cm

Atmospheric pressure head, $H_a = 10.3$ m

The maximum vacuum pressure will be at a throat only

$\therefore$ Pressure head at throat = 8 m (vacuum)

or $H_c = H_a - 8.0$ (absolute)

$= 10.3 - 8.0 = 2.3$ m (abs.)

Let maximum head of water over mouthpiece = H m of water.

The ratio of areas at outlet and throat of a convergent-divergent mouthpiece is given by equation (7.17).

$$\frac{a_1}{a_c} = \sqrt{1 + \frac{H_a - H_c}{H}} \quad \text{or} \quad \frac{\frac{\pi}{4}(d_1)^2}{\frac{\pi}{4}(d_c)^2} = \sqrt{1 + \frac{10.3 - 2.3}{H}}$$

or
$$\frac{10^2}{5^2} = 4 = \sqrt{1 + \frac{8}{H}} \quad \text{or} \quad 16 = 1 + \frac{8}{H} \quad \text{or} \quad 15 = \frac{8}{H}$$

$\therefore$
$$H = \frac{8}{15} = 0.5333 \text{ m of water}$$

$\therefore$ Maximum head of water = **0.533 m. Ans.**

Problem 7.28 *A convergent-divergent mouthpiece is fitted to the side of a tank. The discharge through mouthpiece under a constant head of 1.5 m is 5 litres/s. The head loss in the divergent portion is 0.10 times the kinetic head at outlet. Find the throat and exit diameters, if separation pressure is 2.5 m and atmospheric pressure head = 10.3 m of water.*

Solution. Given :

Constant head, $H = 1.5$ m

Discharge, $Q = 5$ litres $= .005$ m^3/s

h_L or Head loss in divergent $= 0.1 \times$ kinetic head at outlet

H_c or $H_{sep} = 2.5$ (abs.)

$H_a = 10.3$ m of water

Find (*i*) Dia. at throat, d_c

(*ii*) Dia. at outlet, d_1

(*i*) **Dia. at throat (d_c).** Applying Bernoulli's equation to the free water surface and throat section, we get (See Fig. 7.15).

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c$$

Taking the centre line of mouthpiece as datum, we get

$$H_a + 0 + H = H_c + \frac{v_c^2}{2g}$$

$\therefore$
$$\frac{v_c^2}{2g} = H_a + H - H_c = 10.3 + 1.5 - 2.5 = 9.3 \text{ m of water}$$

$\therefore$
$$v_c = \sqrt{2 \times 9.81 \times 9.3} = 13.508 \text{ m/s}$$

Now
$$Q = a_c \times v_c \text{ or } .005 = \frac{\pi}{4} d_c^2 \times 13.508$$

$\therefore$
$$d_c = \sqrt{\frac{.005 \times 4}{\pi \times 13.508}} = \sqrt{.00047} = .0217 \text{ m} = \mathbf{2.17 \text{ cm.}} \text{ Ans.}$$

(*ii*) **Dia. at outlet (d_1).** Applying Bernoulli's equation to the free water surface and outlet of mouthpiece (See Fig. 7.15), we get

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 + h_L$$

$$H_a + 0 + H = H_a + \frac{v_1^2}{2g} + 0 + 0.1 \times \frac{v_1^2}{2g} \qquad \left\{\because \frac{p_1}{\rho g} = H_a\right\}$$

$\therefore$
$$H = \frac{v_1^2}{2g} + .1 \times \frac{v_1^2}{2g} = 1.1\,\frac{v_1^2}{2g}$$

$\therefore$
$$v_1 = \sqrt{\frac{2gH}{1.1}} = \sqrt{\frac{2 \times 9.81 \times 1.5}{1.1}} = 5.1724$$

Now
$$Q = A_1 v_1 \quad \text{or} \quad .005 = \frac{\pi}{4} d_1^{\,2} \times v_1$$

$\therefore$
$$d_1 = \sqrt{\frac{4 \times .005}{\pi \times v_1}} = \sqrt{\frac{4 \times .005}{\pi \times 5.1724}} = 0.035 \text{ m} = \mathbf{3.5\ cm.}\ \textbf{Ans.}$$

▶ 7.15 FLOW THROUGH INTERNAL OR RE-ENTRANT ON BORDA'S MOUTHPIECE

A short cylindrical tube attached to an orifice in such a way that the tube projects inwardly to a tank, is called an internal mouthpiece. It is also called Re-entrant or Borda's mouthpiece. If the length of the tube is equal to its diameter, the jet of liquid comes out from mouthpiece without touching the sides of the tube as shown in Fig. 7.16. The mouthpiece is known as *running free*. But if the length of the tube is about 3 times its diameter, the jet comes out with its diameter equal to the diameter of mouthpiece at outlet as shown in Fig. 7.17. The mouthpiece is said to be *running full*.

(*i*) **Borda's Mouthpiece Running Free.** Fig. 7.16 shows the Borda's mouthpiece running free.

Let H = height of liquid above the mouthpiece,
a = area of mouthpiece,
a_c = area of contracted jet in the mouthpiece,
v_c = velocity through mouthpiece.

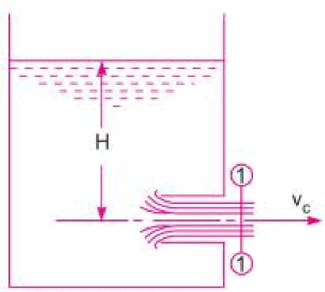


Fig. 7.16

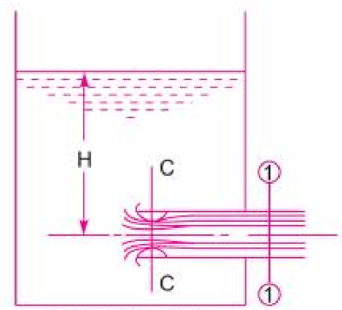


Fig. 7.17

The flow of fluid through mouthpiece is taking place due to the pressure force exerted by the fluid on the entrance section of the mouthpiece. As the area of the mouthpiece is 'a' hence total pressure force on entrance

$$= \rho g \,.\, a.\, h$$

where h = distance of C.G. of area 'a' from free surface = H.

$$= \rho g \,.\, a \,.\, H \qquad \text{...}(i)$$

According to Newton's second law of motion, the net force is equal to the rate of change of momentum.

Now mass of liquid flowing/sec = $\rho \times a_c \times v_c$

The liquid is initially at rest and hence initial velocity is zero but final velocity of fluid is v_c.

$\therefore$ Rate of change of momentum = mass of liquid flowing/sec × [final velocity – initial velocity]

$$= \rho a_c \times v_c[v_c - 0] = \rho a_c\, v_c^2 \qquad \text{...(ii)}$$

Equating (*i*) and (*ii*), we get

$$\rho g\,.a.H. = \rho a_c.\, v_c^2 \qquad \text{...(iii)}$$

Applying Bernoulli's equation to free surface of liquid and section (1)-(1) of Fig. 7.16

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1$$

Taking the centre line of mouthpiece as datum, we have

$$z = H,\ z_1 = 0,\ \frac{p}{\rho g} = \frac{p_1}{\rho g} = p_{atmosp.} = 0,$$

$$v_1 = v_c, \qquad v = 0$$

$$\therefore \qquad 0 + 0 + H = 0 + \frac{v_c^2}{2g} + 0 \ \text{ or } \ H = \frac{v_c^2}{2g}$$

$$\therefore \qquad v_c = \sqrt{2gH}$$

Substituting the value of v_c in (*iii*), we get

$$\rho g\,.\,a\,.\,H. = \rho\,.\,a_c\,.\,2g\,.\,H$$

or
$$a = 2a_c \text{ or } \frac{a_c}{a} = \frac{1}{2} = 0.5$$

$\therefore$ Co-efficient of contraction, $C_c = \dfrac{a_c}{a} = 0.5$

Since there is no loss of head, co-efficient of velocity, $C_v = 1.0$

$\therefore$ Co-efficient of discharge, $C_d = C_c \times C_v = 0.5 \times 1.0 = 0.5$

$\therefore$ Discharge
$$Q = C_d\, a\sqrt{2gH} \qquad \text{...(7.19)}$$
$$= 0.5 \times a\sqrt{2gH}$$

(*ii*) **Borda's Mouthpiece Running Full.** Fig. 7.17 shows Borda's mouthpiece running full.

Let H = height of liquid above the mouthpiece,
v_1 = velocity at outlet or at (1)-(1) of mouthpiece,
a = area of mouthpiece,
a_c = area of the flow at C-C,
v_c = velocity of liquid at vena-contracta or at C-C.

The jet of liquid after passing through C-C, suddenly enlarges at section (1)-(1). Thus there will be a loss of head due to sudden enlargement.

$$\therefore \qquad h_L = \frac{(v_c - v_1)^2}{2g} \qquad \text{...(i)}$$

Now from continuity, we have $a_c \times v_c = a_1 \times v_1$

$$\therefore \quad v_c = \frac{a_1}{a_c} \times v_1 = \frac{v_1}{a_c / a_1} = \frac{v_1}{C_c} = \frac{v_1}{0.5} \qquad \{\because C_c = 0.5\}$$

or $$v_c = 2v_1$$

Substituting this value of v_c in (i), we get $h_L = \dfrac{(2v_1 - v_1)^2}{2g} = \dfrac{v_1^2}{2g}$

Applying Bernoulli's equation to free surface of water in tank and section (1)-(1), we get

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 + h_L$$

Taking datum line passing through the centre line of mouthpiece

$$0 + 0 + H = 0 + \frac{{v_1}^2}{2g} + 0 + \frac{{v_1}^2}{2g}$$

$$\therefore \quad H = \frac{{v_1}^2}{2g} + \frac{v_1^2}{2g} = \frac{{v_1}^2}{g}$$

$$\therefore \quad v_1 = \sqrt{gH}$$

Here v_1 is actual velocity as losses have been taken into consideration,

But theoretical velocity, $v_{th} = \sqrt{2gH}$

$\therefore$ Co-efficient of velocity, $C_v = \dfrac{v_1}{v_{th}} = \dfrac{\sqrt{gH}}{\sqrt{2gH}} = \dfrac{1}{\sqrt{2}} = 0.707$

As the area of the jet at outlet is equal to the area of the mouthpiece, hence co-efficient of contraction = 1

$$\therefore \quad C_d = C_c \times C_v = 1.0 \times .707 = 0.707$$

$\therefore$ Discharge, $$Q = C_d \times a \times \sqrt{2gH} = 0.707 \times a \times \sqrt{2gH} \qquad ...(7.20)$$

Problem 7.29 *An internal mouthpiece of 80 mm diameter is discharging water under a constant head of 8 metres. Find the discharge through mouthpiece, when*

(i) The mouthpiece is running free, and (ii) The mouthpiece is running full.

Solution. Given :

Dia. of mouthpiece, $d = 80$ mm $= 0.08$ m

$\therefore$ Area, $$a = \frac{\pi}{4}(.08)^2 = .005026 \text{ m}^2$$

Constant head, $H = 4$ m.

(i) **Mouthpiece running free.** The discharge, Q is given by equation (7.19) as

$$Q = 0.5 \times a \times \sqrt{2gH}$$
$$= 0.5 \times .005026 \times \sqrt{2 \times 9.81 \times 4.0}$$
$$= 0.02226 \text{ m}^3/\text{s} = \mathbf{22.26 \text{ litres/s.}} \text{ Ans.}$$

(ii) **Mouthpiece running full.** The discharge, Q is given by equation (7.20) as

$$Q = 0.707 \times a \times \sqrt{2gH}$$
$$= 0.707 \times .005026 \times \sqrt{2 \times 9.81 \times 4.0}$$
$$= 0.03147 \text{ m}^3\text{/s} = \textbf{31.47 litre/s. Ans.}$$

HIGHLIGHTS

1. Orifice is a small opening on the side or at the bottom of a tank while mouthpiece is a short length of pipe which is two or three times its diameter in length.
2. Orifices as well as mouthpieces are used for measuring the rate of flow of liquid.
3. Theoretical velocity of jet of water from orifice is given by

$$V = \sqrt{2gH}, \text{ where } H = \text{Height of water from the centre of orifice.}$$

4. There are three hydraulic co-efficients namely :

(a) Co-efficient of velocity, $C_v = \dfrac{\text{Actual velocity at vena - contracta}}{\text{Theoretical velocity}} = \dfrac{x}{\sqrt{4yH}}$

(b) Co-efficient of contraction, $C_c = \dfrac{\text{Area of jet at vena - contracta}}{\text{Area of orifice}}$

(c) Co-efficient of discharge, $C_d = \dfrac{\text{Actual discharge}}{\text{Theoretical discharge}} = C_v \times C_c$

where x and y are the co-ordinates of any point of jet of water from vena-contracta.

5. A large orifice is one, where the head of liquid above the centre of orifice is less than 5 times the depth of orifice. The discharge through a large rectangular orifice is

$$Q = \frac{2}{3} C_d \times b \times \sqrt{2g}\,[H_2^{3/2} - H_1^{3/2}]$$

where b = Width of orifice,
C_d = Co-efficient of discharge for orifice,
H_1 = Height of liquid above top edge of orifice, and
H_2 = Height of liquid above bottom edge of orifice.

6. The discharge through fully sub-merged orifice, $Q = C_d \times b \times (H_2 - H_1) \times \sqrt{2gH}$

where b = Width of orifice,
C_d = Co-efficient of discharge for orifice,
H_2 = Height of liquid above bottom edge of orifice on upstream side,
H_1 = Height of liquid above top edge of orifice on upstream side,
H = Difference of liquid levels on both sides of the orifice.

7. Discharge through partially sub-merged orifice,

$$Q = Q_1 + Q_2$$
$$= C_d b\,(H_2 - H) \times \sqrt{2gH} + 2/3\, C_d b \times \sqrt{2g}\;[H^{3/2} - H_1^{3/2}]$$

where b = Width of orifice
C_d, H_1, H_2 and H are having their usual meaning.

8. Time of emptying a tank through an orifice at its bottom is given by,

$$T = \frac{2A\left[\sqrt{H_1} - \sqrt{H_2}\right]}{C_d \cdot a \cdot \sqrt{2g}}$$

where H_1 = Initial height of liquid in tank,
H_2 = Final height of liquid in tank,

A = Area of tank,
a = Area of orifice,
C_d = Co-efficient of discharge.

If the tank is to be completely emptied, then time T,

$$T = \frac{2A\sqrt{H}}{C_d \cdot a \cdot \sqrt{2g}}.$$

9. Time of emptying a hemispherical tank by an orifice fitted at its bottom,

$$T = \frac{\pi}{C_d \cdot a \cdot \sqrt{2g}}\left[\frac{4}{3}R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right)\right]$$

and for completely emptying the tank, $T = \dfrac{\pi}{C_d \cdot a \cdot \sqrt{2g}}\left[\dfrac{4}{3}RH_1^{3/2} - \dfrac{2}{5}H_1^{5/2}\right]$

where R = Radius of the hemispherical tank,
H_1 = Initial height of liquid,
H_2 = Final height of liquid,
a = Area of orifice, and
C_d = Co-efficient of discharge.

10. Time of emptying a circular horizontal tank by an orifice at the bottom of the tank,

$$T = \frac{4L}{3C_d \cdot a \cdot \sqrt{2g}}\ [(2R - \mathrm{H}_2)^{3/2} - (2R - H_1)^{3/2}]$$

and for completely emptying the tank, $T = \dfrac{4L}{3C_d \cdot a \cdot \sqrt{2g}}\ [(2R)^{3/2} - (2R - H_1)^{3/2}]$

where L = Length of horizontal tank.

11. Co-efficient of discharge for,
(*i*) External mouthpiece, $C_d = 0.855$
(*ii*) Internal mouthpiece, running full, $C_d = 0.707$
(*iii*) Internal mouthpiece, running free, $C_d = 0.50$
(*iv*) Convergent or convergent-divergent, $C_d = 1.0$.

12. For an external mouthpiece, absolute pressure head at vena-contracta

$$H_c = H_a - 0.89\ H$$

where H_a = atmospheric pressure head = 10.3 m of water
H = head of liquid above the mouthpiece.

13. For a convergent-divergent mouthpiece, the ratio of areas at outlet and at vena-contracta is

$$\frac{a_1}{a_c} = \sqrt{1 + \frac{H_a - H_c}{H}}$$

where a_1 = Area of mouthpiece at outlet
a_c = Area of mouthpiece at vena-contracta
H_a = Atmospheric pressure head
H_c = Absolute pressure head at vena-contracta
H = Height of liquid above mouthpiece.

14. In case of internal mouthpieces, if the jet of liquid comes out from mouthpiece without touching its sides it is known as running free. But if the jet touches the sides of the mouthpiece, it is known as running full.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define an orifice and a mouthpiece. What is the difference between the two ?
2. Explain the classification of orifices and mouthpieces based on their shape, size and sharpness ?
3. What are hydraulic co-efficients ? Name them.
4. Define the following co-efficients : (*i*) Co-efficient of velocity, (*ii*) Co-efficient of contraction and (*iii*) Co-efficient of discharge.
5. Derive the expression $C_d = C_v \times C_c$.
6. Define vena-contracta.
7. Differentiate between a large and a small orifice. Obtain an expression for discharge through a large rectangular orifice.
8. What do you understand by the terms wholly sub-merged orifice and partially sub-merged orifice ?
9. Prove that the expression for discharge through an external mouthpiece is given by

$$Q = .855 \times a \times v$$

where a = Area of mouthpiece at outlet and
v = Velocity of jet of water at outlet.

10. Distinguish between : (*i*) External mouthpiece and internal mouthpiece, (*ii*) Mouthpiece running free and mouthpiece running full.
11. Obtain an expression for absolute pressure head at vena-contracta for an external mouthpiece.
12. What is a convergent-divergent mouthpiece ? Obtain an expression for the ratio of diameters at outlet and at vena-contracta for a convergent-divergent 'mouthpiece' in terms of absolute pressure head at vena-contracta, head of liquid above mouthpiece and atmospheric pressure head.
13. The length of the divergent outlet part in a venturimeter is usually made longer compared with that of the converging inlet part. Why ?
14. Justify the statement, "In a convergent-divergent mouthpiece the loss of head is practically eliminated".

(B) NUMERICAL PROBLEMS

1. The head of water over an orifice of diameter 50 mm is 12 m. Find the actual discharge and actual velocity of jet at vena-contracta. Take $C_d = 0.6$ and $C_v = 0.98$. [**Ans.** .018 m^3/s ; 15.04 m/s]
2. The head of water over the centre of an orifice of diameter 30 mm is 1.5 m. The actual discharge through the orifice is 2.35 litres/sec. Find the co-efficient of discharge. [**Ans.** 0.613]
3. A jet of water, issuing from a sharp edged vertical orifice under a constant head of 60 cm, has the horizontal and vertical co-ordinates measured from the vena-contracta at a certain point as 10.0 cm and 0.45 cm respectively. Find the value of C_v. Also find the value of C_v if $C_d = 0.60$. [**Ans.** 0.962, 0.623]
4. The head of water over an orifice of diameter 100 mm is 5 m. The water coming out from orifice is collected in a circular tank of diameter 2 m. The rise of water level in circular tank is .45 m in 30 seconds. Also the co-ordinates of a certain point on the jet, measured from vena-contracta are 100 cm horizontal and 5.2 cm vertical. Find the hydraulic co-efficients C_d, C_v and C_c. [**Ans.** 0.605, 0.98, 0.617]
5. A tank has two identical orifices in one of its vertical sides. The upper orifice is 4 m below the water surface and lower one 6 m below the water surface. If the value of C_v for each orifice is 0.98, find the point of intersection of the two jets. [**Ans.** At a horizontal distance of 9.60 cm]
6. A closed vessel contains water upto a height of 2.0 m and over the water surface there is air having pressure 8.829 N/cm^2 above atmospheric pressure. At the bottom of the vessel there is an orifice of diameter 15 cm. Find the rate of flow of water from orifice. Take $C_d = 0.6$. [**Ans.** 0.15575 m^3/s]

7. A closed tank partially filled with water upto a height of 1 m, having an orifice of diameter 20 mm at the bottom of the tank. Determine the pressure required for a discharge of 3.0 litres/s through the orifice. Take $C_d = 0.62$. [**Ans.** 10.88 N/cm^2]
8. Find the discharge through a rectangular orifice 3.0 m wide and 2 m deep fitted to a water tank. The water level in the tank is 4 m above the top edge of the orifice. Take $C_d = 0.62$ [**Ans.** 36.77 m^3/s]
9. A rectangular orifice, 2.0 m wide and 1.5 m deep is discharging water from a tank. If the water level in the tank is 3.0 m above the top edge of the orifice, find the discharge through the orifice. Take $C_d = 0.6$. [**Ans.** 15.40 m^3/s]
10. A rectangular orifice, 1.0 m wide and 1.5 m deep is discharging water from a vessel. The top edge of the orifice is 0.8 m below the water surface in the vessel. Calculate the discharge through the orifice if $C_d = 0.6$. Also calculate the percentage error if the orifice is treated as a small orifice. [**Ans.** 1.058%]
11. Find the discharge through a fully sub-merged orifice of width 2 m if the difference of water levels on both the sides of the orifice be 800 mm. The height of water from top and bottom of the orifice are 2.5 m and 3 m respectively. Take $C_d = 0.6$. [**Ans.** 2.377 m^3/s]
12. Find the discharge through a totally drowned orifice 1.5 m wide and 1 m deep, if the difference of water levels on both the sides of the orifice be 2.5 m. Take $C_d = 0.62$. [**Ans.** 6.513 m^3/s]
13. A rectangular orifice of 1.5 m wide and 1.2 m deep is fitted in one side of a large tank. The water level on one side of the orifice is 2 m above the top edge of the orifice, while on the other side of the orifice, the water level is 0.4 m below its top edge. Calculate the discharge through the orifice if $C_d = 0.62$. [**Ans.** 7.549 m^3/s]
14. A circular tank of diameter 3 m contains water upto a height of 4 m. The tank is provided with an orifice of diameter 0.4 m at the bottom. Find the time taken by water : (*i*) to fall from 4 m to 2 m and (*ii*) for completely emptying the tank. Take $C_d = 0.6$. [**Ans.** (*i*) 24.8 s, (*ii*) 84.7 s]
15. A circular tank of diameter 1.5 m contains water upto a height of 4 m. An orifice of 40 mm diameter is provided at its bottom. If $C_d = 0.62$, find the height of water above the orifice after 10 minutes. [**Ans.** 2 m]
16. A hemispherical tank of diameter 4 m contains water upto a height of 2.0 m. An orifice of diameter 50 mm is provided at the bottom. Find the time required by water (*i*) to fall from 2.0 m to 1.0 m (*ii*) for completely emptying the tank. Take $C_d = 0.6$ [**Ans.** (*i*) 30 min 14.34 s, (*ii*) 52 min 59 s]
17. A hemispherical cistern of 4 m radius is full of water. It is fitted with a 60 mm diameter sharp edged orifice at the bottom. Calculate the time required to lower the level in the cistern by 2 metres. Take $C_d = 0.6$. [**Ans.** 1 hr 58 min 45.9 s]
18. A cylindrical tank is having a hemispherical base. The height of cylindrical portion is 4 m and diameter is 3 m. At the bottom of this tank an orifice of diameter 300 mm is fitted. Find the time required to completely emptying the tank. Take $C_d = 0.6$. [**Ans.** 2 min 7.37 s]
19. An orifice of diameter 200 mm is fitted at the bottom of a boiler drum of length 6 m and of diameter 2 m. The drum is horizontal and half full of water. Find the time required to empty the boiler, given the value of $C_d = 0.6$ [**Ans.** 2 min 55.20 s]
20. An orifice of diameter 150 mm is fitted at the bottom of a boiler drum of length 6 m and of diameter 2 m. The drum is horizontal and contains water upto a height of 1.8 m. Find the time required to empty the boiler. Take $C_d = 0.6$. [**Ans.** 7 min 46.64 s]
21. Find the discharge from a 80 mm diameter external mouthpiece, fitted to a side of a large vessel if the head over the mouthpiece is 6 m. [**Ans.** 0.0466 m^3/s]
22. An external cylindrical mouthpiece of diameter 100 mm is discharging water under a constant head of 8 m. Determine the discharge and absolute pressure head of water at vena-contracta. Take $C_d = 0.855$ and C_c for vena-contracta = 0.62. Take atmospheric pressure head = 10.3 m of water. [**Ans.** 0.084 m^3/s ; 3.18 m]
23. A convergent-divergent mouthpiece having throat diameter of 60 mm is discharging water under a constant head of 3.0 m. Determine the maximum outlet diameter for maximum discharge. Find maximum discharge also. Take atmospheric pressure head = 10.3 m of water and separation pressure head = 2.5 m of water absolute. [**Ans.** 6.88 cm, $Q_{max} = 0.01506$ m^3/s]

24. The throat and exit diameter of a convergent-divergent mouthpiece are 40 mm and 80 mm respectively. It is fitted to the vertical side of a tank, containing water. Find the maximum head of water for steady flow. The maximum vacuum pressure is 8 m of water. Take atmospheric pressure head = 10.3 m of water. [**Ans.** 0.533 m]

25. The discharge through a convergent-divergent mouthpiece fitted to the side of a tank under a constant head of 2 m is 7 litres/s. The head loss in the divergent portion is 0.10 times the kinetic head at outlet. Find the throat and exit diameters, if separation pressure head = 2.5 m and atmospheric pressure head = 10.3 m of water. [**Ans.** 25.3 mm ; 38.6 mm]

26. An internal mouthpiece of 100 mm diameter is discharging water under a constant head of 5 m. Find the discharge through mouthpiece, when

    (*i*) the mouthpiece is running free, and (*ii*) the mouthpiece is running full.

    [**Ans.** (*i*) 38.8 litres/s, (*ii*) 54.86 litres/s]

8 CHAPTER

NOTCHES AND WEIRS

▶ 8.1 INTRODUCTION

A **notch** is a device used for measuring the rate of flow of a liquid through a small channel or a tank. It may be defined as an opening in the side of a tank or a small channel in such a way that the liquid surface in the tank or channel is below the top edge of the opening.

A **weir** is a concrete or masonary structure, placed in an open channel over which the flow occurs. It is generally in the form of vertical wall, with a sharp edge at the top, running all the way across the open channel. The notch is of small size while the weir is of a bigger size. The notch is generally made of metallic plate while weir is made of concrete or masonary structure.

1. Nappe or Vein. The sheet of water flowing through a notch or over a weir is called Nappe or Vein.

2. Crest or Sill. The bottom edge of a notch or a top of a weir over which the water flows, is known as the sill or crest.

▶ 8.2 CLASSIFICATION OF NOTCHES AND WEIRS

The notches are classified as :

1. According to the shape of the opening :
   (*a*) Rectangular notch,
   (*b*) Triangular notch,
   (*c*) Trapezoidal notch, and
   (*d*) Stepped notch.
2. According to the effect of the sides on the nappe :
   (*a*) Notch with end contraction.
   (*b*) Notch without end contraction or suppressed notch.

Weirs are classified according to the shape of the opening, the shape of the crest, the effect of the sides on the nappe and nature of discharge. The following are important classifications.

(*a*) According to the shape of the opening :
   (*i*) Rectangular weir, (*ii*) Triangular weir, and
   (*iii*) Trapezoidal weir (Cipolletti weir)

(*b*) According to the shape of the crest :
   (*i*) Sharp-crested weir, (*ii*) Broad-crested weir,
   (*iii*) Narrow-crested weir, and (*iv*) Ogee-shaped weir.

(*c*) According to the effect of sides on the emerging nappe :

(*i*) Weir with end contraction, and (*ii*) Weir without end contraction.

▶ 8.3 DISCHARGE OVER A RECTANGULAR NOTCH OR WEIR

The expression for discharge over a rectangular notch or weir is the same.

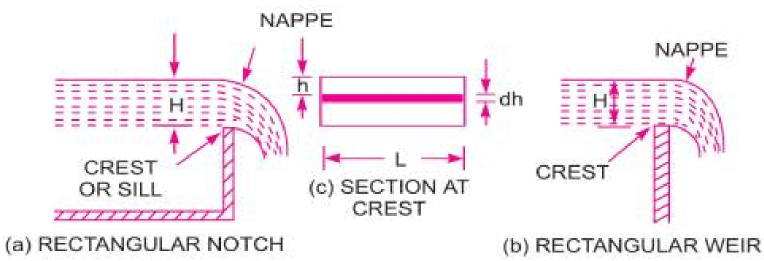


Fig. 8.1 *Rectangular notch and weir.*

Consider a rectangular notch or weir provided in a channel carrying water as shown in Fig. 8.1.

Let H = Head of water over the crest

L = Length of the notch or weir

For finding the discharge of water flowing over the weir or notch, consider an elementary horizontal strip of water of thickness dh and length L at a depth h from the free surface of water as shown in Fig. 8.1(*c*).

The area of strip $= L \times dh$

and theoretical velocity of water flowing through strip $= \sqrt{2gh}$

The discharge dQ, through strip is

$$dQ = C_d \times \text{Area of strip} \times \text{Theoretical velocity}$$

$$= C_d \times L \times dh \times \sqrt{2gh} \qquad ...(i)$$

where C_d = Co-efficient of discharge.

The total discharge, Q, for the whole notch or weir is determined by integrating equation (*i*) between the limits 0 and H.

$$\therefore \quad Q = \int_0^H C_d \,.\, L \,.\, \sqrt{2gh} \,.\, dh = C_d \times L \times \sqrt{2g} \int_0^H h^{1/2}\, dh$$

$$= C_d \times L \times \sqrt{2g} \left[\frac{h^{1/2+1}}{\frac{1}{2}+1} \right]_0^H = C_d \times L \times \sqrt{2g} \left[\frac{h^{3/2}}{3/2} \right]_0^H$$

$$= \frac{2}{3} C_d \times L \times \sqrt{2g}\; [H]^{3/2}. \qquad ...(8.1)$$

Problem 8.1 *Find the discharge of water flowing over a rectangular notch of 2 m length when the constant head over the notch is 300 mm. Take $C_d = 0.60$.*

Solution. Given :

Length of the notch, $L = 2.0$ m

Head over notch, $H = 300 \text{ m} = 0.30 \text{ m}$

$C_d = 0.60$

Discharge,
$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g}\left[H^{3/2}\right]$$
$$= \frac{2}{3} \times 0.6 \times 2.0 \times \sqrt{2 \times 9.81} \times [0.30]^{1.5} \text{ m}^3\text{/s}$$
$$= 3.5435 \times 0.1643 = \mathbf{0.582 \text{ m}^3\text{/s}}. \text{ Ans.}$$

Problem 8.2 *Determine the height of a rectangular weir of length 6 m to be built across a rectangular channel. The maximum depth of water on the upstream side of the weir is 1.8 m and discharge is 2000 litres/s. Take C_d = 0.6 and neglect end contractions.*

Solution. Given :

Length of weir, $L = 6$ m

Depth of water, $H_1 = 1.8$ m

Discharge, $Q = 2000 \text{ lit/s} = 2 \text{ m}^3\text{/s}$

$C_d = 0.6$

Let H is height of water above the crest of weir, and H_2 = height of weir (Fig. 8.2)

The discharge over the weir is given by the equation (8.1) as

$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g}\; H^{3/2}$$

or
$$2.0 = \frac{2}{3} \times 0.6 \times 6.0 \times \sqrt{2 \times 9.81} \times H^{3/2}$$
$$= 10.623\, H^{3/2}$$

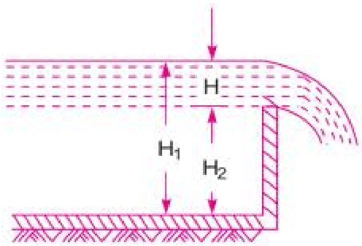


Fig. 8.2

$$\therefore \quad H^{3/2} = \frac{2.0}{10.623}$$

$$\therefore \quad H = \left(\frac{2.0}{10.623}\right)^{2/3} = 0.328 \text{ m}$$

$\therefore$ Height of weir, $H_2 = H_1 - H$

= Depth of water on upstream side $- H$

$= 1.8 - .328 = \mathbf{1.472 \text{ m}}$. **Ans.**

Problem 8.3 *The head of water over a rectangular notch is 900 mm. The discharge is 300 litres/s. Find the length of the notch, when C_d = 0.62.*

Solution. Given :

Head over notch, $H = 90 \text{ cm} = 0.9 \text{ m}$

Discharge, $Q = 300 \text{ lit/s} = 0.3 \text{ m}^3\text{/s}$

$C_d = 0.62$

Let length of notch $= L$

Using equation (8.1), we have

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

or $$0.3 = \frac{2}{3} \times 0.62 \times L \times \sqrt{2 \times 9.81} \times (0.9)^{3/2}$$

$$= 1.83 \times L \times 0.8538$$

$\therefore$ $$L = \frac{0.3}{1.83 \times .8538} = .192 \text{ m} = \textbf{192 mm. Ans.}$$

8.4 DISCHARGE OVER A TRIANGULAR NOTCH OR WEIR

The expression for the discharge over a triangular notch or weir is the same. It is derived as :

Let H = head of water above the V- notch

θ = angle of notch

Consider a horizontal strip of water of thickness 'dh' at a depth of h from the free surface of water as shown in Fig. 8.3.

From Fig. 8.3 (b), we have

$$\tan \frac{\theta}{2} = \frac{AC}{OC} = \frac{AC}{(H - h)}$$

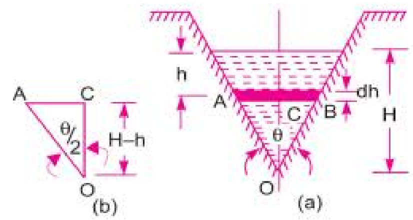


Fig. 8.3 *The triangular notch.*

$\therefore$ $$AC = (H - h) \tan \frac{\theta}{2}$$

Width of strip $$= AB = 2AC = 2\,(H - h) \tan \frac{\theta}{2}$$

$\therefore$ Area of strip $$= 2\,(H - h) \tan \frac{\theta}{2} \times dh$$

The theoretical velocity of water through strip = $\sqrt{2gh}$

$\therefore$ Discharge, through the strip,

$$dQ = C_d \times \text{Area of strip} \times \text{Velocity (theoretical)}$$

$$= C_d \times 2\,(H - h) \tan \frac{\theta}{2} \times dh \times \sqrt{2gh}$$

$$= 2C_d\,(H - h) \tan \frac{\theta}{2} \times \sqrt{2gh} \times dh$$

$\therefore$ Total discharge, $$Q = \int_0^H 2C_d\,(H - h) \tan \frac{\theta}{2} \times \sqrt{2gh} \times dh$$

$$= 2C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \int_0^H (H - h)h^{1/2}\, dh$$

$$= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \int_0^H (Hh^{1/2} - h^{3/2})\, dh$$

$$= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \left[\frac{Hh^{3/2}}{3/2} - \frac{h^{5/2}}{5/2} \right]_0^H$$

$$= 2 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}\left[\frac{2}{3}H.H^{3/2} - \frac{2}{5}H^{5/2}\right]$$

$$= 2 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}\left[\frac{2}{3}H^{5/2} - \frac{2}{5}H^{5/2}\right]$$

$$= 2 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}\left[\frac{4}{15}H^{5/2}\right]$$

$$= \frac{8}{15}C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2} \qquad \text{...(8.2)}$$

For a right-angled V-notch, if $C_d = 0.6$

$$\theta = 90°, \quad \therefore \quad \tan\frac{\theta}{2} = 1$$

Discharge,

$$Q = \frac{8}{15} \times 0.6 \times 1 \times \sqrt{2 \times 9.81} \times H^{5/2} \qquad \text{...(8.3)}$$

$$= 1.417\ H^{5/2}.$$

Problem 8.4 *Find the discharge over a triangular notch of angle 60° when the head over the V-notch is 0.3 m. Assume $C_d = 0.6$.*

Solution. Given :

Angle of V-notch, $\theta = 60°$

Head over notch, $H = 0.3$ m

$C_d = 0.6$

Discharge, Q over a V-notch is given by equation (8.2)

$$Q = \frac{8}{15} \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$$= \frac{8}{15} \times 0.6 \tan\frac{60°}{2} \times \sqrt{2 \times 9.81} \times (0.3)^{5/2}$$

$$= 0.8182 \times 0.0493 = \mathbf{0.040\ m^3/s.}\ \textbf{Ans.}$$

Problem 8.5 *Water flows over a rectangular weir 1 m wide at a depth of 150 mm and afterwards passes through a triangular right-angled weir. Taking C_d for the rectangular and triangular weir as 0.62 and 0.59 respectively, find the depth over the triangular weir.*

Solution. Given :

For rectangular weir, length, $L = 1$ m

Depth of water, $H = 150$ mm $= 0.15$ m

$C_d = 0.62$

For triangular weir, $\theta = 90°$

$C_d = 0.59$

Let depth over triangular weir $= H_1$

The discharge over the rectangular weir is given by equation (8.1) as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 1.0 \times \sqrt{2 \times 9.81} \times (.15)^{3/2} \text{ m}^3\text{/s} = 0.10635 \text{ m}^3\text{/s}$$

The same discharge passes through the triangular right-angled weir. But discharge, Q, is given by equation (8.2) for a triangular weir as

$$Q = \frac{8}{15} \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$\therefore$
$$0.10635 = \frac{8}{15} \times .59 \times \tan\frac{90^\circ}{2} \times \sqrt{2g} \times H_1^{5/2} \qquad \{\because \theta = 90^\circ \text{ and } H = H_1\}$$

$$= \frac{8}{15} \times .59 \times 1 \times 4.429 \times H_1^{5/2} = 1.3936\, H_1^{5/2}$$

$\therefore$
$$H_1^{5/2} = \frac{0.10635}{1.3936} = 0.07631$$

$\therefore$
$$H_1 = (.07631)^{0.4} = \mathbf{0.3572 \text{ m.}} \text{ Ans.}$$

Problem 8.5A *Water flows through a triangular right-angled weir first and then over a rectangular weir of 1 m width. The discharge co-efficients of the triangular and rectangular weirs are 0.6 and 0.7 respectively. If the depth of water over the triangular weir is 360 mm, find the depth of water over the rectangular weir.*

Solution. Given :

For triangular weir : $\theta = 90^\circ$, $C_d = 0.6$, $H = 360$ mm $= 0.36$ m

For rectangular weir : $L = 1$ m, $C_d = 0.7$, $H = ?$

The discharge for a triangular weir is given by equation (8.2) as

$$Q = \frac{8}{15} \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$$= \frac{8}{15} \times 0.6 \times \tan\left(\frac{90^\circ}{2}\right) \times \sqrt{2 \times 9.81} \times (0.36)^{5/2} = 0.1102 \text{ m}^3\text{/s}$$

The same discharge is passing through the rectangular weir. But discharge for a rectangular weir is given by equation (8.1) as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

or
$$0.1102 = \frac{2}{3} \times 0.7 \times 1 \times \sqrt{2 \times 9.81} \times H^{3/2} = 2.067\, H^{3/2}$$

or
$$H^{3/2} = \frac{0.1102}{2.067} = 0.0533$$

$\therefore$
$$H = (0.0533)^{2/3} = 0.1415 \text{ m} = \mathbf{141.5 \text{ mm.}} \text{ Ans.}$$

Problem 8.6 *A rectangular channel 2.0 m wide has a discharge of 250 litres per second, which is measured by a right-angled V-notch weir. Find the position of the apex of the notch from the bed of the channel if maximum depth of water is not to exceed 1.3 m. Take $C_d = 0.62$.*

Solution. Given :

Width of rectangular channel, $L = 2.0$ m

Discharge, $Q = 250$ lit/s $= 0.25$ m^3/s

Depth of water in channel $= 1.3$ m

Let the height of water over V-notch $= H$

The rate of flow through V-notch is given by equation (8.2) as

$$Q = \frac{8}{15} \times C_d \times \sqrt{2g} \times \tan\frac{\theta}{2} \times H^{5/2}$$

where $C_d = 0.62$, $\theta = 90°$

$$\therefore \quad Q = \frac{8}{15} \times .62 \times \sqrt{2 \times 9.81} \times \tan\frac{90°}{2} \times H^{5/2}$$

or
$$0.25 = \frac{8}{15} \times .62 \times 4.429 \times 1 \times H^{5/2}$$

or
$$H^{5/2} = \frac{.25 \times 15}{8 \times .62 \times 4.429} = 0.1707$$

$$\therefore \quad H = (.1707)^{2/5} = (.1707)^{0.4} = 0.493 \text{ m}$$

Position of apex of the notch from the bed of channel

= depth of water in channel–height of water over V-notch

= 1.3 – .493 = **0.807 m. Ans.**

▶ 8.5 ADVANTAGES OF TRIANGULAR NOTCH OR WEIR OVER RECTANGULAR NOTCH OR WEIR

A triangular notch or weir is preferred to a rectangular weir or notch due to following reasons :

1. The expression for discharge for a right-angled V-notch or weir is very simple.
2. For measuring low discharge, a triangular notch gives more accurate results than a rectangular notch.
3. In case of triangular notch, only one reading, *i.e.*, H is required for the computation of discharge.
4. Ventilation of a triangular notch is not necessary.

▶ 8.6 DISCHARGE OVER A TRAPEZOIDAL NOTCH OR WEIR

As shown in Fig. 8.4, a trapezoidal notch or weir is a combination of a rectangular and triangular notch or weir. Thus the total discharge will be equal to the sum of discharge through a rectangular weir or notch and discharge through a triangular notch or weir.

Let H = Height of water over the notch

L = Length of the crest of the notch

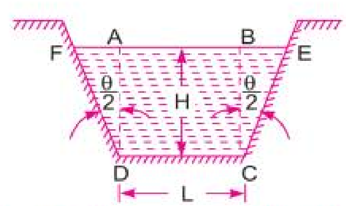


Fig. 8.4 *The trapezoidal notch.*

C_{d_1} = Co-efficient of discharge for rectangular portion *ABCD* of Fig. 8.4.

C_{d_2} = Co-efficient of discharge for triangular portion [*FAD* and *BCE*]

The discharge through rectangular portion *ABCD* is given by (8.1)

or
$$Q_1 = \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g} \times H^{3/2}$$

The discharge through two triangular notches *FDA* and *BCE* is equal to the discharge through a single triangular notch of angle θ and it is given by equation (8.2) as

$$Q_2 = \frac{8}{15} \times C_{d_2} \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

∴ Discharge through trapezoidal notch or weir $FDCEF = Q_1 + Q_2$

$$= \frac{2}{3} C_{d_1} L \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan\theta/2 \times \sqrt{2g} \times H^{5/2}. \quad ...(8.4)$$

Problem 8.7 *Find the discharge through a trapezoidal notch which is 1 m wide at the top and 0.40 m at the bottom and is 30 cm in height. The head of water on the notch is 20 cm. Assume* C_d *for rectangular portion = 0.62 while for triangular portion = 0.60.*

Solution. Given :

Top width, $AE = 1$ m

Base width, $CD = L = 0.4$ m

Head of water, $H = 0.20$ m

For rectangular portion, $C_{d_1} = 0.62$

For triangular portion, $C_{d_2} = 0.60$

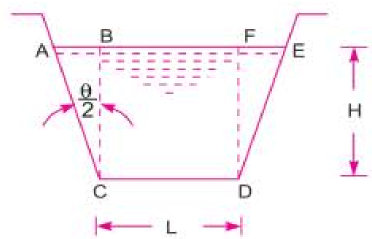


Fig. 8.5

From Δ*ABC*, we have

$$\tan\frac{\theta}{2} = \frac{AB}{BC} = \frac{(AE - CD)/2}{H}$$

$$= \frac{(1.0 - 0.4)/2}{0.3} = \frac{0.6/2}{0.3} = \frac{0.3}{0.3} = 1$$

Discharge through trapezoidal notch is given by equation (8.4)

$$Q = \frac{2}{3} C_{d_1} \times L \times \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$$= \frac{2}{3} \times 0.62 \times 0.4 \times \sqrt{2 \times 9.81} \times (0.2)^{3/2} + \frac{8}{15} \times .60 \times 1 \times \sqrt{2 \times 9.81} \times (0.2)^{5/2}$$

$$= 0.06549 + 0.02535 = 0.09084 \text{ m}^3/\text{s} = \textbf{90.84 litres/s. Ans.}$$

8.7 DISCHARGE OVER A STEPPED NOTCH

A stepped notch is a combination of rectangular notches. The discharge through stepped notch is equal to the sum of the discharges through the different rectangular notches.

Consider a stepped notch as shown in Fig. 8.6.

Let H_1 = Height of water above the crest of notch 1,

L_1 = Length of notch 1,

H_2, L_2 and H_3, L_3 are corresponding values for notches 2 and 3 respectively.

C_d = Co-efficient of discharge for all notches

∴ Total discharge $Q = Q_1 + Q_2 + Q_3$

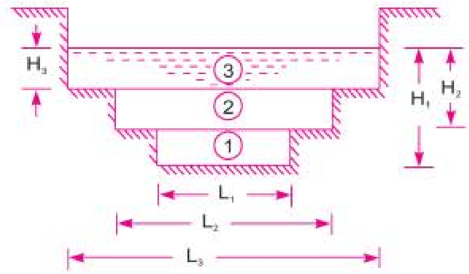


Fig. 8.6 *The stepped notch.*

or
$$Q = \frac{2}{3} \times C_d \times L_1 \times \sqrt{2g}\ [H_1^{3/2} - H_2^{3/2}]$$
$$+ \frac{2}{3} C_d \times L_2 \times \sqrt{2g}\,[H_2^{3/2} - H_3^{3/2}] + \frac{2}{3} C_d \times L_3 \times \sqrt{2g} \times H_3^{3/2}. \quad ...(8.5)$$

Problem 8.8 *Fig. 8.7 shows a stepped notch. Find the discharge through the notch if C_d for all section = 0.62.*

Solution. Given :

L_1 = 40 cm, L_2 = 80 cm,

L_3 = 120 cm

H_1 = 50 + 30 + 15 = 95 cm,

H_2 = 80 cm, H_3 = 50 cm,

C_d = 0.62

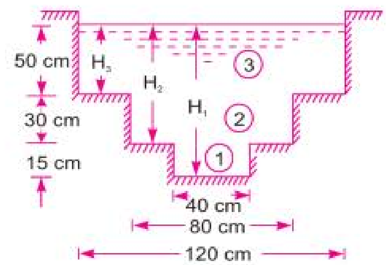


Fig. 8.7

Total discharge, $Q = Q_1 + Q_2 + Q_3$

where
$$Q_1 = \frac{2}{3} \times C_d \times L_1 \times \sqrt{2g}\ [H_1^{3/2} - H_2^{3/2}]$$
$$= \frac{2}{3} \times 0.62 \times 40 \times \sqrt{2 \times 981} \times [95^{3/2} - 80^{3/2}]$$
$$= 732.26[925.94 - 715.54] = 154067 \text{ cm}^3\text{/s} = 154.067 \text{ lit/s}$$
$$Q_2 = \frac{2}{3} \times C_d \times L_2 \times \sqrt{2g} \times [H_2^{3/2} - H_3^{3/2}]$$
$$= \frac{2}{3} \times 0.62 \times 80 \times \sqrt{2 \times 981} \times [80^{3/2} - 50^{3/2}]$$
$$= 1464.52[715.54 - 353.55] \text{ cm}^3\text{/s} = 530141 \text{ cm}^3\text{/s} = 530.144 \text{ lit/s}$$

and
$$Q_3 = \frac{2}{3} \times C_d \times L_3 \times \sqrt{2g} \times H_3^{3/2}$$
$$= \frac{2}{3} \times 0.62 \times 120 \times \sqrt{2 \times 981} \times 50^{3/2} = 776771 \text{ cm}^3\text{/s} = 776.771 \text{ lit/s}$$

∴
$$Q = Q_1 + Q_2 + Q_3 = 154.067 + 530.144 + 776.771$$

= **1460.98 lit/s. Ans.**

8.8 EFFECT ON DISCHARGE OVER A NOTCH OR WEIR DUE TO ERROR IN THE MEASUREMENT OF HEAD

For an accurate value of the discharge over a weir or notch, an accurate measurement of head over the weir or notch is very essential as the discharge over a triangular notch is proportional to $H^{5/2}$ and in case of rectangular notch it is proportional to $H^{3/2}$. A small error in the measurement of head, will affect the discharge considerably. The following cases of error in the measurement of head will be considered :

(*i*) For Rectangular Weir or Notch.

(*ii*) For Triangular Weir or Notch.

8.8.1 For Rectangular Weir or Notch.

The discharge for a rectangular weir or notch is given by equation (8.1) as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$= KH^{3/2} \qquad \text{...}(i)$$

where $K = \frac{2}{3} C_d \times L \times \sqrt{2g}$

Differentiating the above equation, we get

$$dQ = K \times \frac{3}{2} H^{1/2} \, dH \qquad \text{...}(ii)$$

Dividing (*ii*) by (*i*),

$$\frac{dQ}{Q} = \frac{K \times \frac{3}{2} \times H^{1/2} dH}{KH^{3/2}} = \frac{3}{2} \frac{dH}{H} \qquad \text{...(8.6)}$$

Equation (8.6) shows that an error of 1% in measuring H will produce 1.5% error in discharge over a rectangular weir or notch.

8.8.2 For Triangular Weir or Notch.

The discharge over a triangular weir or notch is given by equation (8.2) as

$$Q = \frac{8}{15} C_d . \tan \frac{\theta}{2} \sqrt{2g} \times H^{5/2}$$

$$= KH^{5/2} \qquad \text{...}(iii)$$

where $K = \frac{8}{15} C_d . \tan \frac{\theta}{2} \sqrt{2g}$

Differentiating equation (*iii*), we get

$$dQ = K \frac{5}{2} H^{3/2} \times dH \qquad \text{...}(iv)$$

Dividing (*iv*) by (*iii*), we get

$$\frac{dQ}{Q} = \frac{K \frac{5}{2} H^{3/2} \, dH}{KH^{5/2}} = \frac{5}{2} \frac{dH}{H} \qquad \text{...(8.7)}$$

Equation (8.7) shows that an error of 1% in measuring H will produce 2.5% error in discharge over a triangular weir or notch.

Problem 8.9 *A rectangular notch 40 cm long is used for measuring a discharge of 30 litres per second. An error of 1.5 mm was made, while measuring the head over the notch. Calculate the percentage error in the discharge. Take $C_d = 0.60$.*

Solution. Given :

Length of notch, $L = 40$ cm

Discharge, $Q = 30$ lit/s $= 30000$ cm^3/s

Error in head, $dH = 1.5$ mm $= 0.15$ cm

$C_d = 0.60$

Let the height of water over rectangular notch $= H$

The discharge through a rectangular notch is given by (8.1)

or $$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

or $$30000 = \frac{2}{3} \times 0.60 \times 40 \times \sqrt{2 \times 981} \times H^{3/2}$$

or $$H^{3/2} = \frac{3 \times 30000}{2 \times .60 \times 40 \times \sqrt{2 \times 981}} = 42.33$$

$\therefore$ $$H = (42.33)^{2/3} = 12.16 \text{ cm}$$

Using equation (8.6), we get

$$\frac{dQ}{Q} = \frac{3}{2}\frac{dH}{H} = \frac{3}{2} \times \frac{0.15}{12.16} = 0.0185 = \mathbf{1.85\%}. \text{ Ans.}$$

Problem 8.10 *A right-angled V-notch is used for measuring a discharge of 30 litres/s. An error of 1.5 mm was made while measuring the head over the notch. Calculate the percentage error in the discharge. Take $C_d = 0.62$.*

Solution. Given :

Angle of *V*-notch, $\theta = 90°$

Discharge, $Q = 30$ lit/s $= 30000$ cm^3/s

Error in head, $dH = 1.5$ mm $= 0.15$ cm

$C_d = 0.62$

Let the head over the *V*-notch $= H$

The discharge Q through a triangular notch is given by equation (8.2)

$$Q = \frac{8}{15} C_d \,.\, \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

or $$30000 = \frac{8}{15} \times 0.62 \times \tan\left(\frac{90°}{2}\right) \times \sqrt{2 \times 981} \times H^{5/2}$$

$$= \frac{8}{15} \times .62 \times 1 \times 44.29 \times H^{5/2}$$

$\therefore$ $$H^{5/2} = \frac{30000 \times 15}{8 \times .62 \times 44.29} = 2048.44$$

$\therefore$ $$H = (2048.44)^{2/5} = 21.11 \text{ cm}$$

Using equation (8.7), we get

$$\frac{dQ}{Q} = \frac{5}{2}\frac{dH}{H} = 2.5 \times \frac{0.15}{21.11} = 0.01776 = \mathbf{1.77\%}. \quad \textbf{Ans.}$$

Problem 8.11 *The head of water over a triangular notch of angle 60° is 50 cm and co-efficient of discharge is 0.62. The flow measured by it is to be within an accuracy of 1.5% up or down. Find the limiting values of the head.*

Solution. Given :

Angle of V-notch, $\theta = 60°$

Head of water, $H = 50$ cm

$C_d = 0.62$

$$\frac{dQ}{Q} = \pm\, 1.5\% = \pm\, 0.015$$

The discharge Q over a triangular notch is

$$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\theta}{2} H^{5/2}$$

$$= \frac{8}{15} \times 0.62 \times \sqrt{2 \times 981} \times \tan\frac{60°}{2} \times (50)^{5/2}$$

$$= 14.64 \times 0.5773 \times 17677.67 = 149405.86 \text{ cm}^3\text{/s}$$

Now applying equation (8.7), we get

$$\frac{dQ}{Q} = \frac{5}{2}\frac{dH}{H} \quad \text{or} \quad \pm .015 = 2.5 \frac{dH}{H} \quad \text{or} \quad \frac{dH}{H} = \pm \frac{.015}{2.5}$$

$$\therefore \qquad dH = \pm \frac{.015}{2.5} \times H = \pm \frac{.015}{2.5} \times 50 = \pm\, 0.3$$

$\therefore$ The limiting values of the head

$$= H \pm dH = 50 \pm 0.3 = 50.3 \text{ cm}, 49.7 \text{ cm}$$

$$= \mathbf{50.3 \text{ cm and } 49.7 \text{ cm.}} \quad \textbf{Ans.}$$

▶ 8.9. (a) TIME REQUIRED TO EMPTY A RESERVOIR OR A TANK WITH A RECTANGULAR WEIR OR NOTCH

Consider a reservoir or tank of uniform cross-sectional area A. A rectangular weir or notch is provided in one of its sides.

Let L = Length of crest of the weir or notch

C_d = Co-efficient of discharge

H_1 = Initial height of liquid above the crest of notch

H_2 = Final height of liquid above the crest of notch

T = Time required in seconds to lower the height of liquid from H_1 to H_2.

Let at any instant, the height of liquid surface above the crest of weir or notch be h and in a small time dT, let the liquid surface falls by 'dh'. Then,

$$-\,Adh = Q \times dT$$

–ve sign is taken, as with the increase of T, h decreases.

But
$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g} \times h^{3/2}$$

$\therefore$
$$-Adh = \frac{2}{3} C_d \times L \times \sqrt{2g} \,.\, h^{3/2} \times dT \text{ or } dT = \frac{-Adh}{\frac{2}{3} C_d \times L \times \sqrt{2g} \times h^{3/2}}$$

The total time T is obtained by integrating the above equation between the limits H_1 and H_2.

$$\int_0^T dT = \int_{H_1}^{H_2} \frac{-Adh}{\frac{2}{3} C_d \times L \times \sqrt{2g} \times h^{3/2}}$$

or
$$T = \frac{-A}{\frac{2}{3} C_d \times L \times \sqrt{2g}} \int_{H_1}^{H_2} h^{-3/2}\, dh = \frac{-3A}{2\, C_d \times L \times \sqrt{2g}} \left[\frac{h^{-3/2+1}}{-\frac{3}{2}+1} \right]_{H_1}^{H_2}$$

$$= \frac{-3A}{2\, C_d \times L \times \sqrt{2g}} \left[\frac{h^{-1/2}}{-\frac{1}{2}} \right]_{H_1}^{H_2} = \frac{-3A}{2\, C_d \times L \times \sqrt{2g}} \left(-\frac{2}{1} \right) \left[\frac{1}{\sqrt{h}} \right]_{H_1}^{H_2}$$

$$= \frac{3A}{C_d \times L \times \sqrt{2g}} \left[\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}} \right]. \quad \text{...(8.8)}$$

(b) TIME REQUIRED TO EMPTY A RESERVOIR OR A TANK WITH A TRIANGULAR WEIR OR NOTCH

Consider a reservoir or tank of uniform cross-sectional area A, having a triangular weir or notch in one of its sides.

Let θ = Angle of the notch

C_d = Co-efficient of discharge

H_1 = Initial height of liquid above the apex of notch

H_2 = Final height of liquid above the apex of notch

T = Time required in seconds, to lower the height from H_1 to H_2 above the apex of the notch.

Let at any instant, the height of liquid surface above the apex of weir or notch be h and in a small time dT, let the liquid surface falls by 'dh'. Then

$$-Adh = Q \times dT$$

–ve sign is taken, as with the increase of T, h decreases.

And Q for a triangular notch is

$$Q = \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \sqrt{2g} \times h^{5/2}$$

$\therefore$
$$-Adh = \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times h^{5/2} \times dT$$

$$\therefore \qquad dT = \frac{Adh}{\frac{8}{15} \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g} \times h^{5/2}}$$

The total time T is obtained by integrating the above equation between the limits H_1 and H_2.

$$\therefore \qquad \int_0^T dT = \int_{H_1}^{H_2} \frac{-Adh}{\frac{8}{15} C_d \tan\frac{\theta}{2} \sqrt{2g}\, h^{5/2}}$$

or

$$T = \frac{-A}{\frac{8}{15} C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}} \int_{H_1}^{H_2} h^{-5/2}\, dh$$

$$= \frac{-15A}{8 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}} \left[\frac{h^{-3/2}}{-\frac{3}{2}}\right]_{H_1}^{H_2}$$

$$= \frac{-15A}{8 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}} \times \left(-\frac{2}{3}\right)\left[\frac{1}{h^{3/2}}\right]_{H_1}^{H_2}$$

$$= \frac{5A}{4 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}} \left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right]. \qquad \text{...(8.9)}$$

Problem 8.12 *Find the time required to lower the water level from 3 m to 2 m in a reservoir of dimension 80 m × 80 m, by a rectangular notch of length 1.5 m. Take $C_d = 0.62$.*

Solution. Given :

Initial height of water, $H_1 = 3$ m

Final height of water, $H_2 = 2$ m

Dimension of reservoir = 80 m × 80 m

or Area, $A = 80 \times 80 = 6400$ m^2

Length of notch, $L = 1.5$ m, $C_d = 0.62$

Using the relation given by the equation (8.8)

$$T = \frac{3A}{C_d \times L \times \sqrt{2g}} \left[\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}}\right]$$

$$= \frac{3 \times 6400}{0.62 \times 1.5 \times \sqrt{2 \times 9.81}} \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right]$$

$= 4661.35\ [0.7071 - 0.5773]$ seconds

$= 605.04$ seconds = **10 min 5 sec. Ans.**

Problem 8.13 *If in problem 8.12, instead of a rectangular notch, a right-angled V-notch is used, find the time required. Take all other data same.*

Solution. Given :

Angle of notch, $\theta = 90°$

Initial height of water, $H_1 = 3$ m

Final height of water, $H_2 = 2$ m

Area of reservoir, $A = 80 \times 80 = 6400\ \text{m}^2$

$C_d = 0.62$

Using the relation given by equation (8.9)

$$T = \frac{5A}{4 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}}\left[\frac{1}{H_2^{\,3/2}} - \frac{1}{H_1^{\,3/2}}\right]$$

$$= \frac{5 \times 6400}{4 \times .62 \times \tan\frac{90^\circ}{2} \times \sqrt{2 \times 9.81}}\left[\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right] \qquad \left\{\because\ \tan\frac{90^\circ}{2} = 1\right\}$$

$$= 2913.34 \times \left[\frac{1}{2.8284} - \frac{1}{5.1961}\right]$$

$$= 2913.34\ [0.3535 - 0.1924]\ \text{seconds}$$

$$= 469.33\ \text{seconds} = \mathbf{7\ min\ 49.33\ sec.}\ \textbf{Ans.}$$

Problem 8.14 *A right-angled V-notch is inserted in the side of a tank of length 4 m and width 2.5 m. Initial height of water above the apex of the notch is 30 cm. Find the height of water above the apex if the time required to lower the head in tank from 30 cm to final height is 3 minutes. Take $C_d = 0.60$.*

Solution. Given :

Angle of notch, $\theta = 90°$

Area of tank, $A = \text{Length} \times \text{width} = 4 \times 2.5 = 10.0\ \text{m}^2$

Initial height of water, $H_1 = 30\ \text{cm} = 0.3$ m

Time, $T = 3\ \text{min} = 3 \times 60 = 180$ seconds

$C_d = 0.60$

Let the final height of water above the apex of notch $= H_2$

Using the relation given by equation (8.9)

$$T = \frac{5A}{4 \times C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}}\left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right]$$

$$180 = \frac{5 \times 10}{4 \times .60 \times \tan\left(\frac{90^\circ}{2}\right) \times \sqrt{2 \times 9.81}}\left[\frac{1}{H_2^{3/2}} - \frac{1}{(0.3)^{3/2}}\right]$$

$$= \frac{50}{4 \times .60 \times 1 \times 4.429}\left[\frac{1}{H_2^{\,3/2}} - \frac{1}{(0.3)^{3/2}}\right]$$

or $$\frac{1}{H_2^{\,1.5}} - \frac{1}{0.3^{1.5}} = \frac{180 \times 4 \times 0.60 \times 4.429}{50} = 38.266.$$

or $$\frac{1}{H_2^{1.5}} - 6.0858 = 38.266$$

$\therefore$ $$\frac{1}{H_2^{1.5}} = 38.266 + 6.0858 = 44.35 \text{ or } H_2^{1.5} = \frac{1}{44.35} = 0.0225$$

$\therefore$ $$H_2 = (0.0225)^{1/1.5} = (0.0225)^{.6667} = 0.0822 \text{ m} = \mathbf{8.22 \text{ cm.}} \text{ Ans.}$$

8.10 VELOCITY OF APPROACH

Velocity of approach is defined as the velocity with which the water approaches or reaches the weir or notch before it flows over it. Thus if V_a is the velocity of approach, then an additional head h_a equal to $\frac{V_a^2}{2g}$ due to velocity of approach, is acting on the water flowing over the notch. Then initial height of water over the notch becomes $(H + h_a)$ and final height becomes equal to h_a. Then all the formulae are changed taking into consideration of velocity of approach.

The velocity of approach, V_a is determined by finding the discharge over the notch or weir neglecting velocity of approach. Then dividing the discharge by the cross-sectional area of the channel on the upstream side of the weir or notch, the velocity of approach is obtained. Mathematically,

$$V_a = \frac{Q}{\text{Area of channel}}$$

This velocity of approach is used to find an additional head $\left(h_a = \frac{V_a{}^2}{2g}\right)$. Again the discharge is calculated and above process is repeated for more accurate discharge.

Discharge over a rectangular weir, with velocity of approach

$$= \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ [(H_1 + h_a)^{3/2} - h_a^{3/2}] \quad \text{...(8.10)}$$

Problem 8.15 *Water is flowing in a rectangular channel of 1 m wide and 0.75 m deep. Find the discharge over a rectangular weir of crest length 60 cm, if the head of water over the crest of weir is 20 cm and water from channel flows over the weir. Take $C_d = 0.62$. Neglect end contractions. Take velocity of approach into consideration.*

Solution. Given :

Area of channel, $A = \text{Width} \times \text{depth} = 1.0 \times 0.75 = 0.75 \text{ m}^2$

Length of weir, $L = 60 \text{ cm} = 0.6 \text{ m}$

Head of water, $H_1 = 20 \text{ cm} = 0.2 \text{ m}$

$C_d = 0.62$

Discharge over a rectangular weir without velocity of approach is given by

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H_1{}^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 0.6 \times \sqrt{2 \times 9.81} \times (0.2)^{3/2} \text{ m}^3/\text{s}$$

$$= 1.098 \times 0.0894 = 0.0982 \text{ m}^3/\text{s}$$

Velocity of approach, $V_a = \dfrac{Q}{A} = \dfrac{.0982}{0.75} = 0.1309$ m/s

$\therefore$ Additional head, $h_a = \dfrac{V_a^2}{2g} = (.1309)^2/2 \times 9.81 = .0008733$ m

Then discharge with velocity of approach is given by equation (8.10)

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ [(H_1 + h_a)^{3/2} - h_a^{3/2}]$$

$$= \frac{2}{3} \times 0.62 \times 0.6 \times \sqrt{2 \times 9.81}\ [(0.2 + .00087)^{3/2} - (.00087)^{3/2}]$$

$$= 1.098\ [0.09002 - .00002566]$$

$$= 1.098 \times 0.09017 = \mathbf{.09881\ m^3/s.}$$ **Ans.**

Problem 8.16 *Find the discharge over a rectangular weir of length 100 m. The head of water over the weir is 1.5 m. The velocity of approach is given as 0.5 m/s. Take $C_d = 0.60$.*

Solution. Given :

Length of weir, $L = 100$ m

Head of water, $H_1 = 1.5$ m

Velocity of approach, $V_a = 0.5$ m/s

$C_d = 0.60$

$\therefore$ Additional head, $h_a = \dfrac{V_a^{\,2}}{2g} = \dfrac{0.5 \times 0.5}{2 \times 9.81} = 0.0127$ m

The discharge, Q over a rectangular weir due to velocity of approach is given by equation (8.10)

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ [(H_1 + h_a)^{3/2} - h_a^{3/2}]$$

$$= \frac{2}{3} \times 0.6 \times 100 \times \sqrt{2 \times 9.81}\ [(1.5 + .0127)^{3/2} - .0127^{3/2}]$$

$$= 177.16\ [\ 1.5127^{3/2} - .0127^{3/2}]$$

$$= 177.16\ [1.8605 - .00143] = \mathbf{329.35\ m^3/s.}$$ **Ans.**

Problem 8.17 *A rectangular weir of crest length 50 cm is used to measure the rate of flow of water in a rectangular channel of 80 cm wide and 70 cm deep. Determine the discharge in the channel if the water level is 80 mm above the crest of weir. Take velocity of approach into consideration and value of $C_d = 0.62$.*

Solution. Given :

Length of weir, $L = 50$ cm $= 0.5$ m

Area of channel, A = Width $\times$ depth = 80 cm $\times$ 70 cm = $0.80 \times 0.70 = 0.56$ m^2

Head over weir, $H = 80$ mm $= 0.08$ m

$C_d = 0.62$

The discharge over a rectangular weir without velocity of approach is given by equation (8.1)

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 0.5 \times \sqrt{2 \times 9.81} \times (0.08)^{3/2} \text{ m}^3/\text{s}$$

$$= 0.9153 \times .0226 = .0207 \text{ m}^3/\text{s}$$

Velocity of approach, $V_a = \dfrac{Q}{A} = \dfrac{.0207}{0.56} = .0369$ m/s

$\therefore$ Head due to V_a, $h_a = V_a^2/2g = \dfrac{(.0369)^2}{2 \times 9.81} = .0000697$ m

Discharge with velocity of approach is

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \; [(H_1 + h_a)^{3/2} - h_a^{3/2}]$$

$$= \frac{2}{3} \times 0.62 \times 0.5 \times \sqrt{2 \times 9.81} \; [(.08 + .0000697)^{3/2} - .0000697^{3/2}]$$

$$= 0.9153 \times [.0800697^{1.5} - .0000697^{1.5}]$$

$$= .9153 \; [.02265 - .000000582] = \mathbf{0.2073 \text{ m}^3/\text{s}}. \text{ Ans.}$$

Problem 8.18 *A suppressed rectangular weir is constructed across a channel of 0.77 m width with a head of 0.39 m and the crest 0.6 m above the bed of the channel. Estimate the discharge over it. Consider velocity of approach and assume $C_d = 0.623$.*

Solution. Given :

Width of channel, $b = 0.77$ m

Head over weir, $H = 0.39$ m

Height of crest from bed of channel = 0.6 m

$\therefore$ Depth of channel $= 0.6 + 0.39 = 0.99$

Value of $C_d = 0.623$

Suppressed weir means that the width of channel is equal to width of weir *i.e.*, there is no end contraction.

$\therefore$ Width of channel = Width of weir = 0.77 m

Now area of channel, A = Width of channel × Depth of channel $= 0.77 \times 0.99$

The discharge over a rectangular weir without velocity of approach is given by equation (8.1).

$$\therefore \quad Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g} \times H^{3/2} \qquad (\because \text{ Here } b = L)$$

$$= \frac{2}{3} \times 0.623 \times 0.77 \times \sqrt{2 \times 9.81} \times 0.39^{3/2} = 0.345 \text{ m}^3/\text{s}$$

Now velocity of approach, $V_a = \dfrac{Q}{\text{Area of channel}} = \dfrac{0.345}{0.77 \times 0.99} = 0.4526$ m/s

Head due to velocity of approach,

$$h_a = \frac{V_a^2}{2g} = \frac{0.4526^2}{2 \times 9.81} = 0.0104 \text{ m}$$

Now the discharge with velocity of approach is given by,

$$Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g}\ [(H + h_a)^{3/2} - h_a^{3/2}]$$

$$= \frac{2}{3} \times 0.623 \times 0.77 \times \sqrt{2 \times 9.81}\ [(0.39 + 0.0104)^{3/2} - (0.0104)^{3/2}]$$

$$= \frac{2}{3} \times 0.623 \times 0.77 \times 4.43\ [0.2533 - 0.00106]$$

$$= \mathbf{0.3573\ m^3/s.}\ \textbf{Ans.}$$

Problem 8.19 *A sharp crested rectangular weir of 1 m height extends across a rectangular channel of 3 m width. If the head of water over the weir is 0.45 m, calculate the discharge. Consider velocity of approach and assume $C_d = 0.623$.*

Solution. Given :

Width of channel, $b = 3$ m

Height of weir $= 1$ m

Head of water over weir, $H = 0.45$ m

$\therefore$ Depth of channel = Height of weir + Head of water over weir

$= 1 + 0.45 = 1.45$ m

Value of $C_d = 0.623$

The discharge over a rectangular weir without velocity of approach is given by equation (8.1) as

$$Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g} \times H^{3/2}$$

$$= \frac{2}{3} \times 0.623 \times 3 \times \sqrt{2 \times 9.81} \times 0.45^{3/2} = 1.665 \text{ m}^3/\text{s}$$

Now velocity of approach is given by

$$V_a = \frac{Q}{\text{Area of channel}}$$

$$= \frac{1.665}{\text{Width of channel} \times \text{Depth of channel}} = \frac{1.665}{3 \times 1.45} = 0.382 \text{ m/s}$$

Head due to velocity of approach is given by,

$$h_a = \frac{V_a^2}{2g} = \frac{0.382^2}{2 \times 9.81} = 0.0074 \text{ m}$$

Now the discharge with velocity of approach is given by,

$$Q = \frac{2}{3} \times C_d \times b \times \sqrt{2g}\ [(H + h_a)^{3/2} - (h_a)^{3/2}]$$

$$= \frac{2}{3} \times 0.623 \times 3 \times \sqrt{2 \times 9.81}\ [(0.45 + 0.0074)^{3/2} - (0.0074)^{3/2}]$$

$$= \mathbf{1.703\ m^3/s.}\ \textbf{Ans.}$$

8.11 EMPIRICAL FORMULAE FOR DISCHARGE OVER RECTANGULAR WEIR

The discharge over a rectangular weir is given by

$$Q = \frac{2}{3} C_d \sqrt{2g} \times L \times [H^{3/2}] \text{ without velocity of approach} \quad ...(i)$$

$$= \frac{2}{3} C_d \sqrt{2g} \times L \times [(H + h_a)^{3/2} - h_a^{3/2}] \text{ with velocity of approach} \quad ...(ii)$$

Equations (*i*) and (*ii*) are applicable to the weir or notch for which the crest length is equal to the width of the channel. This type of weir is called *Suppressed weir*. But if the weir is not suppressed, the effect of end contraction will be taken into account.

(*a*) **Francis's Formula.** Francis on the basis of his experiments established that end contraction decreases the effective length of the crest of weir and hence decreases the discharge. Each end contraction reduces the crest length by $0.1 \times H$, where H is the head over the weir. For a rectangular weir there are two end contractions only and hence effective length

$$L = (L - 0.2\ H)$$

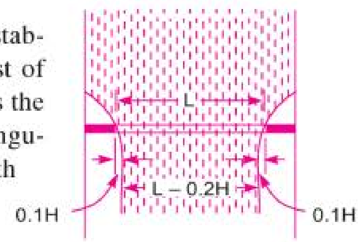


Fig. 8.8

and

$$Q = \frac{2}{3} \times C_d \times [L - 0.2 \times H] \times \sqrt{2g}\ H^{3/2}$$

If $C_d = 0.623$, $g = 9.81$ m/s^2, then

$$Q = \frac{2}{3} \times .623 \times \sqrt{2 \times 9.81} \times [L - 0.2 \times H] \times H^{3/2}$$

$$= 1.84\ [L - 0.2 \times H]H^{3/2} \quad ...(8.11)$$

If end contractions are suppressed, then

$$H = 1.84\ LH^{3/2} \quad ...(8.12)$$

If velocity of approach is considered, then

$$Q = 1.84\ L\ [(H + h_a)^{3/2} - h_a^{3/2}] \quad ...(8.13)$$

(*b*) **Bazin's Formula.** On the basis of results of a series of experiments, Bazin's proposed the following formula for the discharge over a rectangular weir as

$$Q = m \times L \times \sqrt{2g} \times H^{3/2} \quad ...(8.14)$$

where $m = \frac{2}{3} \times C_d = 0.405 + \frac{.003}{H}$

H = height of water over the weir

If velocity of approach is considered, then

$$Q = m_1 \times L \times \sqrt{2g}\ [(H + h_a)^{3/2}] \quad ...(8.15)$$

where $m_1 = 0.405 + \frac{.003}{(H + h_a)}$.

Problem 8.20 *The head of water over a rectangular weir is 40 cm. The length of the crest of the weir with end contraction suppressed is 1.5 m. Find the discharge using the following formulae : (i) Francis's Formula and (ii) Bazin's Formula.*

Solution. Given :

Head of water, $H = 40 \text{ cm} = 0.40 \text{ m}$

Length of weir, $L = 1.5 \text{ m}$

(*i*) Francis's Formula for end contraction suppressed is given by equation (8.12).

$$Q = 1.84\ L \times H^{3/2} = 1.84 \times 1.5 \times (.40)^{3/2}$$
$$= 0.6982 \text{ m}^3\text{/s}$$

(*ii*) Bazin's Formula is given by equation (8.14)

$$Q = m \times L \times \sqrt{2g} \times H^{3/2}$$

where $m = 0.405 + \dfrac{.003}{H} = 0.405 + \dfrac{.003}{.40} = 0.4125$

$$\therefore \quad Q = .4125 \times 1.5 \times \sqrt{2 \times 9.81} \times (.4)^{3/2}$$
$$= \mathbf{0.6932 \text{ m}^3\text{/s.}} \text{ Ans.}$$

Problem 8.21 *A weir 36 metres long is divided into 12 equal bays by vertical posts, each 60 cm wide. Determine the discharge over the weir if the head over the crest is 1.20 m and velocity of approach is 2 metres per second.*

Solution. Given :

Length of weir, $L_1 = 36 \text{ m}$

Number of bays, $= 12$

For 12 bays, no. of vertical post $= 11$

Width of each post $= 60 \text{ cm} = 0.6 \text{ m}$

$\therefore$ Effective length, $L = L_1 - 11 \times 0.6 = 36 - 6.6 = 29.4 \text{ m}$

Head on weir, $H = 1.20 \text{ m}$

Velocity of approach, $V_a = 2 \text{ m/s}$

$\therefore$ Head due to V_a, $h_a = \dfrac{V_a^2}{2g} = \dfrac{2^2}{2 \times 9.81} = 0.2038 \text{ m}$

Number of end contraction, $n = 2 \times 12$ {Each bay has two end contractions}

$= 24$

$\therefore$ Discharge by Francis Formula with end contraction and velocity of approach is

$$Q = 1.84\ [L - 0.1 \times n(H + h_a)][(H + h_a)^{3/2} - h_a^{3/2}]$$
$$= 1.84[29.4 - 0.1 \times 24(1.20 + .2038)] \times [(1.2 + .2038)^{1.5} - .2038^{1.5}]$$
$$= 1.84[29.4 - 3.369][1.663 - .092]$$
$$= \mathbf{75.246 \text{ m}^3\text{/s.}} \text{ Ans.}$$

Problem 8.22 *A discharge of 2000 m³/s is to pass over a rectangular weir. The weir is divided into a number of openings each of span 10 m. If the velocity of approach is 4 m/s, find the number of openings needed in order the head of water over the crest is not to exceed 2 m.*

Solution. Given :

Total discharge, $Q = 2000 \text{ m}^3\text{/s}$

Length of each opening, $L = 10$

Velocity of approach, $V_a = 4$ m/s

Head over weir, $H = 2$ m

Let number of openings $= N$

Head due to velocity of approach,

$$h_a = \frac{V_a^2}{2g} = \frac{4 \times 4}{2 \times 9.81} = 0.8155 \text{ m}$$

For each opening, number of end contractions are two. Hence discharge for each opening considering velocity of approach is given by Francis Formula

i.e.,
$$Q = 1.84[L - 0.1 \times 2 \times (H + h_a)][(H + h_a)^{3/2} - h_a^{3/2}]$$
$$= 1.84[10.0 - 0.2 \times (2 + .8155)][2.8155^{1.5} - .8155^{1.5}]$$
$$= 17.363[4.7242 - 0.7364] = 69.24 \text{ m}^3\text{/s}$$

$\therefore$ Number of opening
$$= \frac{\text{Total discharge}}{\text{Discharge for one opening}} = \frac{2000}{69.24}$$
$$= 28.88 \text{ (say 29)} = \mathbf{29.}\ \textbf{Ans.}$$

▶ 8.12 CIPOLLETTI WEIR OR NOTCH

Cipolletti weir is a trapezoidal weir, which has side slopes of 1 horizontal to 4 vertical as shown in Fig. 8.9. Thus in ΔABC,

$$\tan\frac{\theta}{2} = \frac{AB}{BC} = \frac{H/4}{H} = \frac{1}{4}$$

$$\therefore \qquad \frac{\theta}{2} = \tan\frac{1}{4} = 14° \, 2'.$$

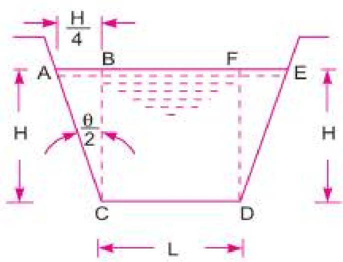


Fig. 8.9 *The cipolletti weir.*

By giving this slope to the sides, an increase in discharge through the triangular portions *ABC* and *DEF* of the weir is obtained. If this slope is not provided the weir would be a rectangular one, and due to end contraction, the discharge would decrease. Thus in case of cipolletti weir, the factor of end contraction is not required which is shown below.

The discharge through a rectangular weir with two end contractions is

$$Q = \frac{2}{3} \times C_d \times (L - 0.2\,H)\,\sqrt{2g} \times H^{3/2}$$
$$= \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ H^{3/2} - \frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$$

Thus due to end contraction, the discharge decreases by $\frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$. This decrease in discharge can be compensated by giving such a slope to the sides that the discharge through two triangular portions is equal to $\frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$. Let the slope is given by $\theta/2$. The discharge through a *V*-notch of angle θ is given by

$$= \frac{8}{15} \times C_d \times \sqrt{2g} \times \tan\frac{\theta}{2}\ H^{5/2}$$

Thus $$\frac{8}{15} \times C_d \times \sqrt{2g} \times \tan\frac{\theta}{2} H^{5/2} = \frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$$

$\therefore$ $$\tan\frac{\theta}{2} = \frac{2}{15} \times \frac{15}{8} = \frac{1}{4} \quad \text{or} \quad \theta/2 = \tan^{-1}\frac{1}{4} = 14° \, 2'.$$

Thus discharge through cipolletti weir is

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ H^{3/2} \qquad ...(8.16)$$

If velocity of approach, V_a is to be taken into consideration,

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ [(H + h_a)^{3/2} - h_a^{3/2}] \qquad ...(8.17)$$

Problem 8.23 *Find the discharge over a cipolletti weir of length 2.0 m when the head over the weir is 1 m. Take $C_d = 0.62$.*

Solution. Given :

Length of weir, $L = 20$ m

Head over weir, $H = 1.0$ m

$C_d = 0.62$

Using equation (8.16), the discharge is given as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 2.0 \times \sqrt{2 \times 9.81} \times (1)^{3/2} = \mathbf{3.661\ m^3/s.}\ \textbf{Ans.}$$

Problem 8.24 *A cipolletti weir of crest length 60 cm discharges water. The head of water over the weir is 360 mm. Find the discharge over the weir if the channel is 80 cm wide and 50 cm deep. Take $C_d = 0.60$.*

Solution. Given :

$C_d = 0.60$

Length of weir, $L = 60$ cm $= 0.60$ m

Head of water, $H = 360$ mm $= 0.36$ m

Channel width $= 80$ cm $= 0.80$ m

Channel depth $= 50$ cm $= 0.50$ m

A = cross-sectional area of channel $= 0.8 \times 0.5 = 0.4\ m^2$

To find velocity of approach, first determine discharge over the weir as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

The velocity of approach, $V_a = \dfrac{Q}{A}$

$\therefore$ $$Q = \frac{2}{3} \times 0.60 \times 0.60 \times \sqrt{2 \times 9.81} \times (0.36)^{3/2}\ m^3/s = 0.2296\ m^3/s$$

$\therefore$ $$V_a = \frac{.2296}{0.40} = 0.574\ m/s$$

Head due to velocity of approach,

$$h_a = V_a^2/2g = \frac{(0.574)^2}{2 \times 9.81} = 0.0168 \text{ m}$$

Thus the discharge is given by equation (8.17) as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g}\ [(H + h_a)^{1.5} - h_a^{1.5}]$$

$$= \frac{2}{3} \times 0.60 \times .6 \times \sqrt{2 \times 9.81}\ [(.36 + .0168)^{1.5} - (.0168)^{1.5}]$$

$$= 1.06296 \times [.2313 - .002177] = \mathbf{0.2435\ m^3/s.}\ \textbf{Ans.}$$

▶ 8.13 DISCHARGE OVER A BROAD-CRESTED WEIR

A weir having a wide crest is known as broad-crested weir.

Let H = height of water above the crest

L = length of the crest

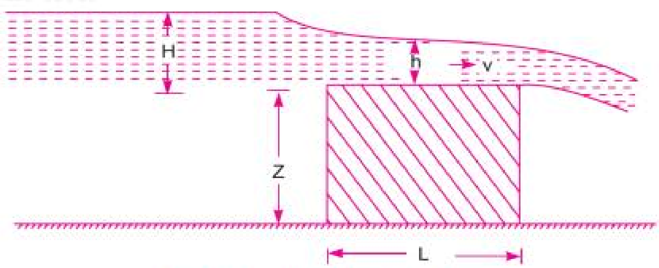


Fig. 8.10 *Broad-crested weir.*

If $2L > H$, the weir is called broad-crested weir

If $2L < H$, the weir is called a narrow-crested weir

Fig. 8.10 shows a broad-crested weir.

Let h = head of water at the middle of weir which is constant

v = velocity of flow over the weir

Applying Bernoulli's equation to the still water surface on the upstream side and running water at the end of weir,

$$0 + 0 + H = 0 + \frac{v^2}{2g} + h$$

$$\therefore \qquad \frac{v^2}{2g} = H - h$$

$$\therefore \qquad v = \sqrt{2g\,(H - h)}$$

$\therefore$ The discharge over weir $Q = C_d \times$ Area of flow $\times$ Velocity

$$= C_d \times L \times h \times \sqrt{2g\,(H - h)}$$

$$= C_d \times L \times \sqrt{2g\left(Hh^2 - h^3\right)} \qquad \text{...(8.18)}$$

The discharge will be maximum, if $(Hh^2 - h^3)$ is maximum

or $$\frac{d}{dh}(Hh^2 - h^3) = 0 \text{ or } 2h \times H - 3h^2 = 0 \text{ or } 2H = 3h$$

$\therefore$ $$h = \frac{2}{3} H$$

$Q_{\max}$ will be obtained by substituting this value of h in equation (8.18) as

$$Q_{max} = C_d \times L \times \sqrt{2g\left[H \times \left(\frac{2}{3}H\right)^2 - \left(\frac{2}{3}H\right)^3\right]}$$

$$= C_d \times L \times \sqrt{2g}\sqrt{H \times \frac{4}{9} \times H^2 - \frac{8}{27}H^3}$$

$$= C_d \times L \times \sqrt{2g}\sqrt{\frac{4}{9}H^3 - \frac{8}{27}H^3} = C_d \times L \times \sqrt{2g}\sqrt{\frac{(12-8)H^3}{27}}$$

$$= C_d \times L \times \sqrt{2g}\sqrt{\frac{4}{27}H^3} = C_d \times L \times \sqrt{2g} \times 0.3849 \times H^{3/2}$$

$$= .3849 \times \sqrt{2 \times 9.81} \times C_d \times L \times H^{3/2} = 1.7047 \times C_d \times L \times H^{3/2}$$

$$= 1.705 \times C_d \times L \times H^{3/2}. \quad \text{...(8.19)}$$

▶ 8.14 DISCHARGE OVER A NARROW-CRESTED WEIR

For a narrow-crested weir, $2L < H$. It is similar to a rectangular weir or notch hence, Q is given by

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \quad \text{...(8.20)}$$

▶ 8.15 DISCHARGE OVER AN OGEE WEIR

Fig. 8.11 shows an Ogee weir, in which the crest of the weir rises upto maximum height of $0.115 \times H$ (where H is the height of water above inlet of the weir) and then falls as shown in Fig. 8.11. The discharge for an Ogee weir is the same as that of a rectangular weir, and it is given by

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \quad \text{...(8.21)}$$

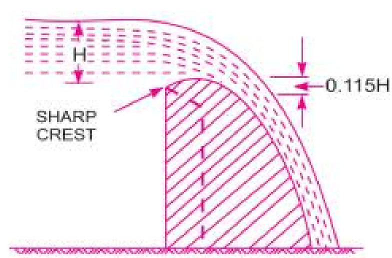


Fig. 8.11 *An Ogee weir.*

▶ 8.16 DISCHARGE OVER SUB-MERGED OR DROWNED WEIR

When the water level on the downstream side of a weir is above the crest of the weir, then the weir is called to be a sub-merged or drowned weir. Fig. 8.12 shows a sub-merged weir. The total discharge, over the weir is obtained by dividing the weir into two parts. The portion between upstream and downstream water surface may be treated as free weir and portion between downstream water surface and crest of weir as a drowned weir.

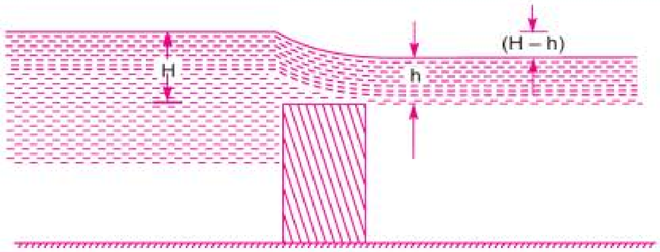


Fig. 8.12 *Sub-merged weir.*

Let H = height of water on the upstream side of the weir

h = height of water on the downstream side of the weir

Then Q_1 = discharge over upper portion

$$= \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g}\ [H - h]^{3/2}$$

Q_2 = discharge through drowned portion

$$= C_{d_2} \times \text{Area of flow} \times \text{Velocity of flow}$$

$$= C_{d_2} \times L \times h \times \sqrt{2g(H - h)}$$

$\therefore$ Total discharge, $Q = Q_1 + Q_2$

$$= \frac{2}{3} C_{d_1} \times L \times \sqrt{2g}\ [H - h]^{3/2} + C_{d_2} \times L \times h \times \sqrt{2g(H - h)}\,. \quad \text{...(8.22)}$$

Problem 8.25 *(a) A broad-crested weir of 50 m length, has 50 cm height of water above its crest. Find the maximum discharge. Take C_d = 0.60. Neglect velocity of approach. (b) If the velocity of approach is to be taken into consideration, find the maximum discharge when the channel has a cross-sectional area of 50 m² on the upstream side.*

Solution. Given :

Length of weir, $L = 50$ m

Head of water, $H = 50$ cm $= 0.5$ m

$C_d = 0.60$

(*i*) **Neglecting velocity of approach.** Maximum discharge is given by equation (8.19) as

$$Q_{max} = 1.705 \times C_d \times L \times H^{3/2}$$

$$= 1.705 \times 0.60 \times 50 \times (.5)^{3/2} = \mathbf{18.084\ m^3/s.}\ \textbf{Ans.}$$

(*ii*) **Taking velocity of approach into consideration**

Area of channel, $A = 50\ m^2$

Velocity of approach, $V_a = \frac{Q}{A} = \frac{18.084}{50} = 0.36$ m/s

$\therefore$ Head due to V_a, $h_a = \frac{V_a^2}{2g} = \frac{0.36 \times .36}{2 \times 9.81} = .0066$ m

Maximum discharge, Q_{max} is given by

$$Q_{max} = 1.705 \times C_d \times L \times [(H + h_a)^{3/2} - h_a^{\ 3/2}]$$

$$= 1.705 \times 0.6 \times 50 \times [(.50 + .0066)^{1.5} - (.0066)^{1.5}]$$

$$= 51.15[0.3605 - .000536] = \mathbf{18.412\ m^3/s.}\ \textbf{Ans.}$$

Problem 8.26 *An Ogee weir 5 metres long has a head of 40 cm of water. If $C_d = 0.6$, find the discharge over the weir.*

Solution. Given :

Length of weir, $L = 5$ m

Head of water, $H = 40$ cm $= 0.40$ m

$C_d = 0.6$

Discharge over Ogee weir is given by equation (8.21) as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$= \frac{2}{3} \times 0.60 \times 5.0 \times \sqrt{2 \times 9.81} \times (0.4)^{3/2} = \mathbf{2.2409\ m^3/s.}\ \textbf{Ans.}$$

Problem 8.27 *The heights of water on the upstream and downstream side of a sub-merged weir of 3 m length are 20 cm and 10 cm respectively. If C_d for free and drowned portions are 0.6 and 0.8 respectively, find the discharge over the weir.*

Solution. Given :

Height of water on upstream side, $H = 20$ cm $= 0.20$ m

Height of water on downstream side, $h = 10$ cm $= 0.10$ m

Length of weir, $L = 3$ m

$C_{d_1} = 0.6$

$C_{d_2} = 0.8$

Total discharge Q is the sum of discharge through free portion and discharge through the drowned portion. This is given by equation (8.22) as

$$Q = \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g}\ [H - h]^{3/2} + C_{d_2} \times L \times h \times \sqrt{2g\,(H - h)}$$

$$= \frac{2}{3} \times 0.6 \times 3 \times \sqrt{2 \times 9.81}\ [.20 - .10]^{1.5} + 0.8 \times 3 \times .10 \times \sqrt{2 \times 9.81\,(.2 - .1)}$$

$$= 0.168 + 0.336 = \mathbf{0.504\ m^3/s.}\ \textbf{Ans.}$$

HIGHLIGHTS

1. A notch is a device used for measuring the rate of flow of a liquid through a small channel. A weir is a concrete or masonary structure placed in the open channel over which the flow occurs.
2. The discharge through a rectangular notch or weir is given by

$$Q = \frac{2}{3}\ C_d \times L \times H^{3/2}$$

where C_d = Co-efficient of discharge,

L = Length of notch or weir,

H = Head of water over the notch or weir.

3. The discharge over a triangular notch or weir is given by

$$Q = \frac{8}{15}\ C_d \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

where θ = total angle of triangular notch.

4. The discharge through a trapezoidal notch or weir is equal to the sum of discharge through a rectangular notch and the discharge through a triangular notch. It is given as

$$Q = \frac{2}{3} C_{d_1} \times L \times \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan\frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

where C_{d_1} = co-efficient of discharge for rectangular notch,

C_{d_2} = co-efficient of discharge for triangular notch,

$\theta/2$ = slope of the side of trapezoidal notch.

5. The error in discharge due to the error in the measurement of head over a rectangular and triangular notch or weir is given by

$$\frac{dQ}{Q} = \frac{3}{2}\frac{dH}{H} \quad \text{... For a rectangular weir or notch}$$

$$= \frac{5}{2}\frac{dH}{H} \quad \text{... For a triangular weir or notch}$$

where Q = discharge through rectangular or triangular notch or weir

H = head over the notch or weir.

6. The time required to empty a reservoir or a tank by a rectangular or a triangular notch is given by

$$H = \frac{3A}{C_d L \sqrt{2g}}\left[\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}}\right] \quad \text{... By a rectangular notch}$$

$$= \frac{5A}{4C_d \tan\frac{\theta}{2} \times \sqrt{2g}}\left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right] \quad \text{... By a triangular notch}$$

where A = cross-sectional area of a tank or a reservoir

H_1 = initial height of liquid above the crest or apex of notch

H_2 = final height of liquid above the crest or apex of notch.

7. The velocity with which the water approaches the weir or notch is called the velocity of approach. It is denoted by V_a and is given by

$$V_a = \frac{\text{Discharge over the notch or weir}}{\text{Cross-sectional area of channel}}.$$

8. The head due to velocity of approach is given by $h_a = \dfrac{V_a^2}{2g}$.

9. Discharge over a rectangular weir, with velocity of approach,

$$Q = \frac{2}{3} C_d L \sqrt{2g}\ [(H_1 + h_a)^{3/2} - h_a^{3/2}].$$

10. Francis's Formula for a rectangular weir is given by

$$Q = 1.84[L - 0.2\,H]\,H^{3/2} \quad \text{... For two end contractions}$$

$$= 1.84\,L\,H^{3/2} \quad \text{... If end contractions are suppressed}$$

$$= 1.84\,L\,[(H + h_a)^{3/2} - h_a^{3/2}] \quad \text{... If velocity of approach is considered}$$

where L = length of weir,

H = height of water above the crest of the weir,

h_a = head due to velocity of approach.

11. Bazin's Formula for discharge over a rectangular weir,

$$Q = m\,L\,\sqrt{2g}H^{3/2} \quad \text{... without velocity of approach}$$

$$= m\,L\,\sqrt{2g}\;[(H + h_a)^{3/2}] \quad \text{... with velocity of approach}$$

where $$m = \frac{2}{3}\,C_d = 0.405 + \frac{.003}{H} \quad \text{... without velocity of approach}$$

$$= 0.405 + \frac{.003}{(H + h_a)} \quad \text{... with velocity of approach.}$$

12. *A* trapezoidal weir, with side slope or 1 horizontal to 4 vertical, is called Cipolletti weir. The discharge through Cipolletti weir is given by

$$Q = \frac{2}{3}\,C_d \times L \times \sqrt{2g}\;H^{3/2} \quad \text{... without velocity of approach}$$

$$= \frac{2}{3}\,C_d \times L \times \sqrt{2g}\;[(H + h_a)^{3/2} - h_a^{\;3/2}] \quad \text{... with velocity of approach.}$$

13. The discharge over a broad-crested weir is given by,

$$Q = C_d L\,\sqrt{2g\left(Hh^2 - h^3\right)}$$

where H = height of water above the crest
h = head of water at the middle of the weir which is constant
L = length of the weir.

14. The condition for maximum discharge over a broad-crested weir is $h = \frac{2}{3}\,H$

and maximum discharge is given by $Q_{max} = 1.705\,C_d\,L\,H^{3/2}$.

15. The discharge over an Ogee weir is given by $Q = \frac{2}{3}\,C_d L \times \sqrt{2g} \times H^{3/2}$.

16. The discharge over sub-merged or drowned weir is given by

Q = discharge over upper portion + discharge through downed portion

$$= \frac{2}{3}\,C_{d_1}\,L \times \sqrt{2g}\;(H - h)^{3/2} + C_{d_2}\,Lh \times \sqrt{2g\,(H - h)}$$

where H = height of water on the upstream side of the weir,
h = height of water on the downstream side of the weir.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define the terms : notch, weir, nappe and crest.
2. How are the weirs and notches classified ?
3. Find an expression for the discharge over a rectangular weir in terms of head of water over the crest of the weir.
4. Prove that the discharge through a triangular notch or weir is given by

$$Q = \frac{8}{15}\,C_d \times \tan\frac{\theta}{2} \times \sqrt{2g}\;H^{3/2}$$

where H = head of water over the notch or weir
θ = angle of notch or weir.

5. What are the advantages of triangular notch or weir over rectangular notch ?
6. Prove that the error in discharge due to the error in the measurement of head over a rectangular notch is given by

$$\frac{dQ}{Q} = \frac{3}{2}\frac{dH}{H}$$

where Q = discharge through rectangular notch
and H = head over the rectangular notch.
7. Find an expression for the time required to empty a tank of area of cross-section A, with a rectangular notch.
8. What do you understand by 'Velocity of Approach' ? Find an expression for the discharge over a rectangular weir with velocity of approach.
9. Define 'end contraction' of a weir. What is the effect of end contraction on the discharge through a weir ?
10. What is a Cipolletti Weir ? Prove that the discharge through Cipolletti weir is given by

$$Q = \frac{2}{3} C_d L \sqrt{2g}\, H^{3/2}$$

where L = length of weir, and H = head of water over weir.
11. Differentiate between Broad-crested weir and Narrow-crested weir. Find the condition for maximum discharge over a Broad-crested weir and hence derive an expression for maximum discharge over a broad-crested weir.
12. What do you mean by a drowned weir ? How will you determine the discharge for the downed weir ?
13. Discuss 'end contraction' of a weir.
14. State the different devices that can be used to measure the discharge through a pipe also through an open channel. Describe one of such devices with a neat sketch and explain how one can obtain the actual discharge with its help.
15. What is the difference between a notch and a weir ?
16. Define velocity of approach. How does the velocity of approach affect the discharge over a weir ?

(B) NUMERICAL PROBLEMS

1. Find the discharge of water flowing over rectangular notch of 3 m length when the constant head of water over the notch is 40 cm. Take $C_d = 0.6$. [**Ans.** 1.344 m^3/s]
2. Determine the height of a rectangular weir of length 5 m to be built across a rectangular channel. The maximum depth of water on the upstream side of the weir is 1.5 m and discharge is 1.5 m^3 per second. Take $C_d = 0.6$ and neglect end contractions. [**Ans.** 1.194 m]
3. Find the discharge over a triangular notch of angle 60° when the head over the triangular notch is 0.20 m. Take $C_d = 0.6$. [**Ans.** 0.0164 m^3/s]
4. A rectangular channel 1.5 m wide has a discharge of 200 litres per second, which is measured by a right-angled V-notch weir. Find the position of the apex of the notch from the bed of the channel if maximum depth of water is not be exceed 1 m. Take $C_d = 0.62$. [**Ans.** .549 m]
5. Find the discharge through a trapezoidal notch which is 1.2 m wide at the top and 0.50 m at the bottom and is 40 cm in height. The head of water on the notch is 30 cm. Assume C_d for rectangular portion as 0.62 while for triangular portion = 0.60. [**Ans.** 0.22 m^3/s]
6. A rectangular notch 50 cm long is used for measuring a discharge of 40 litres per second. An error of 2 mm was made in measuring the head over the notch. Calculate the percentage error in the discharge. Take $C_d = 0.6$. [**Ans.** 2.37%]
7. A right-angled V-notch is used for measuring a discharge of 30 litres/s. An error of 2 mm was made in measuring the head over the notch. Calculate the percentage error in the discharge. Take $C_d = 0.62$. [**Ans.** 2.37%]

8. Find the time required to lower the water level from 3 m to 1.5 m in a reservoir of dimension 70 m × 70 m, by a rectangular notch of length 2.0 m. Take $C_d = 0.60$. [**Ans.** 11 min 1 s]
9. If in the problem 8, instead of a rectangular notch, a right angled *V*-notch is used, find the time required. Take all other data same. [**Ans.** 13 min 31 s]
10. Water is flowing in a rectangular channel of 1.2 m wide and 0.8 m deep. Find the discharge over a rectangular weir of crest length 70 cm if the head of water over the crest of weir is 25 cm and water from channel flows over the weir. Take $C_d = 0.60$. Neglect end contractions but consider velocity of approach. [**Ans.** 0.1557 m^3/s]
11. Find the discharge over a rectangular weir of length 80 m. The head of water over the weir is 1.2 m. The velocity of approach is given as 1.5 m/s. Take $C_d = 3.6$. [**Ans.** 208.11 m^3/s]
12. The head of water over a rectangular weir is 50 cm. The length of the crest of the weir with end contraction suppressed is 1.4 m. Find the discharge using following formulae : (*i*) Francis's Formula and (*ii*) Bazin's Formula. [**Ans.** (*i*) 0.91 m^3/s, (*ii*) .901 m^3/s]
13. A discharge of 1500 m^3/s is to pass over a rectangular weir. The weir is divided into a number of openings each of span 7.5 m. If the velocity of approach is 3 m/s, find the number of openings needed in order the head of water over the crest is not to exceed 1.8. [**Ans.** 37.5 say 38]
14. Find the discharge over a cipolletti weir of length 1.8 m when the head over the weir is 1.2 m. Take $C_d = 0.62$ [**Ans.** 4.331 m^3/s]
15. (*a*) A broad-crested weir of length 40 m, has 400 mm height of water above its crest. Find the maximum discharge. Take $C_d = 0.6$. Neglect velocity of approach. [**Ans.** 10.352 m^3/s]
    (*b*) If the velocity of approach is to be taken into consideration, find the maximum discharge when the channel has a cross-sectional area of 40 m^2 on the upstream side. [**Ans.** 10.475 m^3/s]
16. An Ogee weir 4 m long has a head of 500 mm of water. If $C_d = 0.6$, find the discharge over the weir. [**Ans.** 2.505 m^3/s]
17. The heights of water on the upstream and downstream side of a sub-merged weir of length 3.5 m are 300 mm and 150 mm respectively. If C_d for free and drowned portion are 0.6 and 0.8 respectively, find the discharge over the weir. [**Ans.** 1.0807 m^3/s]

9 CHAPTER

VISCOUS FLOW

▶ 9.1 INTRODUCTION

This chapter deals with the flow of fluids which are viscous and flowing at very low velocity. At low velocity the fluid moves in layers. Each layer of fluid slides over the adjacent layer. Due to relative velocity between two layers the velocity gradient $\frac{du}{dy}$ exists and hence a shear stress $\tau = \mu \frac{du}{dy}$ acts on the layers.

The following cases will be considered in this chapter :

1. Flow of viscous fluid through circular pipe.
2. Flow of viscous fluid between two parallel plates.
3. Kinetic energy correction and momentum correction factors.
4. Power absorbed in viscous flow through
   (*a*) Journal bearings, (*b*) Foot-step bearings, and (*c*) Collar bearings.

▶ 9.2 FLOW OF VISCOUS FLUID THROUGH CIRCULAR PIPE

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum velocity to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through the circular pipe will be viscous or laminar, if the Reynolds number (R_e*) is less than 2000. The expression for Reynold number is given by

$$R_e = \frac{\rho V D}{\mu}$$

where ρ = Density of fluid flowing through pipe
V = Average velocity of fluid
D = Diameter of pipe and
μ = Viscosity of fluid.

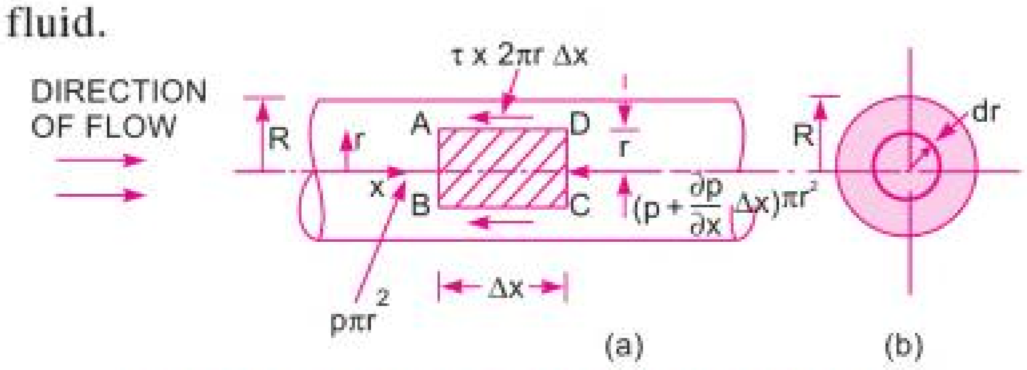


Fig. 9.1 *Viscous flow through a pipe.*

* For derivation, please refer to Art. 12.8.1.

Consider a horizontal pipe of radius R. The viscous fluid is flowing from left to right in the pipe as shown in Fig. 9.1 (a). Consider a fluid element of radius r, sliding in a cylindrical fluid element of radius $(r + dr)$. Let the length of fluid element be Δx. If 'p' is the intensity of pressure on the face AB, then the intensity of pressure on face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$. Then the forces acting on the fluid element are :

1. The pressure force, $p \times \pi r^2$ on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2$ on face CD.
3. The shear force, $\tau \times 2\pi r \Delta x$ on the surface of fluid element. As there is no acceleration, hence the summation of all forces in the direction of flow must be zero *i.e.*,

$$p\pi r^2 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or
$$-\frac{\partial p}{\partial x}\Delta x \pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or
$$-\frac{\partial p}{\partial x} . r - 2\tau = 0$$

$\therefore$
$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \qquad \text{...(9.1)}$$

The shear stress τ across a section varies with 'r' as $\frac{\partial p}{\partial x}$ across a section is constant. Hence shear stress distribution across a section is linear as shown in Fig. 9.2 (a).

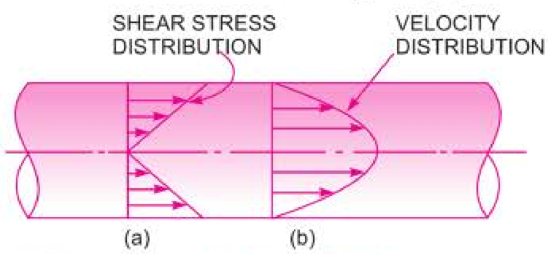


Fig. 9.2 *Shear stress and velocity distribution across a section.*

(i) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \frac{du}{dy}$ is substituted in equation (9.1).

But in the relation $\tau = \mu \frac{du}{dy}$, y is measured from the pipe wall. Hence

$$y = R - r \quad \text{and} \quad dy = -dr$$

$\therefore$
$$\tau = \mu \frac{du}{-dr} = -\mu \frac{du}{dr}$$

Substituting this value in (9.1), we get

$$-\mu \frac{du}{dr} = -\frac{\partial p}{\partial x}\frac{r}{2} \quad \text{or} \quad \frac{du}{dr} = \frac{1}{2\mu}\frac{\partial p}{\partial x} r$$

Integrating this above equation w.r.t. 'r', we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 + C \qquad \text{...(9.2)}$$

where C is the constant of integration and its value is obtained from the boundary condition that at $r = R$, $u = 0$.

$$\therefore \qquad 0 = \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 + C$$

$$\therefore \qquad C = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

Substituting this value of C in equation (9.2), we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \qquad \text{...(9.3)}$$

In equation (9.3), values of μ, $\frac{\partial p}{\partial x}$ and R are constant, which means the velocity, u varies with the square of r. Thus equation (9.3) is a equation of parabola. This shows that the velocity distribution across the section of a pipe is parabolic. This velocity distribution is shown in Fig. 9.2 (*b*).

(*ii*) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when $r = 0$ in equation (9.3). Thus maximum velocity, U_{max} is obtained as

$$U_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 \qquad \text{...(9.4)}$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge of the fluid across the section by the area of the pipe (πR^2). The discharge (Q) across the section is obtained by considering the flow through a circular ring element of radius r and thickness dr as shown in Fig. 9.1 (*b*). The fluid flowing per second through this elementary ring

$$dQ = \text{velocity at a radius } r \times \text{area of ring element}$$

$$= u \times 2\pi r\, dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \times 2\pi r\, dr$$

$$\therefore \qquad Q = \int_0^R dQ = \int_0^R -\frac{1}{4\mu}\frac{\partial p}{\partial x}(R^2 - r^2) \times 2\pi r\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \int_0^R (R^2 - r^2)\, r dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \int_0^R (R^2 r - r^3)\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \times \frac{R^4}{4} = \frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^4$$

∴ Average velocity, $$\bar{u} = \frac{Q}{\text{Area}} = \frac{\frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^4}{\pi R^2}$$

or $$\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^2 \quad \text{...(9.5)}$$

Dividing equation (9.4) by equation (9.5),

$$\frac{U_{\max}}{\bar{u}} = \frac{-\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2}{\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2} = 2.0$$

∴ Ratio of maximum velocity to average velocity = 2.0.

(*iii*) **Drop of Pressure for a given Length (*L*) of a pipe**

From equation (9.5), we have

$$\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^2 \quad \text{or} \quad \left(\frac{-\partial p}{\partial x}\right) = \frac{8\mu\bar{u}}{R^2}$$

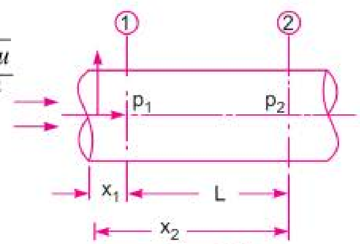


Fig. 9.3

Integrating the above equation w.r.t. x, we get

$$-\int_2^1 dp = \int_2^1 \frac{8\mu\bar{u}}{R^2}\, dx$$

∴ $$-[p_1 - p_2] = \frac{8\mu\bar{u}}{R^2}[x_1 - x_2] \text{ or } (p_1 - p_2) = \frac{8\mu\bar{u}}{R^2}[x_2 - x_1]$$

$$= \frac{8\mu\bar{u}}{R^2} L \qquad \{\because\ x_2 - x_1 = L \text{ from Fig. 9.3}\}$$

$$= \frac{8\mu\bar{u}L}{(D/2)^2} \qquad \left\{\because\ R = \frac{D}{2}\right\}$$

or $$(p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2},$$ where $p_1 - p_2$ is the drop of pressure.

∴ Loss of pressure head $= \dfrac{p_1 - p_2}{\rho g}$

∴ $$\frac{p_1 - p_2}{\rho g} = h_f = \frac{32\mu\bar{u}L}{\rho g D^2} \quad \text{...(9.6)}$$

Equation (9.6) is called **Hagen Poiseuille Formula.**

Problem 9.1 *A crude oil of viscosity 0.97 poise and relative density 0.9 is flowing through a horizontal circular pipe of diameter 100 mm and of length 10 m. Calculate the difference of pressure at the two ends of the pipe, if 100 kg of the oil is collected in a tank in 30 seconds.*

Solution. Given : $\mu = 0.97 \text{ poise} = \frac{0.97}{10} = 0.097 \text{ Ns/m}^2$

Relative density $= 0.9$

$\therefore$ ρ_0, or density, $= 0.9 \times 1000 = 900 \text{ kg/m}^3$

Dia. of pipe, $D = 100 \text{ mm} = 0.1 \text{ m}$

$L = 10 \text{ m}$

Mass of oil collected, $M = 100 \text{ kg}$

Time, $t = 30 \text{ seconds}$

Calculate difference of pressure or $(p_1 - p_2)$.

The difference of pressure $(p_1 - p_2)$ for viscous or laminar flow is given by

$$p_1 - p_2 = \frac{32\mu \bar{u} L}{D^2}, \text{ where } \bar{u} = \text{average velocity} = \frac{Q}{\text{Area}}$$

Now, mass of oil/sec $= \frac{100}{30}$ kg/s

$$= \rho_0 \times Q = 900 \times Q \qquad (\because \rho_0 = 900)$$

$$\therefore \qquad \frac{100}{30} = 900 \times Q$$

$$\therefore \qquad Q = \frac{100}{30} \times \frac{1}{900} = 0.0037 \text{ m}^3/\text{s}$$

$$\therefore \qquad \bar{u} = \frac{Q}{\text{Area}} = \frac{.0037}{\frac{\pi}{4} D^2} = \frac{.0037}{\frac{\pi}{4} (.1)^2} = 0.471 \text{ m/s.}$$

For laminar or viscous flow, the Reynolds number (R_e) is less than 2000. Let us calculate the Reynolds number for this problem.

Reynolds number, $R_e{}^* = \frac{\rho V D}{\mu}$

where $\rho = \rho_0 = 900$, $V = \bar{u} = 0.471$, $D = 0.1$ m, $\mu = 0.097$

$$\therefore \qquad R_e = 900 \times \frac{.471 \times 0.1}{0.097} = 436.91$$

As Reynolds number is less than 2000, the flow is laminar.

$$\therefore \qquad p_1 - p_2 = \frac{32\mu \bar{u} L}{D^2} = \frac{32 \times 0.097 \times .471 \times 10}{(.1)^2} \text{ N/m}^2$$

$$= 1462.28 \text{ N/m}^2 = 1462.28 \times 10^{-4} \text{ N/cm}^2 = \mathbf{0.1462 \text{ N/cm}^2}. \textbf{ Ans.}$$

* For derivation, please refer to Art. 12.8.1

Problem 9.2 *An oil of viscosity 0.1 Ns/m² and relative density 0.9 is flowing through a circular pipe of diameter 50 mm and of length 300 m. The rate of flow of fluid through the pipe is 3.5 litres/s. Find the pressure drop in a length of 300 m and also the shear stress at the pipe wall.*

Solution. Given : Viscosity, $\mu = 0.1$ Ns/m²

Relative density $= 0.9$

$\therefore$ ρ_0 or density of oil $= 0.9 \times 1000 = 900$ kg/m³ ($\because$ Density of water = 1000 kg/m³)

$$D = 50 \text{ mm} = .05 \text{ m}$$

$$L = 300 \text{ m}$$

$$Q = 3.5 \text{ litres/s} = \frac{3.5}{1000} = .0035 \text{ m}^3\text{/s}$$

Find (*i*) Pressure drop, $p_1 - p_2$

(*ii*) Shear stress at pipe wall, τ_0

(*i*) **Pressure drop ($\mathbf{p_1 - p_2}$)** $= \dfrac{32\mu\bar{u}L}{D^2}$, where $\bar{u} = \dfrac{Q}{\text{Area}} = \dfrac{.0035}{\frac{\pi}{4}D^2} = \dfrac{.0035}{\frac{\pi}{4}(.05)^2} = 1.782$ m/s

The Reynolds number (R_e) is given by, $R_e = \dfrac{\rho VD}{\mu}$

where $\rho = 900$ kg/m³, V = average velocity $= \bar{u} = 1.782$ m/s

$$\therefore \qquad R_e = 900 \times \frac{1.782 \times .05}{0.1} = 801.9$$

As Reynolds number is less than 2000, the flow is viscous or laminar

$$\therefore \qquad p_1 - p_2 = \frac{32 \times 0.1 \times 1.782 \times 3000}{(.05)^2}$$

$= 684288$ N/m² $= 68428 \times 10^{-4}$ N/cm² $=$ **68.43 N/cm². Ans.**

(*ii*) **Shear Stress at the pipe wall (τ_0)**

The shear stress at any radius r is given by the equation (9.1)

i.e.,
$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2}$$

$\therefore$ Shear stress at pipe wall, where $r = R$ is given by

$$\tau_0 = \frac{-\partial p}{\partial x}\frac{R}{2}$$

Now
$$\frac{-\partial p}{\partial x} = \frac{-(p_2 - p_1)}{x_2 - x_1} = \frac{p_1 - p_2}{x_2 - x_1} = \frac{p_1 - p_2}{L}$$

$$= \frac{684288}{300}\frac{\text{N/m}^2}{\text{m}} = 2280.96 \text{ N/m}^3$$

and
$$R = \frac{D}{2} = \frac{.05}{2} = .025 \text{ m}$$

$$\tau_0 = 2280.96 \times \frac{.025}{2}\frac{\text{N}}{\text{m}^2} = \mathbf{28.512\ N/m^2}. \textbf{ Ans.}$$

Problem 9.3 *A laminar flow is taking place in a pipe of diameter 200 mm. The maximum velocity is 1.5 m/s. Find the mean velocity and the radius at which this occurs. Also calculate the velocity at 4 cm from the wall of the pipe.*

Solution. Given : Dia. of pipe, $D = 200$ mm $= 0.20$ m

$$U_{max} = 1.5 \text{ m/s}$$

Find (*i*) Mean velocity, $\bar{u}$

(*ii*) Radius at which $\bar{u}$ occurs

(*iii*) Velocity at 4 cm from the wall.

(*i*) **Mean velocity,** $\bar{u}$

Ratio of $$\frac{U_{\max}}{\bar{u}} = 2.0 \quad \text{or} \quad \frac{1.5}{\bar{u}} = 2.0 \quad \therefore \quad \bar{u} = \frac{1.5}{2.0} = \mathbf{0.75\ m/s.}\ \textbf{Ans.}$$

(*ii*) **Radius at which** $\bar{u}$ occurs

The velocity, u, at any radius 'r' is given by (9.3)

$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 \left[1 - \frac{r^2}{R^2}\right]$$

But from equation (9.4) $U_{\max}$ is given by

$$U_{\max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 \quad \therefore \quad u = U_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{...(1)}$$

Now, the radius r at which $u = \bar{u} = 0.75$ m/s

$$\therefore \quad 0.75 = 1.5\left[1 - \left(\frac{r}{D/2}\right)^2\right]$$

$$= 1.5\left[1 - \left(\frac{r}{0.2/2}\right)^2\right] = 1.5\left[1 - \left(\frac{r}{0.1}\right)^2\right]$$

$$\therefore \quad \frac{0.75}{0.50} = 1 - \left(\frac{r}{0.1}\right)^2$$

$$\therefore \quad \left(\frac{r}{0.1}\right)^2 = 1 - \frac{.75}{1.50} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \quad \frac{r}{0.1} = \sqrt{\frac{1}{2}} = \sqrt{0.5}$$

$$\therefore \quad r = 0.1 \times \sqrt{.5} = 0.1 \times .707 = .0707 \text{ m}$$

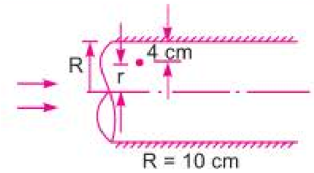
$$= \mathbf{70.7\ mm.}\ \textbf{Ans.}$$

(*iii*) **Velocity at 4 cm from the wall**

$$r = R - 4.0 = 10 - 4.0 = 6.0 \text{ cm} = 0.06 \text{ m}$$

$\therefore$ The velocity at a radius = 0.06 m

or 4 cm from pipe wall is given by equation (1)



$$= U_{max}\left[1-\left(\frac{r}{R}\right)^2\right] = 1.5\left[1-\left(\frac{.06}{.1}\right)^2\right]$$

$= 1.5[1.0 - .36] = 1.5 \times .64 =$ **0.96 m/s. Ans.**

Fig. 9.4

Problem 9.4 *Crude oil of μ = 1.5 poise and relative density 0.9 flows through a 20 mm diameter vertical pipe. The pressure gauges fixed 20 m apart read 58.86 N/cm² and 19.62 N/cm² as shown in Fig. 9.5. Find the direction and rate of flow through the pipe.*

Solution. Given :

$$\mu = 1.5 \text{ poise} = \frac{1.5}{10} = 0.15 \text{ Ns/m}^2$$

Relative density $= 0.9$

$\therefore$ Density of oil $= 0.9 \times 1000 = 900 \text{ kg/m}^3$

Dia. of pipe, $D = 20 \text{ mm} = 0.02 \text{ m}$

$L = 20 \text{ m}$

$p_A = 58.86 \text{ N/cm}^2 = 58.86 \times 10^4 \text{ N/m}^2$

$p_B = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$.

Find (*i*) Direction of flow

(*ii*) Rate of flow.

(*i*) **Direction of flow.** To find the direction of flow, the total energy $\left(\frac{p}{\rho g}+\frac{v^2}{2g}+Z\right)$ at the lower end A and at the upper end B is to be calculated. The direction of flow will be given from the higher energy to the lower energy. As here the diameter of the pipe is same and hence kinetic energy at A and B will be same. Hence to find the direction of flow, calculate $\left(\frac{p}{\rho g}+Z\right)$ at A and B.

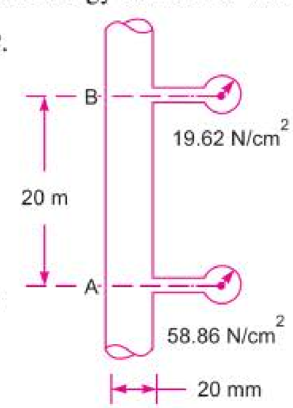


Fig. 9.5

Taking the level at A as datum. The value of $\left(\frac{p_A}{\rho g}+Z\right)$ at

$$A = \frac{p_A}{\rho g} + Z_A$$

$$= \frac{6 \times 10^4 \times 9.81}{900 \times 9.81} + 0 \quad \{\because \ r = 900 \text{ kg/cm}^2\}$$

$$= 66.67 \text{ m}$$

The value of $\left(\frac{p}{\rho g}+Z\right)$ at $B = \frac{p_B}{\rho g} + Z_B$

$$= \frac{2 \times 10^4 \times 9.81}{900 \times 9.81} + 20 = 22.22 + 20 = 42.22 \text{ m}$$

As the value of $\left(\frac{p}{\rho g}+Z\right)$ is higher at A and hence flow takes place from A to B. **Ans.**

(*ii*) **Rate of flow.** The loss of pressure head for viscous flow through circular pipe is given by

$$h_f = \frac{32\mu \bar{u} L}{\rho g D^2}$$

For a vertical pipe h_f = Loss of peizometric head

$$= \left(\frac{p_A}{\rho g} + Z_A\right) - \left(\frac{p_B}{\rho g} + Z_B\right) = 66.67 - 42.22 = 24.45 \text{ m}$$

$\therefore$
$$24.45 = \frac{32 \times 0.15 \times \bar{u} \times 20.0}{900 \times 9.81 \times (.02)^2}$$

or
$$\bar{u} = \frac{24.45 \times 900 \times 9.81 \times .0004}{32 \times 0.15 \times 20.0} = 0.889 \simeq 0.9 \text{ m/s.}$$

The Reynolds number should be calculated. If Reynolds number is less than 2000, the flow will be laminar and the above expression for loss of pressure head for laminar flow can be used.

Now Reynolds number $= \dfrac{\rho V D}{\mu}$

where $\rho = 900$ kg/m^3 and $V = \bar{u}$

$\therefore$ Reynolds number $= 900 \times \dfrac{0.9 \times .02}{0.15} = 108$

As Reynolds number is less than 2000, the flow is laminar.

$\therefore$ Rate of flow = average velocity × area

$$= \bar{u} \times \frac{\pi}{4} D^2 = 0.9 \times \frac{\pi}{4} \times (.02)^2 \text{ m}^3\text{/s} = 2.827 \times 10^{-4} \text{ m}^3\text{/s}$$

= **0.2827 litres/s. Ans.** ($\because$ 10^{-3} m^3 = 1 litre)

Problem 9.5 *A fluid of viscosity 0.7 Ns/m^2 and specific gravity 1.3 is flowing through a circular pipe of diameter 100 mm. The maximum shear stress at the pipe wall is given as 196.2 N/m^2, find (i) the pressure gradient, (ii) the average velocity, and (iii) Reynolds number of the flow.*

Solution. Given : $\mu = 0.7 \dfrac{\text{Ns}}{\text{m}^2}$

Sp. gr. = 1.3

$\therefore$ Density = 1.3 × 1000 = 1300 kg/m^3

Dia. of pipe, D = 100 mm = 0.1 m

Shear stress, τ_0 = 196.2 N/m^2

Find (*i*) Pressure gradient, $\dfrac{dp}{dx}$

(*ii*) Average velocity, $\bar{u}$

(*iii*) Reynolds number, R_e

(*i*) **Pressure gradient,** $\dfrac{dp}{dx}$

The maximum shear stress (τ_0) is given by

$$\tau_0 = -\frac{\partial p}{\partial x}\frac{R}{2} \text{ or } 196.2 = -\frac{\partial p}{\partial x} \times \frac{D}{4} = -\frac{\partial p}{\partial x} \times \frac{0.1}{4}$$

$$\therefore \qquad \frac{\partial p}{\partial x} = -\frac{196.2 \times 4}{0.1} = -7848 \text{ N/m}^2 \text{ per m}$$

$\therefore$ Pressure Gradient = **– 7848 N/m² per m. Ans.**

(*ii*) **Average velocity,** $\bar{u}$

$$\bar{u} = \frac{1}{2} U_{max} = \frac{1}{2}\left[-\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2\right] \qquad \left\{\because \; U_{max} = -\frac{1}{8\mu}\frac{\partial p}{\partial x} R^2\right\}$$

$$= \frac{1}{8\mu} \times \left(-\frac{\partial p}{\partial x}\right) R^2$$

$$= \frac{1}{8 \times 0.7} \times (7848) \times (.05)^2 \qquad \left\{\because \; R = \frac{D}{2} = \frac{1}{2} = .05\right\}$$

$$= 3.50 \text{ m/s}$$

(*iii*) **Reynolds number,** R_e

$$R_e = \frac{\bar{u} \times D}{\nu} = \frac{\bar{u} \times D}{\mu / \rho} = \frac{\rho \times \bar{u} \times D}{\mu}$$

$$= 1300 \times \frac{3.50 \times 0.1}{0.7} = \mathbf{650.00.} \textbf{ Ans.}$$

Problem 9.6 *What power is required per kilometre of a line to overcome the viscous resistance to the flow of glycerine through a horizontal pipe of diameter 100 mm at the rate of 10 litres/s ? Take* $\mu = 8$ *poise and kinematic viscosity* (ν) = *6.0 stokes.*

Solution. Given :

Length of pipe, $L = 1 \text{ km} = 1000 \text{ m}$

Dia. of pipe, $D = 100 \text{ mm} = 0.1 \text{ m}$

Discharge, $Q = 10 \text{ litres/s} = \dfrac{10}{1000} \text{ m}^3\text{/s} = .01 \text{ m}^3\text{/s}$

Viscosity, $\mu = 8 \text{ poise} = \dfrac{8}{10}\dfrac{\text{Ns}}{\text{m}^2} = 0.8 \text{ N s/m}^2$

Kinematic Viscosity, $\nu = 6.0$ stokes $\qquad \left(\because \; 1 \text{ poise} = \dfrac{1}{10} \text{Ns/m}^2\right)$

$$= 6.0 \text{ cm}^2\text{/s} = 6.0 \times 10^{-4} \text{ m}^2\text{/s}$$

Loss of pressure head is given by equation (9.6) as $h_f = \dfrac{32\mu\bar{u}L}{\rho g D^2}$

Power required = $W \times h_f$ watts ...(*i*)

where W = weight of oil flowing per sec = $\rho g \times Q$

Substituting the values of W and h_f in equation (*i*),

Power required $= (\rho g \times Q) \times \dfrac{(32\,\mu\bar{u}L)}{\rho g D^2}$ watts $= \dfrac{Q \times 32\,\mu\bar{u}L}{D^2}$ (cancelling ρg)

But $\bar{u} = \dfrac{Q}{\text{Area}} = \dfrac{.01}{\frac{\pi}{4}D^2} = \dfrac{.01 \times 4}{\pi \times (.1)^2} = 1.273 \text{ m/s}$

$\therefore$ Power required $= \dfrac{.01 \times 32 \times 0.8 \times 1.273 \times 1000}{(.1)^2}$

$= 32588.8$ W $=$ **32.588 kW. Ans.**

9.3 FLOW OF VISCOUS FLUID BETWEEN TWO PARALLEL PLATES

In this case also, the shear stress distribution, the velocity distribution across a section ; the ratio of maximum velocity to average velocity and difference of pressure head for a given length of parallel plates, are to be calculated.

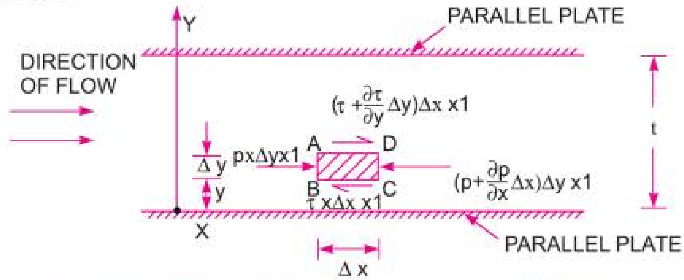


Fig. 9.6 Viscous flow between two parallel plates.

Consider two parallel fixed plates kept at a distance 't' apart as shown in Fig. 9.6. A viscous fluid is flowing between these two plates from left to right. Consider a fluid element of length Δx and thickness Δy at a distance y from the lower fixed plate. If p is the intensity of pressure on the face AB of the fluid element then intensity of pressure on the face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$. Let τ is the shear stress acting on the face BC then the shear stress on the face AD will be $\left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)$. If the width of the element in the direction perpendicular to the paper is unity then the forces acting on the fluid element are :

1. The pressure force, $p \times \Delta y \times 1$ on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y \times 1$ on face CD.
3. The shear force, $\tau \times \Delta x \times 1$ on face BC.
4. The shear force, $\left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)\Delta x \times 1$ on face AD.

For steady and uniform flow, there is no acceleration and hence the resultant force in the direction of flow is zero.

$$\therefore \quad p\Delta y \times 1 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y \times 1 - \tau\Delta x \times 1 + \left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)\Delta x \times 1 = 0$$

or
$$-\frac{\partial p}{\partial x}\Delta x\Delta y + \frac{\partial \tau}{\partial x}\Delta y\Delta x = 0$$

Dividing by $\Delta x\Delta y$, we get $-\dfrac{\partial p}{\partial x} + \dfrac{\partial \tau}{\partial y} = 0$ or $\dfrac{\partial p}{\partial x} = \dfrac{\partial \tau}{\partial y}$ ...(9.7) (*i*) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \dfrac{du}{dy}$ from Newton's law of viscosity for laminar flow is substituted in equation (9.7).

$$\therefore \qquad \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{du}{dy}\right) = \mu \frac{\partial^2 u}{dy^2}$$

$$\therefore \qquad \frac{\partial^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating the above equation w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1 \qquad \left\{\because \frac{\partial p}{\partial x} \text{ is constant}\right\}$$

Integrating again
$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + C_1 y + C_2 \qquad \text{...(9.8)}$$

where C_1 and C_2 are constants of integration. Their values are obtained from the two boundary conditions that is (*i*) at $y = 0$, $u = 0$ (*ii*) at $y = t$, $u = 0$.

The substitution of $y = 0$, $u = 0$ in equation (9.8) gives

$$0 = 0 + C_1 \times 0 + C_2 \text{ or } C_2 = 0$$

The substitution of $y = t$, $u = 0$ in equation (9.8) gives

$$0 = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{t^2}{2} + C_1 \times t + 0$$

$$\therefore \qquad C_1 = -\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{t^2}{2 \times t} = -\frac{1}{2\mu}\frac{\partial p}{\partial x} t$$

Substituting the values of C_1 and C_2 in equation (9.8)

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + y\left(-\frac{1}{2\mu}\frac{\partial p}{\partial x} t\right)$$

or
$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2] \qquad \text{...(9.9)}$$

In the above equation, μ, $\dfrac{\partial p}{\partial x}$ and t are constant. It means u varies with the square of y. Hence equation (9.9) is a equation of a parabola. Hence velocity distribution across a section of the parallel plate is parabolic. This velocity distribution is shown in Fig. 9.7 (*a*).

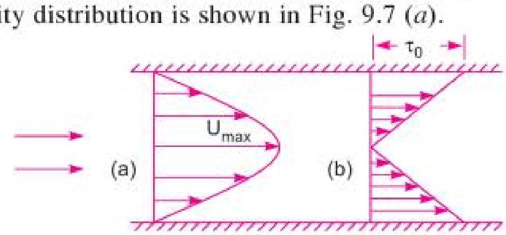


Fig. 9.7 *Velocity distribution and shear stress distribution across a section of parallel plates.*

(*ii*) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when $y = t/2$. Substituting this value in equation (9.9), we get

$$U_{max} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[t \times \frac{t}{2} - \left(\frac{t}{2}\right)^2\right]$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{t^2}{2} - \frac{t^2}{4}\right] = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{t^2}{4} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2 \qquad \text{...(9.10)}$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge (Q) across the section by the area of the section ($t \times 1$). And the discharge Q is obtained by considering the rate of flow of fluid through the strip of thickness dy and integrating it. The rate of flow through strip is

$$dQ = \text{Velocity at a distance } y \times \text{Area of strip}$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2] \times dy \times 1$$

$$\therefore \quad Q = \int_0^t dQ = \int_0^t -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2]\, dy$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{ty^2}{2} - \frac{y^3}{3}\right]_0^t = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{t^3}{2} - \frac{t^3}{3}\right]$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{t^3}{6} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^3$$

$$\therefore \quad \bar{u} = \frac{Q}{\text{Area}} = -\frac{\frac{1}{12\mu}\frac{\partial p}{\partial x}.t^3}{t \times 1} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2 \qquad \text{...(9.11)}$$

Dividing equation (9.10) by equation (9.11), we get

$$\frac{U_{max}}{\bar{u}} = \frac{-\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2}{-\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2} = \frac{12}{8} = \frac{3}{2} \qquad \text{...(9.12)}$$

(*iii*) **Drop of Pressure head for a given Length.** From equation (9.11), we have

$$\bar{u} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2 \quad \text{or} \quad \frac{\partial p}{\partial x} = -\frac{12\mu\bar{u}}{t^2}$$

Integrating this equation w.r.t. x, we get

$$\int_2^1 dp = \int_2^1 -\frac{12\mu\bar{u}}{t^2}\, dx$$

or

$$p_1 - p_2 = -\frac{12\mu\bar{u}}{t^2}[x_1 - x_2] = \frac{12\mu\bar{u}}{t^2}[x_2 - x_1]$$

or $$p_1 - p_2 = \frac{12\mu \bar{u} L}{t^2} \qquad [\because x_1 - x_2 = L]$$

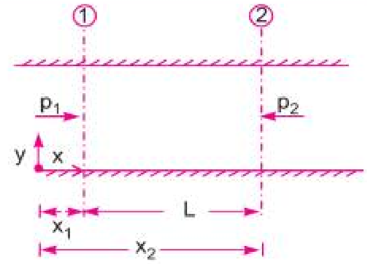


Fig. 9.8

If h_f is the drop of pressure head, then

$$h_f = \frac{p_1 - p_2}{\rho g} = \frac{12\mu \bar{u} L}{\rho g t^2} \qquad \text{...(9.13)}$$

(*iv*) **Shear Stress Distribution.** It is obtained by substituting the value of u from equation (9.9) into

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$\therefore \qquad \tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial}{\partial y}\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(ty - y^2\right)\right] = \mu\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}(t - 2y)\right]$$

$$\tau = -\frac{1}{2}\frac{\partial p}{\partial x}[t - 2y] \qquad \text{...(9.14)}$$

In equation (9.14), $\frac{\partial p}{\partial x}$ and t are constant. Hence τ varies linearly with y. The shear stress distribution is shown in Fig. 9.7 (*b*). Shear stress is maximum, when $y = 0$ or t at the walls of the plates. Shear stress is zero, when $y = t/2$ that is at the centre line between the two plates. Max. shear stress (τ_0) is given by

$$\tau_0 = -\frac{1}{2}\frac{\partial p}{\partial x} t. \qquad \text{...(9.15)}$$

Problem 9.7 *Calculate : (i) the pressure gradient along flow, (ii) the average velocity, and (iii) the discharge for an oil of viscosity 0.02 Ns/m² flowing between two stationary parallel plates 1 m wide maintained 10 mm apart. The velocity midway between the plates is 2 m/s.*

Solution. Given :

Viscosity, $\mu = .02 \text{ N s/m}^2$

Width, $b = 1$ m

Distance between plates, $t = 10$ mm $= .01$ m

Velocity midway between the plates, $U_{max} = 2$ m/s.

(*i*) **Pressure gradient** $\left(\frac{dp}{dx}\right)$

Using equation (9.10), $U_{max} = -\frac{1}{8\mu}\frac{dp}{dx}t^2$ or $2.0 = -\frac{1}{8 \times .02}\left(\frac{dp}{dx}\right)(.01)^2$

$$\therefore \qquad \frac{dp}{dx} = -\frac{2.0 \times 8 \times .02}{.01 \times .01} = \mathbf{-3200\ N/m^2\ per\ m.}\ \text{Ans.}$$

(*ii*) **Average velocity** ($\bar{u}$)

Using equation (9.12), $\frac{U_{max}}{\bar{u}} = \frac{3}{2} \quad \therefore \quad \bar{u} = \frac{2U_{max}}{3} = \frac{2 \times 2}{3} = \mathbf{1.33\ m/s.}$ **Ans.**

(*iii*) **Discharge** (Q) $= \text{Area of flow} \times \bar{u} = b \times t \times \bar{u} = 1 \times .01 \times 1.33 = \mathbf{.0133\ m^3/s.}$ **Ans.**

Problem 9.8 *Determine (a) the pressure gradient, (b) the shear stress at the two horizontal parallel plates and (c) the discharge per metre width for the laminar flow of oil with a maximum velocity of 2 m/s between two horizontal parallel fixed plates which are 100 mm apart. Given $\mu = 2.4525$ N s/m^2.*

Solution. Given :

$$U_{max} = 2 \text{ m/s}, t = 100 \text{ mm} = 0.1 \text{ m}, \mu = 2.4525 \text{ N/m}^2$$

Find (*i*) Pressure gradient, $\frac{dp}{dx}$

(*ii*) Shear stress at the wall, τ_0

(*iii*) Discharge per metre width, Q.

(*i*) **Pressure gradient,** $\frac{dp}{dx}$

Maximum velocity, U_{max}, is given by equation (9.10)

$$U_{max} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2$$

Substituting the values

or $$2.0 = -\frac{1}{8 \times 2.4525} \times \frac{\partial p}{\partial x} \times (.1)^2$$

$\therefore$ $$\frac{\partial p}{\partial x} = -\frac{2.0 \times 8 \times 2.4525}{.1 \times .1} = -\ \mathbf{3924\ N/m^2\ per\ m.\ Ans.}$$

(*ii*) **Shear stress at the wall,** τ_0

τ_0 is given by equation (9.15) as $\tau_0 = -\frac{1}{2}\frac{\partial p}{\partial x} \times t = -\frac{1}{2}(-3924) \times 0.1 = \mathbf{196.2\ N/m^2.\ Ans.}$

(*iii*) **Discharge per metre width,** Q

$$= \text{Mean velocity} \times \text{Area}$$

$$= \frac{2}{3}U_{max} \times (t \times 1) = \frac{2}{3} \times 2.0 \times 0.1 \times 1 = \mathbf{0.133\ m^3/s.\ Ans.}$$

Problem 9.9 *An oil of viscosity 10 poise flows between two parallel fixed plates which are kept at a distance of 50 mm apart. Find the rate of flow of oil between the plates if the drop of pressure in a length of 1.2 m be 0.3 N/cm^3. The width of the plates is 200 mm.*

Solution. Given :

$$\mu = 10 \text{ poise}$$

$$= \frac{10}{10} \text{ N s/m}^2 = 1 \text{ N s/m}^2 \qquad \left(\because 1 \text{ poise} = \frac{1}{10}\frac{\text{Ns}}{\text{m}^2}\right)$$

$$t = 50 \text{ mm} = 0.05 \text{ m}$$

$$p_1 - p_2 = 0.3 \text{ N/m}^2 = 0.3 \times 10^4 \text{ N/m}^2$$

$$L = 1.20 \text{ m}$$

Width, $B = 200$ mm $= 0.20$ m.

Find Q, rate of flow

The difference of pressure is given by equation (9.13)

$$p_1 - p_2 = \frac{12\mu \bar{u} L}{t^2}$$

Substituting the values, we get

$$0.3 \times 10^4 = 12 \times 1.0 \times \frac{\bar{u} \times 1.20}{.05 \times .05}$$

$$\therefore \qquad \bar{u} = \frac{0.3 \times 10^4 \times 1.0 \times .05 \times .05}{12 \times 1.20} = 0.52 \text{ m/s}$$

$\therefore$ Rate of flow $= \bar{u} \times \text{Area} = 0.52 \times (B \times t)$

$= 0.52 \times 0.20 \times .05 \text{ m}^3\text{/s} = .0052 \text{ m}^3\text{/s}$

$= 0.0052 \times 10^3$ litre/s = **5.2 litre/s. Ans.**

Problem 9.10 *Water at 15°C flows between two large parallel plates at a distance of 1.6 mm apart. Determine (i) the maximum velocity (ii) the pressure drop per unit length and (iii) the shear stress at the walls of the plates if the average velocity is 0.2 m/s. The viscosity of water at 15°C is given as 0.01 poise.*

Solution. Given : $t = 1.6 \text{ mm} = 1.6 \times 10^{-3}$ m

$= 0.0016$ m

$\bar{u} = 0.2$ m/sec, $\mu = .01 \text{ poise} = \frac{.01}{10} = 0.001 \text{ N s/m}^2$

(*i*) **Maximum velocity**, U_{max} is given by equation (9.12)

i.e., $U_{max} = \frac{3}{2}\bar{u} = 1.5 \times 0.2 =$ **0.3 m/s. Ans.**

(*ii*) **The pressure drop**, $(p_1 - p_2)$ is given by equation (9.13)

$$p_1 - p_2 = \frac{12\mu \bar{u} L}{t^2}$$

or pressure drop per unit length $= \frac{12\mu \bar{u}}{t^2}$

or

$$\frac{\partial p}{\partial x} = 12 \times \frac{.01}{10} \times \frac{0.2}{(.0016)^2} = 937.44 \text{ N/m}^2 \text{ per m.}$$

(*iii*) **Shear stress at the walls** is given by equation (9.15)

$$\tau_0 = -\frac{1}{2}\frac{\partial p}{\partial x} \times t = \frac{1}{2} \times 937.44 \times .0016 = \mathbf{0.749 \text{ N/m}^2}. \text{ Ans.}$$

Problem 9.11 *There is a horizontal crack 40 mm wide and 2.5 mm deep in a wall of thickness 100 mm. Water leaks through the crack. Find the rate of leakage of water through the crack if the difference of pressure between the two ends of the crack is 0.02943 N/cm². Take the viscosity of water equal to 0.01 poise.*

Solution. Given :

Width of crack, $b = 40 \text{ mm} = 0.04$ m

Depth of crack, $t = 2.5 \text{ mm} = .0025$ m

Length of crack, $L = 100 \text{ mm} = 0.1$ m

$$p_1 - p_2 = 0.02943 \text{ N/cm}^2 = 0.02943 \times 10^4 \text{ N/m}^2 = 294.3 \text{ N/m}^2$$

$$\mu = .01 \text{ poise} = \frac{.01}{10} \frac{\text{Ns}}{\text{m}^2}$$

Find rate of leakage (Q)

$(p_1 - p_2)$ is given by equation (9.13) as

$$p_1 - p_2 = \frac{12\mu \bar{u} L}{t^2} \quad \text{or} \quad 294.3 = 12 \times \frac{.01}{10} \times \frac{\bar{u} \times 0.1}{(.0025 \times .0025)}$$

$$\therefore \qquad \bar{u} = \frac{294.3 \times 10 \times .0025 \times .0025}{12 \times .01 \times 0.1} = 1.5328 \text{ m/s}$$

$$\therefore \qquad \text{Rate of leakage} = \bar{u} \times \text{area of cross-section of crack}$$

$$= 1.538 \times (b \times t)$$

$$= 1.538 \times .04 \times .0025 \text{ m}^3\text{/s} = 1.538 \times 10^{-4} \text{ m}^3\text{/s}$$

$$= 1.538 \times 10^{-4} \times 10^3 \text{ litre/s} = \mathbf{0.1538 \text{ litre/s.}}$$ **Ans.**

Problem 9.12 *The radial clearance between a hydraulic plunger and the cylinder walls is 0.1 mm; the length of the plunger is 300 mm and diameter 100 mm. Find the velocity of leakage and rate of leakage past the plunger at an instant when the difference of the pressure between the two ends of the plunger is 9 m of water. Take μ = 0.0127 poise.*

Solution. Given :

The flow through the clearance area will be the same as the flow between two parallel surfaces,

$$t = 0.1 \text{ mm} = 0.0001 \text{ m}$$

$$L = 300 \text{ mm} = 0.3 \text{ m}$$

Diameter, $D = 100 \text{ mm} = 0.1 \text{ m}$

Difference of pressure $= \dfrac{p_1 - p_2}{\rho g} = 9$ m of water

$$\therefore \qquad p_1 - p_2 = 9 \times 1000 \times 9.81 \text{ N/m}^2 = 88290 \text{ N/m}^2$$

Viscosity, $\mu = .0127 \text{ poise} = \dfrac{.0127}{10} \dfrac{\text{Ns}}{\text{m}^2}$

Find (*i*) Velocity of leakage, *i.e.*, mean velocity $\bar{u}$

(*ii*) Rate of leakage, Q

(*i*) **Velocity of leakage** $(\bar{u})$. The average velocity $(\bar{u})$ is given by equation (9.11)

$$\bar{u} = -\frac{1}{12\mu} \frac{\partial p}{\partial x} t^2$$

$$= \frac{1}{12 \times \frac{.0127}{10}} \times \frac{p_1 - p_2}{L} \times (.0001) \times (.0001)$$

$$= \frac{1}{12 \times .0127} \times \frac{88290}{0.3} \times (.0001) \times (.0001)$$

$$= .193 \text{ m/s} = \mathbf{19.3 \text{ cm/s.}}$$ **Ans.**

(*ii*) **Rate of leakage,** Q

$$Q = \bar{u} \times \text{area of flow}$$
$$= 0.193 \times \pi D \times t \text{ m}^3\text{/s} = 0.193 \times \pi \times .1 \times .0001 \text{ m}^3\text{/s}$$
$$= 6.06 \times 10^{-6} \text{ m}^3\text{/s} = 6.06 \times 10^{-6} \times 10^3 \text{ litre/s}$$
$$= \mathbf{6.06 \times 10^{-3} \text{ litre/s.}} \text{ Ans.}$$

▶ 9.4 KINETIC ENERGY CORRECTION AND MOMENTUM CORRECTION FACTORS

Kinetic energy correction factor is defined as the ratio of the kinetic energy of the flow per second based on actual velocity across a section to the kinetic energy of the flow per second based on average velocity across the same section. It is denoted by α. Hence mathematically,

$$\alpha = \frac{\text{K.E./sec based on actual velocity}}{\text{K.E./sec based on average velocity}} \quad ...(9.16)$$

Momentum Correction Factor. It is defined as the ratio of momentum of the flow per second based on actual velocity to the momentum of the flow per second based on average velocity across a section. It is denoted by β. Hence mathematically,

$$\beta = \frac{\text{Momentum per second based on actual velocity}}{\text{Momentum per second based on average velocity}}. \quad ...(9.17)$$

Problem 9.13 *Show that the momentum correction factor and energy correction factor for laminar flow through a circular pipe are 4/3 and 2.0 respectively.*

Solution. (*i*) **Momentum Correction Factor or** β

The velocity distribution through a circular pipe for laminar flow at any radius r is given by equation (9.3)

or $$u = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)(R^2 - r^2) \quad ...(1)$$

Consider an elementary area dA in the form of a ring at a radius r and of width dr, then

$$dA = 2\pi r\, dr$$

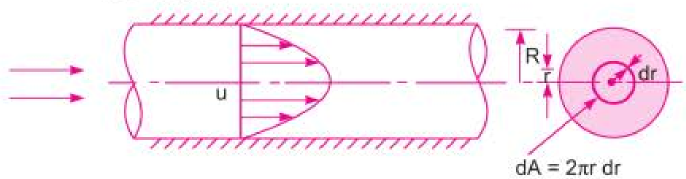


Fig. 9.9

Rate of fluid flowing through the ring

$$= dQ = \text{velocity} \times \text{area of ring element}$$
$$= u \times 2\pi r\, dr$$

Momentum of the fluid through ring per second

$$= \text{mass} \times \text{velocity}$$
$$= \rho \times dQ \times u = \rho \times 2\pi r\, dr \times u \times u = 2\pi\rho\, u^2 r\, dr$$

$\therefore$ Total actual momentum of the fluid per second across the section

$$= \int_0^R 2\pi\rho\, u^2 r\, dr$$

Substituting the value of u from (1)

$$= 2\pi\rho \int_0^R \left[\frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right)\left(R^2 - r^2\right)\right]^2 r dr$$

$$= 2\pi\rho\left[\frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right)\right]^2 \int_0^R [R^2 - r^2]^2 \, r dr$$

$$= 2\pi\rho \frac{1}{\left(16\mu^2\right)}\left(\frac{\partial p}{\partial x}\right)^2 \int_0^R (R^4 + r^4 - 2R^2r^2) \, r dr$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \int_0^R (R^4 r + r^5 - 2R^2r^3) \, dr$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2\left[\frac{R^4r^2}{2} + \frac{r^6}{6} - \frac{2R^2r^4}{4}\right]_0^R = \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2\left[\frac{R^6}{2} + \frac{R^6}{6} - \frac{2R^6}{4}\right]$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \frac{6R^6 + 2R^6 - 6R^6}{12}$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \times \frac{R^6}{6} = \frac{\pi\rho}{48\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6 \qquad \text{...(2)}$$

Momentum of the fluid per second based on average velocity

$$= \frac{\text{mass of fluid}}{\text{sec}} \times \text{average velocity}$$

$$= \rho A\bar{u} \times \bar{u} = \rho A\bar{u}^2$$

where A= Area of cross-section = πR^2, $\bar{u}$ = average velocity = $\frac{U_{\max}}{2}$

$$= \frac{1}{2} \times \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) R^2 \qquad \left\{\because \quad U_{\max} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) R^2\right\}$$

$$= \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2$$

$\therefore$ Momentum/sec based on average velocity

$$= \rho \times \pi R^2 \times \left[\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2\right]^2 = \rho \times \pi R^2 \times \frac{1}{64\mu^2}\left(-\frac{\partial p}{\partial x}\right)^2 R^4$$

$$= \frac{\rho\pi\left(-\frac{\partial p}{\partial x}\right)^2 R^6}{64\mu^2} \qquad \text{...(3)}$$

$$\therefore \qquad \beta = \frac{\text{Momentum / sec based on actual velocity}}{\text{Momentum / sec based on average velocity}}$$

$$= \frac{\frac{\pi\rho}{48\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6}{\frac{\pi\rho}{64\mu^2}\left(-\frac{\partial p}{\partial x}\right)^2 R^6} = \frac{64}{48} = \frac{4}{3}. \textbf{ Ans.}$$

(*ii*) **Energy Correction Factor**, α. Kinetic energy of the fluid flowing through the elementary ring of radius 'r' and of width 'dr' per sec

$$= \frac{1}{2} \times \text{mass} \times u^2 = \frac{1}{2} \times \rho dQ \times u^2$$

$$= \frac{1}{2} \times \rho \times (u \times 2\pi r\, dr) \times u^2 = \frac{1}{2}\rho \times 2\pi r u^3\, dr = \pi\rho r u^3\, dr$$

$\therefore$ Total actual kinetic energy of flow per second

$$= \int_0^R \pi\rho r u^3\, dr = \int_0^R \pi\rho r \left[\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(R^2 - r^2\right)\right]^3 dr$$

$$= \pi\rho \times \left[\frac{1}{4\mu}\left(-\frac{\partial p}{\partial \mu}\right)\right]^3 \int_0^R [R^2 - r^2]^3\, r dr$$

$$= \pi\rho \times \frac{1}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \int_0^R (R^6 - r^6 - 3R^4 r^2 + 3R^6 r^4) r\, dr$$

$$= \frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \int_0^R (R^6 r - r^7 - 3R^4 r^3 + 3R^2 r^5)\, dr$$

$$= \frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \left[\frac{R^6 r^2}{2} - \frac{r^8}{8} - \frac{3R^4 r^4}{4} + \frac{3R^2 r^6}{6}\right]_0^R$$

$$= \frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \left[\frac{R^8}{2} - \frac{R^8}{8} - \frac{3R^8}{4} + \frac{3R^6}{6}\right]$$

$$= \frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 R^8 \left[\frac{12 - 3 - 18 + 12}{24}\right]$$

$$= \frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \frac{R^8}{8} \qquad \text{...(4)}$$

Kinetic energy of the flow based on average velocity

$$= \frac{1}{2} \times \text{mass} \times \bar{u}^2 = \frac{1}{2} \times \rho A\bar{u} \times \bar{u}^2 = \frac{1}{2} \times \rho A\bar{u}^3$$

Substituting the value of $A = (\pi R^2)$

and $\bar{u} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2$

∴ Kinetic energy of the flow/sec

$$= \frac{1}{2} \times \rho \times \pi R^2 \times \left[\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2\right]^3$$

$$= \frac{1}{2} \times \rho \times \pi R^2 \times \frac{1}{64 \times 8\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \times R^6$$

$$= \frac{\rho\pi}{128 \times 8\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \times R^8 \quad \text{...(5)}$$

$$\therefore \quad \alpha = \frac{\text{K.E./sec based on actual velocity}}{\text{K.E./sec based on average velocity}} = \frac{\text{Equation (4)}}{\text{Equation (5)}}$$

$$= \frac{\frac{\pi\rho}{64\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \times \frac{R^8}{8}}{\frac{\rho}{128 \times 8\mu^3}\left(-\frac{\partial p}{\partial x}\right)^3 \times R^8} = \frac{128 \times 8}{64 \times 8} = \textbf{2.0. Ans.}$$

▶ 9.5 POWER ABSORBED IN VISCOUS FLOW

For the lubrication of the machine parts, an oil is used. Flow of oil in bearings is an example of viscous flow. If a highly viscous oil is used for lubrication of bearings, it will offer great resistance and thus a greater power loss will take place. But if a light oil is used, a required film between the rotating part and stationary metal surface will not be possible. Hence, the wear of the two surface will take place. Hence an oil of correct viscosity should be used for lubrication. The power required to overcome the viscous resistance in the following cases will be determined :

1. Viscous resistance of Journal Bearings,
2. Viscous resistance of Foot-step Bearings,
3. Viscous resistance of Collar Bearings.

9.5.1 Viscous Resistance of Journal Bearings. Consider a shaft of diameter D rotating in a journal bearing. The clearance between the shaft and journal bearing is filled with a viscous oil. The oil film in contact with the shaft rotates as the same speed as that of shaft while the oil film in contact with journal bearing is stationary. Thus the viscous resistance will be offered by the oil to the rotating shaft.

Let N = speed of shaft in r.p.m.

t = thickness of oil film

L = length of oil film

∴ Angular speed of the shaft, $\omega = \frac{2\pi N}{60}$

∴ Tangential speed of the shaft $= \omega \times R$ or $V = \frac{2\pi N}{60} \times \frac{D}{2} = \frac{\pi D N}{60}$

The shear stress in the oil is given by, $\tau = \mu \frac{du}{dy}$

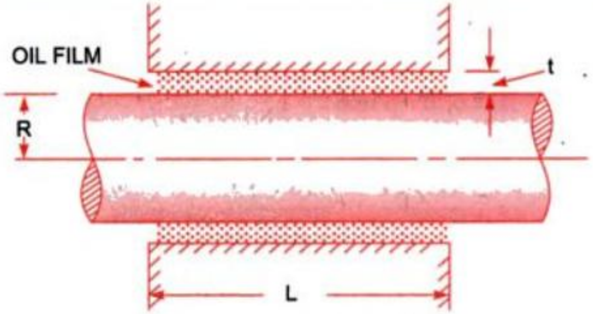


Fig. 9.10 *Journal bearing.*

As the thickness of oil film is very small, the velocity distribution in the oil film can be assumed as linear.

Hence $$\frac{du}{dy} = \frac{V-0}{t} = \frac{V}{t} = \frac{\pi DN}{60 \times t}$$

$\therefore$ $$\tau = \mu \frac{\pi DN}{60 \times t}$$

$\therefore$ Shear force or viscous resistance = $\tau \times$ Area of surface of shaft

$$= \frac{\mu \pi DN}{60t} \times \pi DL = \frac{\mu \pi^2 D^2 NL}{60t}$$

$\therefore$ Torque required to overcome the viscous resistance,

$$T = \text{Viscous resistance} \times \frac{D}{2}$$

$$= \frac{\mu \pi^2 D^2 NL}{60t} \times \frac{D}{2} = \frac{\mu \pi^2 D^3 NL}{120t}$$

$\therefore$ Power absorbed in overcoming the viscous resistance

$$*P = \frac{2\pi NT}{60} = \frac{2\pi N}{60} \times \frac{\mu \pi^2 D^3 NL}{120t}$$

$$= \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t} \text{ watts. Ans.} \quad \text{...(9.18)}$$

Problem 9.14 *A shaft having a diameter of 50 mm rotates centrally in a journal bearing having a diameter of 50.15 mm and length 100 mm. The angular space between the shaft and the bearing is filled with oil having viscosity of 0.9 poise. Determine the power absorbed in the bearing when the speed of rotation is 60 r.p.m.*

Solution. Given :

Dia. of shaft, $D = 50$ mm or .05 m

Dia. of bearing, $D_1 = 50.15$ mm or 0.05015 m

Length, $L = 100$ mm or 0.1 m

*Power, $P = T \times \omega = T \times \frac{2\pi N}{60} = \frac{2\pi NT}{60}$ watts $= \frac{2\pi NT}{60{,}000}$ kW.

$$\mu \text{ of oil} = 0.9 \text{ poise} = \frac{0.9}{10} \frac{\text{Ns}}{\text{m}^2}$$

$$N = 600 \text{ r.p.m.}$$

$$\text{Power} = ?$$

∴ Thickness of oil film, $t = \dfrac{D_1 - D}{2} = \dfrac{50.15 - 50}{2}$

$$= \frac{0.15}{2} = 0.075 \text{ mm} = 0.075 \times 10^{-3} \text{ m}$$

Tangential speed of shaft, $V = \dfrac{\pi DN}{60} = \dfrac{\pi \times 0.05 \times 600}{60} = 0.5 \times \pi \text{ m/s}$

Shear stress $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{V}{t} = \dfrac{0.9}{10} \times \dfrac{0.5 \times \pi}{0.075 \times 10^{-3}} = 1883.52 \text{ N/m}^2$

∴ Shear force (F) $= \tau \times \text{Area} = 1883.52 \times \pi D \times L$

$$= 1883.52 \times \pi \times .05 \times 0.1 = 29.586 \text{ N}$$

Resistance torque $T = F \times \dfrac{D}{2} = 29.586 \times \dfrac{.05}{2} = 0.7387 \text{ Nm}$

Power $= \dfrac{2\pi NT}{60} = \dfrac{2\pi \times 600 \times 0.7387}{60} = \mathbf{46.41 \text{ W}}$. **Ans.**

Problem 9.15 *A shaft of 100 mm, diameter rotates at 60 r.p.m. in a 200 mm long bearing. Taking that the two surfaces are uniformly separated by a distance of 0.5 mm and taking linear velocity distribution in the lubricating oil having dynamic viscosity of 4 centipoises, find the power absorbed in the bearing.*

Solution. Given :

Dia. of shaft, $D = 100 \text{ mm} = 0.1 \text{ m}$

Length of bearing, $L = 200 \text{ mm} = 0.2 \text{ m}$

$$t = 0.5 \text{ mm} = .5 \times 10^{-3} \text{ m}$$

$$\mu = 4 \text{ centipoise} = .04 \text{ poise} = \frac{0.04}{10} \frac{\text{Ns}}{\text{m}^2}$$

$$N = 60 \text{ r.p.m.}$$

Find power absorbed

Using equation (9.18), $P = \dfrac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t}$

$$= \frac{.04}{10} \times \frac{\pi^3 \times (.1)^3 \times (60)^2 \times 0.2}{60 \times 60 \times 0.5 \times 10^{-3}} = \mathbf{4.961 \times 10^{-2} \text{ W}}. \textbf{ Ans.}$$

Problem 9.16 *A shaft of diameter 0.35 m rotates at 200 r.p.m. inside a sleeve 100 mm long. The dynamic viscosity of lubricating oil in the 2 mm gap between sleeve and shaft is 8 poises. Calculate the power lost in the bearing.*

Solution. Given :

Dia. of shaft, $D = 0.35 \text{ m}$

Speed of shaft, $N = 200$ r.p.m.

Length of sleeve, $L = 100$ mm $= 0.1$ m

Distance between sleeve and shaft, $t = 2$ mm $= 2 \times 10^{-3}$ m

Viscosity, $\mu = 8 \text{ poise} = \frac{8}{10} \frac{\text{Ns}}{\text{m}^2}$

The power lost in the bearing is given by equation (9.18) as

$$P = \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t} \text{ watts}$$

$$= \frac{8}{10} \times \frac{\pi^3 \times (.35)^3 \times (200)^2 \times 0.1}{60 \times 60 \times 2 \times 10^{-3}} = \mathbf{590.8\ W = 0.59\ kW.}\ \textbf{Ans.}$$

Problem 9.17 *A sleeve, in which a shaft of diameter 75 mm, is running at 1200 r.p.m., is having a radial clearance of 0.1 mm. Calculate the torque resistance if the length of sleeve is 100 mm and the space is filled with oil of dynamic viscosity 0.96 poise.*

Solution. Given :

Dia. of shaft, $D = 75$ mm $= 0.075$ m

$N = 1200$ r.p.m.

$t = 0.1$ mm $= 0.1 \times 10^{-3}$ m

Length of sleeve, $L = 100$ mm $= 0.1$ m

$\mu = 0.96 \text{ poise} = \frac{0.96}{10} \frac{\text{Ns}}{\text{m}^2}$

Tangential velocity of shaft, $V = \frac{\pi DN}{60} = \frac{\pi \times .075 \times 1200}{60} = 4.712$ m/s

Shear stress, $\tau = \mu \frac{V}{t} = \frac{.96}{10} \times \frac{4.712}{.1 \times 10^{-3}} = 4523.5$ N/m^2

Shear force, $F = \tau \times \pi DL$

$= 4523.5 \times \pi \times .075 \times .1 = 106.575$ N

$\therefore$ Torque resistance $= F \times \frac{D}{2}$

$$= 106.575 \times \frac{.075}{2} = \mathbf{3.996\ Nm.}\ \textbf{Ans.}$$

Problem 9.18 *A shaft of 100 mm diameter runs in a bearing of length 200 mm with a radial clearance of 0.025 mm at 30 r.p.m. Find the velocity of the oil, if the power required to overcome the viscous resistance is 183.94 watts.*

Solution. Given :

$D = 100$ mm $= 0.1$ m

$L = 200$ mm $= 0.2$ m

$t = .025$ mm $= 0.025 \times 10^{-3}$ m

$N = 30$ r.p.m. ; H.P. $= 0.25$

Find viscosity of oil, μ.

The h.p. is given by equation (9.18) as

$$P = \frac{\mu\pi^3 D^3 N^2 L}{60 \times 60 \times t} \quad \text{or} \quad 183.94 = \frac{\mu\pi^3 \times (.1)^3 \times (30)^2 \times 0.2}{60 \times 60 \times 0.025 \times 10^{-3}}$$

$$\therefore \quad \mu = \frac{183.94 \times 60 \times 60 \times .025 \times 10^{-3}}{\pi^3 \times .001 \times 900 \times 0.2} \frac{\text{Ns}}{\text{m}^2}$$

$$= 2.96 \frac{\text{Ns}}{\text{m}^2} = 2.96 \times 10 = \mathbf{29.6 \text{ poise.}} \text{ Ans.}$$

9.5.2 Viscous Resistance of Foot-Step Bearing.

Fig. 9.11 shows the foot-step bearing, in which a vertical shaft is rotating. An oil film between the bottom surface of the shaft and bearing is provided, to reduce the wear and tear. The viscous resistance is offered by the oil to the shaft. In this case the radius of the surface of the shaft in contact with oil is not constant as in the case of the journal bearing. Hence, viscous resistance in foot-step bearing is calculated by considering an elementary circular ring of radius r and thickness dr as shown in Fig. 9.11.

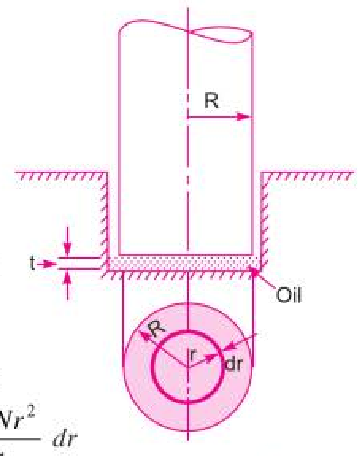


Fig. 9.11 *Foot-step bearing.*

Let N = speed of the shaft

t = thickness of oil film

R = radius of the shaft

Area of the elementary ring $= 2\pi r dr$

Now shear stress is given by $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{V}{t}$

where V is the tangential velocity of shaft at radius r and is equal to

$$\omega \times r = \frac{2\pi N}{60} \times r$$

$\therefore$ Shear force on the ring $= dF = \tau \times$ area of elementary ring

$$= \mu \times \frac{2\pi N}{60} \times \frac{r}{t} \times 2\pi r\, dr = \frac{\mu}{15} \frac{\pi^2 N r^2}{t} dr$$

$\therefore$ Torque required to overcome the viscous resistance,

$$dT = dF \times r$$

$$= \frac{\mu}{15t} \pi^2 N r^2\, dr \times r = \frac{\mu}{15t} \pi^2 N r^3\, dr \qquad \text{...(9.19)}$$

$\therefore$ Total torque required to overcome the viscous resistance,

$$T = \int_0^R dT = \int_0^R \frac{\mu}{15t} \pi^2 N r^3 dr$$

$$= \frac{\mu}{15t} \pi^2 N \int_0^R r^3 dr = \frac{\mu}{15t} \pi^2 N \left[\frac{r^4}{4}\right]_0^R = \frac{\mu}{15t} \pi^2 N \frac{R^4}{4}$$

$$= \frac{\mu}{60t} \pi^2 N R^4 \qquad \text{...(9.19}A\text{)}$$

$\therefore$ Power absorbed, $$P = \frac{2\pi N T}{60} \text{ watts}$$

$$= \frac{2\pi N}{60} \times \frac{\mu}{60t} \pi^2 N R^4 = \frac{\mu\pi^3 N^2 R^4}{60 \times 30t} \qquad \text{...(9.20)}$$

Problem 9.19 *Find the torque required to rotate a vertical shaft of diameter 100 mm at 750 r.p.m. The lower end of the shaft rests in a foot-step bearing. The end of the shaft and surface of the bearing are both flat and are separated by an oil film of thickness 0.5 mm. The viscosity of the oil is given 1.5 poise.*

Solution. Given :

Dia. of shaft, $D = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore$ $R = \dfrac{D}{2} = \dfrac{0.1}{2} = 0.05 \text{ m}$

$N = 750$ r.p.m.

Thickness of oil film, $t = 0.5 \text{ mm} = 0.0005 \text{ m}$

$\mu = 1.5 \text{ poise} = \dfrac{1.5}{10} \dfrac{\text{Ns}}{\text{m}^2}$

The torque required is given by equation (9.19) as

$$T = \frac{\mu}{60t} \pi^2 N R^4 \text{ Nm}$$

$$= \frac{1.5}{10} \times \frac{\pi^2 \times 750 \times (.05)^4}{60 \times .0005} = \textbf{0.2305 Nm. Ans.}$$

Problem 9.20 *Find the power required to rotate a circular disc of diameter 200 mm at 1000 r.p.m. The circular disc has a clearance of 0.4 mm from the bottom flat plate and the clearance contains oil of viscosity 1.05 poise.*

Solution. Given :

Dia. of disc, $D = 200 \text{ mm} = 0.2 \text{ m}$

$\therefore$ $R = \dfrac{D}{2} = \dfrac{0.2}{2} = 0.1 \text{ m}$

$N = 1000$ r.p.m.

Thickness of oil film, $t = 0.4 \text{ mm} = 0.0004 \text{ m}$

$\mu = 1.05 \text{ poise} = \dfrac{1.05}{10} \text{ N s/m}^2$

The power required to rotate the disc is given by equation (9.20) as

$$P = \frac{\mu \pi^3 N^2 R^4}{60 \times 30 \times t} \text{ watts}$$

$$= \frac{1.05}{10} \times \frac{\pi^3 \times 1000^2 \times (0.1)^4}{60 \times 30 \times .0004} = \textbf{452.1 W. Ans.}$$

9.5.3 Viscous Resistance of Collar Bearing.

Fig. 9.12 shows the collar bearing, where the face of the collar is separated from bearing surface by an oil film of uniform thickness.

Let N = Speed of the shaft in r.p.m.

R_1 = Internal radius of the collar

R_2 = External radius of the collar

t = Thickness of oil film.

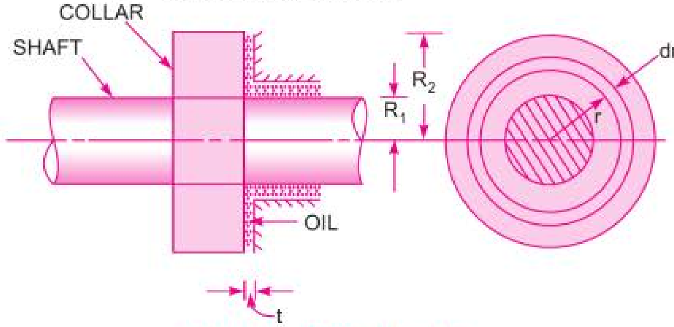


Fig. 9.12 *Collar bearing.*

Consider an elementary circular ring of radius 'r' and width dr of the bearing surface. Then the torque (dT) required to overcome the viscous resistance on the elementary circular ring is the same as given by equation (9.19A) or

$$dT = \frac{\mu}{15t} \pi^2 N r^3 \, dr$$

$\therefore$ Total torque, required to overcome the viscous resistance, on the whole collar is

$$T = \int_{R_1}^{R_2} dT = \int_{R_1}^{R_2} \frac{\mu}{15t} \pi^2 N r^3 \, dr = \frac{\mu}{15t} \pi^2 N \left[\frac{r^4}{4}\right]_{R_1}^{R_2}$$

$$= \frac{\mu}{15t \times 4} \pi^2 N [R_2^4 - R_1^4] = \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4] \quad \text{...(9.21)}$$

$\therefore$ Power absorbed in overcoming viscous resistance

$$P = \frac{2\pi NT}{60} = \frac{2\pi N}{60} \times \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4]$$

$$= \frac{\mu \pi^3 N^2}{60 \times 30t} [R_2^4 - R_1^4] \text{ watts.} \quad \text{...(9.22)}$$

Problem 9.21 *A collar bearing having external and internal diameters 150 mm and 100 mm respectively is used to take the thrust of a shaft. An oil film of thickness 0.25 mm is maintained between the collar surface and the bearing. Find the power lost in overcoming the viscous resistance when the shaft rotates at 300 r.p.m. Take μ = 0.91 poise.*

Solution. Given :

External Dia. of collar, $D_2 = 150 \text{ mm} = 0.15 \text{ m}$

$\therefore$ $R_2 = \frac{D_2}{2} = \frac{.15}{2} = 0.075 \text{ m}$

Internal Dia. of collar, $D_1 = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore$ $R_1 = \frac{D_1}{2} = \frac{0.1}{2} = 0.05 \text{ m}$

Thickness of oil film, $t = 0.25 \text{ mm} = 0.00025 \text{ m}$

$N = 300 \text{ r.p.m.}$

$\mu = 0.91 \text{ poise} = \frac{0.91}{10} \frac{\text{Ns}}{\text{m}^2}$

The power required is given by equation (9.22) or

$$P = \frac{\mu\pi^3 N^2}{60 \times 30 \times t} [R_2^4 - R_1^4]$$

$$= \frac{0.91}{10} \times \frac{\pi^3 \times 300^2 \times \left[.075^4 - .05^4\right]}{60 \times 30 \times .00025}$$

$$= 564314 \, [.00003164 - .00000625]$$

$$= 564314 \times .00002539 = \mathbf{14.327 \ W.} \ \textbf{Ans.}$$

Problem 9.22 *The external and internal diameters of a collar bearing are 200 mm and 150 mm respectively. Between the collar surface and the bearing, an oil film of thickness 0.25 mm and of viscosity 0.9 poise, is maintained. Find the torque and the power lost in overcoming the viscous resistance of the oil when the shaft is running at 250 r.p.m.*

Solution. Given :

$$D_2 = 200 \text{ mm} = 0.2 \text{ m}$$

$$\therefore \quad R_2 = \frac{D_2}{2} = \frac{0.2}{2} = 0.1 \text{ m}$$

$$D_1 = 150 \text{ mm} = 0.15 \text{ m}$$

$$\therefore \quad R_1 = \frac{D_1}{2} = \frac{0.15}{2} = .075 \text{ m}$$

$$t = 0.25 \text{ mm} = .00025 \text{ m}$$

$$\mu = 0.9 \text{ poise} = \frac{0.9}{10} \frac{\text{Ns}}{\text{m}^2}$$

Torque required is given by equation (9.21)

$$\therefore \quad T = \frac{\mu}{60t} \pi^2 N \, [R_2^4 - R_1^4] = \frac{0.9}{10} \times \frac{\pi^2 \times 250 \left[0.1^4 - .075^4\right]}{60 \times 0.00025} \text{ Nm}$$

$$= 14804.4 \, [\,.0001 - .00003164] = \mathbf{1.0114 \ Nm.} \ \textbf{Ans.}$$

$\therefore$ Power lost in viscous resistance

$$= \frac{2\pi NT}{60} = \frac{2\pi \times 250 \times 1.0114}{60} = \mathbf{26.48 \ W.} \ \textbf{Ans.}$$

▶ 9.6 LOSS OF HEAD DUE TO FRICTION IN VISCOUS FLOW

The loss of pressure head, h_f in a pipe of diameter D, in which a viscous fluid of viscosity μ is flowing with a velocity $\bar{u}$ is given by Hagen Poiseuille formula *i.e.*, by equation (9.6) as

$$h_f = \frac{32\mu\bar{u}L}{\rho g D^2} \qquad \text{...}(i)$$

where L = length of pipe

The loss of head due to friction* is given by

$$h_f = \frac{4 \, . \, f \, . \, L \, . \, V^2}{D \times 2g} = \frac{4 \, . \, f \, . \, L \, . \, \bar{u}^2}{D \times 2g} \qquad \text{...}(ii)$$

$\{\because$ velocity in pipe is always average velocity $\quad \therefore \quad V = \bar{u}\}$

*For derivation, please refer to Art. 10.3.1.

where f = co-efficient of friction between the pipe and fluid.

Equating (*i*) and (*ii*), we get $\dfrac{32\mu\bar{u}L}{\rho g D^2} = \dfrac{4 \cdot f \cdot L \cdot \bar{u}^2}{D \times 2g}$

$$f = \frac{32\mu\bar{u}L \times D \times 2g}{4 \cdot L \cdot \bar{u}^2 \cdot \rho g \cdot D^2} = \frac{16\mu}{\bar{u} \cdot \rho \cdot D} \qquad \{\because \bar{u} = V\}$$

$$= 16 \times \frac{\mu}{\rho V D} = 16 \times \frac{1}{R_e}$$

where $\dfrac{\mu}{\rho V D} = \dfrac{1}{R_e}$ and R_e = Reynolds number = $\dfrac{\rho V D}{\mu}$

$$\therefore \qquad f = \frac{16}{R_e}. \qquad \text{...(9.23)}$$

Problem 9.23 *Water is flowing through a 200 mm diameter pipe with coefficient of friction $f = 0.04$. The shear stress at a point 40 mm from the pipe axis is 0.00981 N/cm^2. Calculate the shear stress at the pipe wall.*

Solution. Given :

Dia. of pipe, D = 200 mm = 0.20 m

Co-efficient of friction, f = 0.04

Shear stress at r = 40 mm, τ = 0.00981 N/cm^2

Let the shear stress at pipe wall = τ_0.

First find whether the flow is viscous or not. The flow will be viscous if Reynolds number R_e is less than 2000.

Using equation (9.23), we get $f = \dfrac{16}{R_e}$ or $.04 = \dfrac{16}{R_e}$

$$\therefore \qquad R_e = \frac{16}{.04} = 400$$

This means flow is viscous. The shear stress in case of viscous flow through a pipe is given by the equation (9.1) as

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2}$$

But $\dfrac{\partial p}{\partial x}$ is constant across a section. Across a section, there is no variation of x and there is no variation of p.

$$\therefore \qquad \tau \propto r$$

At the pipe wall, radius = 100 mm and shear stress is τ_0

$$\therefore \qquad \frac{\tau}{r} = \frac{\tau_0}{100} \quad \text{or} \quad \frac{0.00981}{40} = \frac{\tau_0}{100}$$

$$\therefore \qquad \tau_0 = \frac{100 \times 0.00981}{40} = \mathbf{0.0245\ N/cm^2}. \ \textbf{Ans.}$$

Problem 9.24 *A pipe of diameter 20 cm and length 10^4 m is laid at a slope of 1 in 200. An oil of sp. gr. 0.9 and viscosity 1.5 poise is pumped up at the rate of 20 litres per second. Find the head lost due to friction. Also calculate the power required to pump the oil.*

Solution. Given :

Dia. of pipe, $D = 20 \text{ cm} = 2.50 \text{ m}$

Length of pipe, $L = 10000 \text{ m}$

Slope of pipe, $i = 1 \text{ in } 200 = \dfrac{1}{200}$

Sp. gr. of oil, $S = 0.9$

$\therefore$ Density of oil, $\rho = 0.9 \times 1000 = 900 \text{ kg/m}^3$

Viscosity of oil, $\mu = 1.5 \text{ poise} = \dfrac{1.5}{10} \dfrac{\text{Ns}}{\text{m}^2}$

Discharge, $Q = 20 \text{ litre/s} = 0.02 \text{ m}^3\text{/s}$ $\{\because$ 1000 litres = 1 $\text{m}^3\}$

$\therefore$ Velocity of flow, $\bar{u} = \dfrac{Q}{\text{Area}} = \dfrac{0.020}{\frac{\pi}{4} D^2} = \dfrac{0.020}{\frac{\pi}{4}(.2)^2} = 0.6366 \text{ m/s}$

$\therefore$ R_e = Reynolds number

$$= \frac{\rho V D}{\mu} = \frac{900 \times 0.6366 \times .2}{\frac{1.5}{10}}$$

$$= \frac{900 \times .6366 \times .2 \times 10}{1.5} \qquad \{\because V = \bar{u} = 0.6366\}$$

$$= 763.89$$

As the Reynolds number is less than 2000, the flow is viscous. The co-efficient of friction for viscous flow is given by equation (9.23) as

$$f = \frac{16}{R_e} = \frac{16}{763.89} = 0.02094$$

$\therefore$ Head lost due to friction, $h_f = \dfrac{4 \,.\, f \,.\, L \,.\, \bar{u}^2}{D \times 2g}$

$$= \frac{4 \times .02094 \times 10000 \times (.6366)^2}{0.2 \times 2 \times 9.81} \text{ m} = \mathbf{86.50 \text{ m}}. \textbf{ Ans.}$$

Due to slope of pipe 1 in 200, the height through which oil is to be raised by pump

$$= \text{Slope} \times \text{Length of pipe}$$

$$= i \times L = \frac{1}{200} \times 10000 = 50 \text{ m}$$

$\therefore$ Total head against which pump is to work,

$$H = h_f + i \times L = 86.50 + 50 = 136.50 \text{ m}$$

$\therefore$ Power required to pump the oil

$$= \frac{\rho g \cdot Q \cdot H}{1000} = \frac{900 \times 9.81 \times 0.20 \times 136.50}{1000} = \mathbf{24.1\ kW.}\ \textbf{Ans.}$$

▶ 9.7 MOVEMENT OF PISTON IN DASH-POT

Consider a piston moving in a vertical dash-pot containing oil as shown in Fig. 9.13.

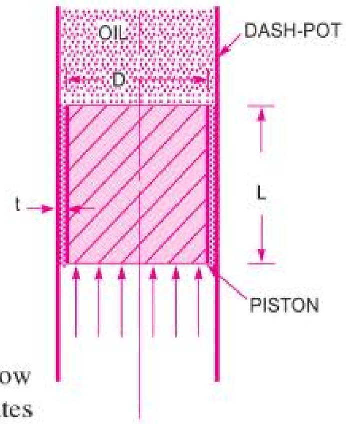


Fig. 9.13

Let D = Diameter of piston,

L = Length of piston,

W = Weight of piston,

μ = Viscosity of oil,

V = Velocity of piston,

$\bar{u}$ = Average velocity of oil in the clearance,

l = Clearance between the dash-pot and piston,

Δp = Difference of pressure intensities between the two ends of the piston.

The flow of oil through clearance is similar to the viscous flow between two parallel plates. The difference of pressure for parallel plates for length 'L' is given by

$$\Delta p = \frac{12 \mu \bar{u} L}{t^2} \qquad ...(i)$$

Also the difference of pressure at the two ends of piston is given by,

$$\Delta p = \frac{\text{Weight of piston}}{\text{Area of piston}} = \frac{W}{\frac{\pi}{4} D^2} = \frac{4W}{\pi D^2} \qquad ...(ii)$$

Equating (*i*) and (*ii*), we get $\dfrac{12 \mu \bar{u} L}{t^2} = \dfrac{4W}{\pi D^2}$

$$\therefore \qquad \bar{u} = \frac{4W}{\pi D^2} \times \frac{t^2}{12 \mu L} = \frac{W t^2}{3 \pi \mu L D^2} \qquad ...(iii)$$

V is the velocity of piston or the velocity of oil in dash-pot in contact with piston. The rate of flow of oil in dash-pot

$$= \text{velocity} \times \text{area of dash-pot} = V \times \frac{\pi}{4} D^2$$

Rate of flow through clearance = velocity through clearance × area of clearance = $\bar{u} \times \pi D \times t$

Due to continuity equation, rate of flow through clearance must be equal to rate of flow through dash-pot.

$$\therefore \qquad \bar{u} \times \pi D \times t = V \times \frac{\pi}{4} D^2$$

$$\therefore \qquad \bar{u} = V \times \frac{\pi}{4} D^2 \times \frac{1}{\pi D \times t} = \frac{VD}{4t} \qquad ...(iv)$$

Equating the value of $\bar{u}$ from (*iii*) and (*iv*), we get

$$\frac{W t^2}{3 \pi \mu L D^2} = \frac{VD}{4t}$$

$$\mu = \frac{4t^3W}{3\pi LD^3V} = \frac{4Wt^3}{3\pi LD^3V}. \quad ...(9.24)$$

Problem 9.25 *An oil dash-pot consists of a piston moving in a cylinder having oil. This arrangement is used to damp out the vibrations. The piston falls with uniform speed and covers 5 cm in 100 seconds. If an additional weight of 1.36 N is placed on the top of the piston, it falls through 5 cm in 86 seconds with uniform speed. The diameter of the piston is 7.5 cm and its length is 10 cm. The clearance between the piston and the cylinder is 0.12 cm which is uniform throughout. Find the viscosity of oil.*

Solution. Given :

Distance covered by piston due to self weight, = 5 cm

Time taken, = 100 sec

Additional weight, = 1.36 N

Time taken to cover 5 cm due to additional weight, = 86 sec

Dia. of piston, $D = 7.5$ cm = 0.075 m

Length of piston, $L = 10$ cm = 0.1 m

Clearance, $t = 0.12$ cm = 0.0012 m

Let the viscosity of oil $= \mu$

W = Weight of piston,

V = Velocity of piston without additional weight,

V^* = Velocity of piston with additional weight.

Using equation (9.24), we have

$$\mu = \frac{4Wt^3}{3\pi D^3LV} = \frac{4[W+1.36]t^3}{3\pi D^3LV^*}$$

or
$$\frac{W}{V} = \frac{W+1.36}{V^*} \quad \left(\text{Cancelling } \frac{4Wt^3}{3\pi D^3L}\right)$$

or
$$\frac{V}{V^*} = \frac{W}{W+1.36} \quad ...(i)$$

But V = Velocity of piston due to self weight of piston

$$= \frac{\text{Distance covered}}{\text{Time taken}} = \frac{5}{100} \text{ cm/s}$$

Similarly,
$$V^* = \frac{\text{Distance covered due to self weight + additional weight}}{\text{Time taken}}$$

$$= \frac{5}{86} \text{ cm/s}$$

$\therefore$
$$\frac{V}{V^*} = \frac{5}{100} \times \frac{86}{5} = 0.86 \quad ...(ii)$$

Equating (*i*) and (*ii*), we get $\dfrac{W}{W+1.36} = 0.86$

or
$$W = 0.86\text{ W} + .86 \times 1.36$$

or
$$W - 0.86\ W = 0.14\text{ W} = .86 \times 1.36$$

$\therefore \qquad W = \dfrac{0.86 \times 1.36}{0.14} = 8.354 \text{ N}$

Using equation (9.24), we get $\mu = \dfrac{4Wt^3}{3\pi D^3 LV}$

$$= \frac{4 \times 8.354 \times (.0012)^3}{3\pi \times (0.075)^3 \times .10 \times \left(\frac{5}{100} \times \frac{1}{100}\right)} \quad \left\{\because V = \frac{5}{100} \text{ cm/s} = \frac{5}{100} \times \frac{1}{100} \text{ m/s}\right\}$$

$= 0.29 \text{ N s/m}^2 = 0.29 \times 10$ poise = **2.9 poise. Ans.**

▶ 9.8 METHODS OF DETERMINATION OF CO-EFFICIENT OF VISCOSITY

The following are the experimental methods of determining the co-efficient of viscosity of a liquid:

1. Capillary tube method,
2. Falling sphere resistance method,
3. By rotating cylinder method, and
4. Orifice type viscometer.

The apparatus used for determining the viscosity of a liquid is called viscometer.

9.8.1 Capillary Tube Method. In capillary tube method, the viscosity of a liquid is calculated by measuring the pressure difference for a given length of the capillary tube. The Hagen Poiseuille law is used for calculating viscosity.

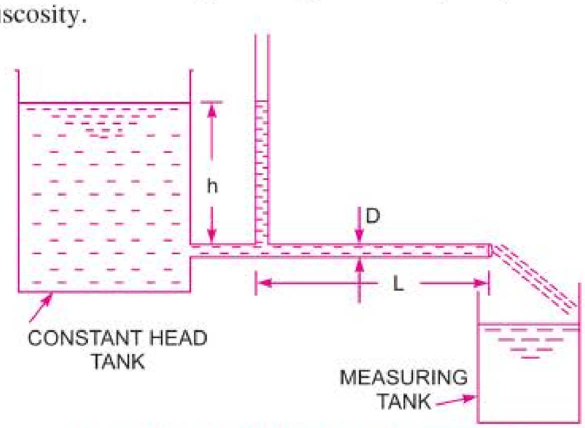


Fig. 9.14 *Capillary tube viscometer.*

Fig. 9.14 shows the capillary tube viscometer. The liquid whose viscosity is to be determined is filled in a constant head tank. The liquid is maintained at constant temperature and is allowed to pass through the capillary tube from the constant head tank. Then, the liquid is collected in a measuring tank for a given time. Then the rate of liquid collected in the tank per second is determined. The pressure head 'h' is measured at a point far away from the tank as shown in Fig. 9.14.

Then $\qquad h$ = Difference of pressure head for length L.

The pressure at outlet is atmospheric.

Let $\qquad D$ = Diameter of capillary tube,

L = Length of tube for which difference of pressure head is known,

ρ = Density of fluid,

and μ = Co-efficient of viscosity.

Using Hagen Poiseuille's Formula, $h = \dfrac{32\mu \bar{u} L}{\rho g D^2}$

But $$\bar{u} = \frac{Q}{\text{Area}} = \frac{Q}{\frac{\pi}{4}D^2}$$

where Q is rate of liquid flowing through tube.

$$h = \frac{32\mu \times \dfrac{Q}{\frac{\pi}{4}D^2} \times L}{\rho g D^2} = \frac{128\,\mu\, Q\,.\,L}{\pi \rho g D^4}$$

or $$\mu = \frac{\pi \rho g h \overline{D}^4}{128\, Q\,.\,L} \qquad \text{...(9.25)}$$

Measurement of D should be done very accurately.

9.8.2 Falling Sphere Resistance Method.

Theory. This method is based on Stoke's law, according to which the drag force, F on a small sphere moving with a constant velocity, U through a viscous fluid of viscosity, μ for viscous conditions is given by

$$F = 3\pi\mu U d \qquad \text{...}(i)$$

where d = diameter of sphere

U = velocity of sphere.

When the sphere attains a constant velocity U, the drag force is the difference between the weight of sphere and buoyant force acting on it.

Let L = distance travelled by sphere in viscous fluid,

t = time taken by sphere to cover distance l,

ρ_s = density of sphere,

ρ_f = density of fluid,

W = weight of sphere,

and F_B = buoyant force acting on sphere.

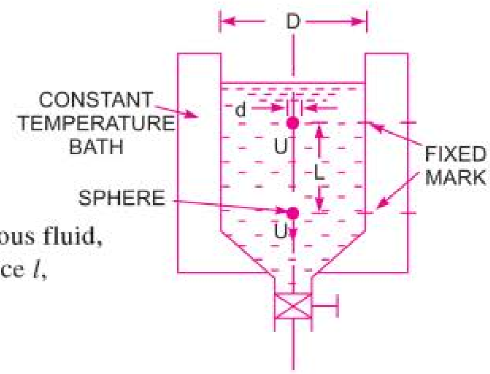


Fig. 9.15 *Falling sphere resistance method.*

Then constant velocity of sphere, $U = \dfrac{L}{t}$

Weight of sphere, W = volume $\times$ density of sphere $\times g$

$$= \frac{\pi}{6} d^3 \times \rho_s \times g \qquad \left\{\because \text{ volume of sphere} = \frac{\pi}{6} d^3\right\}$$

and buoyant force, F_B = weight of fluid displaced

= volume of liquid displaced $\times$ density of fluid $\times g$

$$= \frac{\pi}{6} d^3 \times \rho_f \times g \qquad \{\text{volume of liquid displaced} = \text{volume of sphere}\}$$

For equilibrium,

$$\text{Drag force} = \text{Weight of sphere} - \text{buoyant force}$$

or
$$F = W - F_B$$

Substituting the values of F, W and F_B, we get

$$3\pi\mu U d = \frac{\pi}{6}\, d^3 \times \rho_s \times g - \frac{\pi}{6}\, d^3 \times \rho_f \times g = \frac{\pi}{6}\, d^3 \times g\, [\rho_s - \rho_f]$$

or
$$\mu = \frac{\pi}{6}\, \frac{d^3 \times g[\rho_s - \rho_f]}{3\pi U d} = \frac{g d^2}{18U}\, [\rho_s - \rho_f] \qquad \text{...(9.26)}$$

where ρ_f = Density of liquid

Hence in equation (9.26), the values of d, U, ρ_s and ρ_f are known and hence the viscosity of liquid can be determined.

Method. Thus this method consists of a tall vertical transparent cylindrical tank, which is filled with the liquid whose viscosity is to be determined. This tank is surrounded by another transparent tank to keep the temperature of the liquid in the cylindrical tank to be constant.

A spherical ball of small diameter 'd' is placed on the surface of liquid. Provision is made to release this ball. After a short distance of travel, the ball attains a constant velocity. The time to travel a known vertical distance between two fixed marks on the cylindrical tank is noted to calculate the constant velocity U of the ball. Then with the known values of d, ρ_s, ρ_f the viscosity μ of the fluid is calculated by using equation (9.26).

9.8.3 Rotating Cylinder Method.

This method consists of two concentric cylinders of radii R_1 and R_2 as shown in Fig. 9.16. The narrow space between the two cylinders is filled with the liquid whose viscosity is to be determined. The inner cylinder is held stationary by means of a torsional spring while outer cylinder is rotated at constant angular speed ω. The torque T acting on the inner cylinder is measured by the torsional spring. The torque on the inner cylinder must be equal and opposite to the torque applied on the outer cylinder.

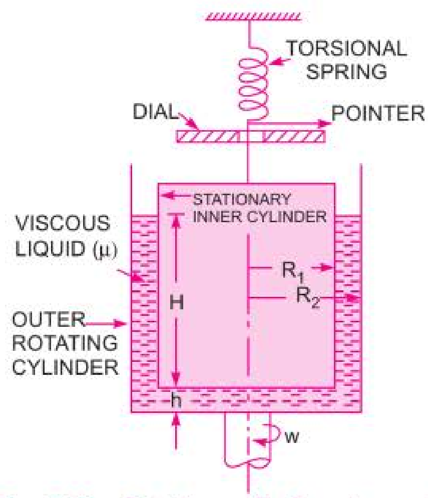


Fig. 9.16 Rotating cylinder viscometer.

The torque applied on the outer cylinder is due to viscous resistance provided by liquid in the annular space and at the bottom of the inner cylinder.

Let ω = angular speed of outer cylinder.

Tangential (peripheral) speed of outer cylinder

$$= \omega \times R_2$$

Tangential velocity of liquid layer in contact with outer cylinder will be equal to the tangential velocity of outer cylinder.

$\therefore$ Velocity of liquid layer with outer cylinder $= \omega \times R_2$

Velocity of liquid layer with inner cylinder $= 0$ {$\because$ Inner cylinder is stationary}

$\therefore$ Velocity gradient over the radial distance $(R_2 - R_1)$

$$= \frac{du}{dy} = \frac{\omega R_2 - 0}{R_2 - R_1} = \frac{\omega R_2}{R_2 - R_1}$$

$\therefore$ Shear stress (τ)
$$= \mu\, \frac{du}{dy} = \mu\, \frac{\omega R_2}{(R_2 - R_1)}$$

$\therefore$ Shear force (F) $= \text{shear stress} \times \text{area of surface}$

$$= \tau \times 2\pi R_1 H$$

$\{\because$ shear stress is acting on surface area $= 2\pi R_1 \times H\}$

$$= \mu \frac{\omega R_2}{(R_2 - R_1)} \times 2\pi R_1 H$$

The torque T_1 on the inner cylinder due to shearing action of the liquid in the annular space is

$$T_1 = \text{shear force} \times \text{radius}$$

$$= \mu \frac{\omega R_2}{(R_2 - R_1)} \times 2\pi R_1 H \times R_1$$

$$= \frac{2\pi\mu\omega H R_1^2 R_2}{(R_2 - R_1)} \quad \text{...(i)}$$

If the gap between the bottom of the two cylinders is 'h', then the torque applied on inner cylinder (T_2) is given by equation (9.19A) as

$$T_2 = \frac{\mu}{60t} \pi^2 N R^4$$

But here $R = R_1$, $t = h$ then $T_2 = \dfrac{\mu}{60h} \pi^2 N R_1{}^4$

$$\omega = \frac{2\pi N}{60} \text{ or } N = \frac{60\omega}{2\pi}$$

$$\therefore \quad T_2 = \frac{\mu}{60h} \times \pi^2 \times \frac{60\omega}{2\pi} \times R_1{}^4 = \frac{\pi\mu\omega}{2h} R_1{}^4 \quad \text{...(ii)}$$

$\therefore$ Total torque T acting on the inner cylinder is

$$T = T_1 + T_2$$

$$= \frac{2\pi\mu\omega H R_1^2 R_2}{(R_2 - R_1)} + \frac{\pi\mu\omega}{2h} R_1{}^4 = 2\pi\mu R_1{}^2 \left[\frac{R_2 H}{R_2 - R_1} + \frac{R_1^2}{4h}\right] \times \omega$$

$$\therefore \quad \mu = \frac{2(R_2 - R_1)hT}{\pi R_1^2 \omega \left[4HhR_2 + R_1^2 (R_2 - R_1)\right]} \quad \text{...(9.27)}$$

where T = torque measured by the strain of the torsional spring,
R_1, R_2 = radii of inner and outer cylinder,
h = clearance at the bottom of cylinders,
H = height of liquid in annular space,
μ = co-efficient of viscosity to be determined.

Hence, the value of μ can be calculated from equation (9.27).

9.8.4 Orifice Type Viscometer. In this method, the time taken by a certain quantity of the liquid whose viscosity is to be determined, to flow through a short capillary tube is noted down. The co-efficient of viscosity is then obtained by comparing with the co-efficient of viscosity of a liquid whose viscosity is known or by the use conversion factors.

Viscometers such as Saybolt, Redwood or Engler are usually used. The principle for all the three viscometer is same. In the United Kingdom, Redwood viscometer is used while in U.S.A., Saybolt viscometer is commonly used.

Fig. 9.17 shows that Saybolt viscometer, which consists of a tank at the bottom of which a short capillary tube is fitted. In this tank the liquid whose viscosity is to be determined is filled. This tank is surrounded by another tank, called constant temperature bath. The liquid is allowed to flow through capillary tube at a standard temperature. The time taken by 60 c.c. of the liquid to flow through the capillary tube is noted down. The initial height of liquid in the tank is previously adjusted to a standard height. From the time measurement, the kinematic viscosity of liquid is known from the relation,

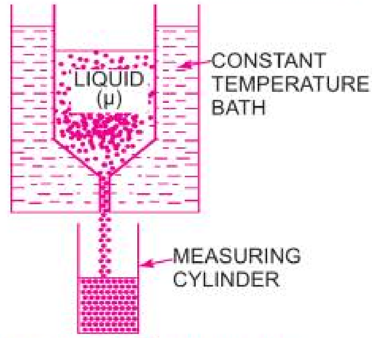


Fig. 9.17 *Saybolt viscometer.*

$$\nu = At - \frac{B}{t}$$

where $A = 0.24$, $B = 190$, t = time noted in seconds, ν = kinematic viscosity in stokes.

Problem 9.26 *The viscosity of an oil of sp. gr. 0.9 is measured by a capillary tube of diameter 50 mm. The difference of pressure head between two points 2 m apart is 0.5 m of water. The mass of oil collected in a measuring tank is 60 kg in 100 seconds. Find the viscosity of oil.*

Solution. Given :

Sp. gr. of oil $= 0.9$

Dia. of capillary tube, $D = 50 \text{ mm} = 5 \text{ cm} = 0.05 \text{ m}$

Length of tube, $L = 2 \text{ m}$

Difference of pressure head, $h = 0.5 \text{ m}$

Mass of oil, $M = 60 \text{ kg}$

Time, $t = 100 \text{ s}$

Mass of oil per second $= \dfrac{60}{100} = 0.6 \text{ kg/s}$

Density of oil, $\rho = \text{sp. gr. of oil} \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$

$\therefore$ Discharge, $Q = \dfrac{\text{Mass of oil / s}}{\text{Density}} = \dfrac{0.6}{900} \text{ m}^3\text{/s} = 0.000667 \text{ m}^3\text{/s}$

Using equation (9.25), we get viscosity,

$$\mu = \frac{\pi \rho g h D^4}{128\, Q \,.\, L} \qquad [\text{here } h = h_f = 0.5]$$

$$= \frac{\pi \times 900 \times 9.81 \times 0.5 \times (.05)^4}{128 \times 0.000667 \times 2.0} = 0.5075 \text{ (SI Units) N s/m}^2$$

$= 0.5075 \times 10$ poise $=$ **5.075 poise. Ans.**

Problem 9.27 *A capillary tube of diameter 2 mm and length 100 mm is used for measuring viscosity of a liquid. The difference of pressure between the two ends of the tube is 0.6867 N/cm² and the viscosity of liquid is 0.25 poise. Find the rate of flow of liquid through the tube.*

Solution. Given :

Dia. of capillary tube, $D = 2 \text{ m} = 2 \times 10^{-3} \text{ m}$

Length of tube, $L = 100 \text{ mm} = 10 \text{ cm} = 0.1 \text{ m}$

Difference of pressure, $\Delta p = 0.6867 \text{ N/cm}^2 = 0.6867 \times 10^4 \text{ N/m}^2$

$\therefore$ Difference of pressure head, $h = \dfrac{\Delta p}{\rho g} = \dfrac{0.6867 \times 10^4}{\rho g}$

Viscosity, $\mu = 0.25$ poise

$$= \frac{0.25}{10} \text{ Ns/m}^2$$

Let the rate of flow of liquid = Q

Using equation (9.25), we get $\mu = \dfrac{\pi \rho g h D^4}{128 \,.\, Q \,.\, L} = \pi \rho g \times \dfrac{\dfrac{0.6867 \times 10^4}{\rho g} \times \left(2 \times 10^{-3}\right)^4}{128 \times Q \times 0.1}$

or
$$\frac{0.25}{10} = \frac{\pi \times 0.6867 \times 10^4 \times \left(2 \times 10^{-3}\right)^4}{128 \times Q \times 0.1}$$

or
$$Q = \frac{\pi \times 0.6867 \times 10^4 \times 2^4 \times 10^{-12} \times 10}{128 \times 0.1 \times 0.25} \text{ m}^3\text{/s}$$

$$= 107.86 \times 10^{-8} \text{ m}^3\text{/s} = 107.86 \times 10^{-8} \times 10^6 \text{ cm}^3\text{/s}$$

$$= 107.86 \times 10^{-2} \text{ cm}^3\text{/s} = \mathbf{1.078 \text{ cm}^3\text{/s}}. \text{ Ans.}$$

Problem 9.28 *A sphere of diameter 2 mm falls 150 mm in 20 seconds in a viscous liquid. The density of the sphere is 7500 kg/m³ and of liquid is 900 kg/m³. Find the co-efficient of viscosity of the liquid.*

Solution. Given :

Dia. of sphere, $d = 2 \text{ mm} = 2 \times 10^{-3}$ m

Distance travelled by sphere = 150 mm = 0.15 m

Time taken, $t = 20$ seconds

Velocity of sphere, $U = \dfrac{\text{Distance}}{\text{Time}} = \dfrac{0.15}{20} = .0075$ m/s

Density of sphere, $\rho_s = 7500$ kg/m³

Density of liquid, $\rho_f = 900$ kg/m³

Using relation (9.26), we get $\mu = \dfrac{gd^2}{18U}\,[\rho_s - \rho_f] = \dfrac{9.81 \times \left[2 \times 10^{-3}\right]^2}{18 \times 0.0075}\,[7500 - 900]$

$$= \frac{9.81 \times 4 \times 10^{-6} \times 6600}{18 \times 0.0075} = 1.917 \, \frac{\text{Ns}}{\text{m}^2}$$

$$= 1.917 \times 10 = \mathbf{19.17 \text{ poise}}. \text{ Ans.}$$

Problem 9.29 *Find the viscosity of a liquid of sp. gr. 0.8, when a gas bubble of diameter 10 mm rises steadily through the liquid at a velocity of 1.2 cm/s. Neglect the weight of the bubble.*

Solution. Given :

Sp. gr. of liquid = 0.8

∴ Density of liquid, $\rho_f = 0.8 \times 1000 = 800$ kg/m³

Dia. of gas bubble, $D = 10 \text{ mm} = 1 \text{ cm} = 0.01$ m

Velocity of bubble, $U = 1.2 \text{ cm/s} = .012$ m/s

As weight of bubble is neglected and density of bubble

$$\rho_s = 0$$

Now using the relation, $\mu = \frac{gd^2}{18U} [\rho_s - \rho_f]$ which is for a falling sphere.

For a rising bubble, the relation will become as

$$\mu = \frac{gd^2}{18U} [\rho_f - \rho_s]$$

Substituting the values, we get $\mu = \frac{9.81 \times .01 \times .01}{18 \times .012} [800 - 0] \frac{\text{Ns}}{\text{m}^2} = 3.63 \frac{\text{Ns}}{\text{m}^2}$

$$= 3.63 \times 10 = \textbf{36.3 poise. Ans.}$$

Problem 9.30 *The viscosity of a liquid is determined by rotating cylinder method, in which case the inner cylinder of diameter 20 cm is stationary. The outer cylinder of diameter 20.5 cm, contains the liquid upto a height of 30 cm. The clearance at the bottom of the two cylinders is 0.5 cm. The outer cylinder is rotated at 400 r.p.m. The torque registered on the torsion meter attached to the inner cylinder is 5.886 Nm. Find the viscosity of fluid.*

Solution. Given :

Dia. of inner cylinder, $D_1 = 20$ cm

$\therefore$ Radius of inner cylinder, $R_1 = 10$ cm $= 0.1$ m

Dia. of outer cylinder, $D_2 = 20.5$ cm

$\therefore$ Radius of outer cylinder, $R_2 = \frac{20.5}{2} = 10.25$ cm $= .1025$ m

Height of liquid from bottom of outer cylinder = 30 cm

Clearance at the bottom of two cylinders, $h = 0.5$ cm $= .005$ m

$\therefore$ Height of inner cylinder immersed in liquid

$$= 30 - h = 30 - 0.5 = 29.5 \text{ m}$$

or $H = 29.5$ cm $= .295$ m

Speed of outer cylinder, $N = 400$ r.p.m.

$$\therefore \quad \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 400}{60} = 41.88$$

Torque measured, $T = 5.886$ Nm

Using equation (9.27), we get $\mu = \frac{2(R_2 - R_1) \times h \times T}{\pi R_1^2 \omega \left[4HhR_2 + R_1^2 (R_2 - R_1)\right]}$

$$= \frac{2(.1025 - 0.1) \times .005 \times 5.886}{\pi \times (.1)^2 \times 41.88 \left[4 \times .295 \times .005 \times .1025 + .1^2\ (.1025 - .1)\right]}$$

$$= \frac{2 \times .0025 \times .005 \times 5.886}{\pi \times .01 \times 41.88 \left[.0006047 - .000025\right]}$$

$$= 0.19286 \text{ Ns/m}^2 = 0.19286 \times 10 = \textbf{1.9286 poise. Ans.}$$

Problem 9.31 *A sphere of diameter 1 mm falls through 335 m in 100 seconds in a viscous fluid. If the relative densities of the sphere and the liquid are 7.0 and 0.96 respectively, determine the dynamic viscosity of the liquid.*

Solution. Given :

Dia. of sphere, $d = 1 \text{ mm} = 0.001 \text{ m}$

Distance travelled by sphere $= 335 \text{ mm} = 0.335 \text{ m}$

Time taken, $t = 100$ seconds

$\therefore$ Velocity of sphere, $U = \dfrac{\text{Distance}}{\text{Time}} = \dfrac{0.335}{100} = 0.00335 \text{ m/sec}$

Relative density of sphere $= 7$

$\therefore$ Density of sphere, $\rho_s = 7 \times 1000 = 7000 \text{ kg/m}^3$

Relative density of liquid $= 0.96$

$\therefore$ Density of liquid, $\rho_f = 0.96 \times 1000 = 960 \text{ kg/m}^3$

Using the relation (9.26), we get $\mu = \dfrac{gd^2}{18U} [\rho_s - \rho_f] = \dfrac{9.81 \times 0.001^2}{18 \times 0.00335} [7000 - 960]$

$$= \frac{0.00000981 \times 6040}{18 \times 0.00335} = 0.981 \text{ Ns/m}^2$$

$$= 0.981 \times 10 = \mathbf{9.81 \text{ poise.}} \text{ Ans.}$$

Problem 9.32 *Determine the fall velocity of 0.06 mm sand particle (specific gravity = 2.65) in water at 20°C, take $\mu = 10^{-3}$ kg/ms.*

Solution. Given :

Dia. of sand particle, $d = 0.06 \text{ mm} = 0.06 \times 10^{-3} \text{ m}$

Specific gravity of sand $= 2.65$

$\therefore$ Density of sand, $\rho_s = 2.65 \times 1000 \text{ kg/m}^3$ $\quad (\because \rho$ for water in S.I. unit $= 1000 \text{ kg/m}^3)$

$= 2650 \text{ kg/m}^3$

Viscosity of water, $\mu^* = 10^{-3} \text{ kg/ms} = 10^{-3} \text{ Ns/m}^2$ $\quad \left[\because \dfrac{\text{Ns}}{\text{m}^2} = \left(\text{kg} \times \dfrac{\text{m}}{\text{s}^2}\right) \times \dfrac{\text{s}}{\text{m}^2} = \dfrac{\text{kg}}{\text{ms}}\right]$

Density of water, $\rho_f = 1000 \text{ kg/m}^3$

Sand particle is just like a sphere.

For equilibrium of sand particle,

Drag force = Weight of sand particle – buoyant force

or $\quad F_D = W - F_B \quad$ ...(*i*)

But $\quad F_D = 3\pi\mu \times U \times d$, where U = Velocity of particle

$= 3\pi \times 10^{-3} \times U \times 0.06 \times 10^{-3}$ N

W = Weight of sand particle

$$= \frac{\pi}{6} \times d^3 \times \rho_s \times g = \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 2650 \times 9.81 \text{ N}$$

F_B = Buoyant force = Weight of water displaced

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*Viscosity in S.I. unit $= \text{N s/m}^2$. But $1 \text{ N} = 1 \text{ kg} \times 1 \text{ m/s}^2$

Hence viscosity $= \left(\dfrac{1 \text{ kg} \times 1 \text{ m}}{\text{s}^2}\right) \times \dfrac{\text{s}}{\text{m}^2} = \text{kg/ms}$. Hence $\text{kg/ms} = \dfrac{\text{Ns}}{\text{m}^2}$.

$$= \frac{\pi}{6} \times d^3 \times \rho_f \times g = \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 1000 \times 9.81 \text{ N}$$

Substituting the above values in equation (*i*), we get

$$3\pi \times 10^{-3} \times U \times 0.06 \times 10^{-3} = \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 2650 \times 9.81 - \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 1000 \times 9.81$$

Cancelling $(\pi \times 0.06 \times 10^{-3})^2$ throughout, we get

$$3 \times U = \frac{1}{6} \times 0.06^2 \times 10^{-3} \times 2650 \times 9.81 - \frac{1}{6} \times 0.06^2 \times 10^{-3} \times 1000 \times 9.81$$

$$= \frac{1}{6} \times 0.06^2 \times 10^{-3} \times 9.81\ (2650 - 1000)$$

$$= \frac{1}{6} \times 0.0036 \times 10^{-3} \times 9.81 \times 1650 = 0.009712$$

$\therefore$ $U = 0.009712/3 =$ **0.00323 m/sec. Ans.**

HIGHLIGHTS

1. A flow is said to be viscous if the Reynolds number is less than 2000, or the fluid flows in layers.
2. For the viscous flow through circular pipes,

   (*i*) Shear stress ...... $\tau = -\frac{\partial p}{\partial x}\frac{r}{2}$

   (*ii*) Velocity ......$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2]$

   (*iii*) Ratio of velocities $\frac{U_{max}}{\bar{u}} = 2.0$

   (*iv*) Loss of pressure head, $h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$

   where $\frac{\partial p}{\partial x}$ = pressure gradient, r = radius at any point,

   R = radius of the pipe, U_{max} = maximum velocity or velocity at $r = 0$,

   $\bar{u}$ = average velocity $= \frac{Q}{\pi R^2}$, μ = co-efficient of viscosity,

   D = diameter of the pipe.
3. For the viscous flow between two parallel plates,

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}(ty - y^2) \quad \text{... Velocity distribution}$$

$$\frac{U_{max}}{\bar{u}} = 1.5 \quad \text{... Ratio of maximum and average velocity}$$

$$h_f = \frac{12\mu\bar{u}L}{\rho g t^2} \quad \text{... Loss of pressure head}$$

$$\tau = -\frac{1}{2}\frac{\partial p}{\partial x}[t - 2y] \quad \text{... Shear stress distribution}$$

where t = thickness or distance between two plates,

y = distance in the vertical direction from the lower plate,

τ = shear stress at any point in flow.

4. The kinetic energy correction factor α is given as

$$\alpha = \frac{\text{K.E. per second based on actual velocity}}{\text{K.E. per second based on average velocity}}$$

$$= 2.0 \text{ ... for a circular pipe.}$$

5. Momentum correction factor, β is given by

$$\beta = \frac{\text{Momentum per second based on actual velocity}}{\text{Momentum per second based on average velocity}}$$

$$= \frac{4}{3} \text{ ... for a circular pipe.}$$

6. For the viscous resistance of Journal Bearing.

$$V = \frac{\pi DN}{60}, \frac{du}{dy} = \frac{V}{t} = \frac{\pi DN}{60t}$$

$$\tau = \frac{\mu \pi dN}{60t}, \text{ Shear force} = \frac{\mu \pi^2 D^2 NL}{60t}$$

Torque, $$T = \frac{\mu \pi^2 D^3 NL}{120t} \text{ and power} = \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t}$$

where L = length of bearing, N = speed of shaft

t = clearance between the shaft and bearing.

7. For the Foot-Step Bearing, the shear force, torque and h.p. absorbed are given as :

Shear force, $$F = \frac{\mu}{15} \frac{\pi^2 N}{t} \frac{R^3}{3}$$

Torque, $$T = \frac{\mu}{60t} \pi^2 NR^4$$

Power $$= \frac{\mu \pi^3 N^2 R^4}{60 \times 30 \times t}$$

where R = radius of the shaft, N = speed of the shaft.

8. For the collar bearing the torque and power absorbed are given as

$$T = \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4], \qquad P = \frac{\mu \pi^3 N^2}{60 \times 30t} [R_2^4 - R_1^4]$$

where R_1 = internal radius of the collar, R_2 = external radius of the collar,

t = thickness of oil film, P = power in watts.

9. For the viscous flow the co-efficient of friction is given by, $f = \dfrac{16}{R_e}$

where R_e = the Reynolds number $= \dfrac{\rho VD}{\mu} = \dfrac{VD}{\nu}$.

10. The co-efficient of viscosity is determined by dash-pot arrangement as $\mu = \dfrac{4Wt^3}{3\pi LD^3 V}$

where W = weight of the piston, t = clearance between dash-pot and piston,

L = length of the piston, D = diameter of the piston,

V = velocity of the piston.

11. The co-efficient of viscosity of a liquid is also determined experimentally by the following method :

(*i*) Capillary tube method, $\mu = \dfrac{\pi \rho g h D^4}{128\, QL}$

(*ii*) Falling sphere method, $\mu = \dfrac{gd^2 \left[\rho_s - \rho_f\right]}{18U}$

(*iii*) Rotating cylinder method, $\mu = \dfrac{2\left(R_2 - R_1\right)hT}{\pi R_1^2 \omega \left[4HhR_2 + R_1^2\left(R_2 - R_1\right)\right]}$

where w = specific weight of fluid,
D = diameter of the capillary tube,
d = diameter of the sphere,
ρ_f = density of fluid,
R_2 = radius of outer rotating cylinder,
T = torque.
L = length of the tube,
Q = rate of flow of fluid through capillary tube,
ρ_s = density of sphere,
U = velocity of sphere,
R_1 = radius of inner stationary cylinder,

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define the terms : Viscosity, kinematic viscosity, velocity gradient and pressure gradient.
2. What do you mean by 'Viscous Flow'?
3. Derive an expression for the velocity distribution for viscous flow through a circular pipe. Also sketch the velocity distribution and shear stress distribution across a section of the pipe.
4. Prove that the maximum velocity in a circular pipe for viscous flow is equal to two times the average velocity of the flow. (*Delhi University, December 2002*)
5. Find an expression for the loss of head of a viscous fluid flowing through a circular pipe.
6. What is Hagen Poiseuille's Formula ? Derive an expression for Hagen Poiseuille's Formula.
7. Prove that the velocity distribution for viscous flow between two parallel plates when both plates are fixed across a section is parabolic in nature. Also prove that maximum velocity is equal to one and a half times the average velocity.
8. Show that the difference of pressure head for a given length of the two parallel plates which are fixed and through which viscous fluid is flowing is given by

$$h_f = \frac{12 \mu \bar{u} L}{\rho g t^2}$$

where μ = Viscosity of fluid, $\bar{u}$ = Average velocity,
t = Distance between the two parallel plates, L = Length of the plates.

9. Define the terms : Kinetic energy correction factor and momentum correction factor.
10. Prove that for viscous flow through a circular pipe the kinetic energy correction factor is equal to 2 while momentum correction factor $= \dfrac{4}{3}$.
11. A shaft is rotating in a journal bearing. The clearance between the shaft and the bearing is filled with a viscous oil. Find an expression for the power absorbed in overcoming viscous resistance.
12. Prove that power absorbed in overcoming viscous resistance in foot-step bearing is given by

$$P = \frac{\mu \pi^3 N^2 R^4}{60 \times 30 t}$$

where R = Radius of the shaft, N = Speed of the shaft,
t = Clearance between shaft and foot-step bearing, μ = Viscosity of fluid.

13. Show that the value of the co-efficient of friction for viscous flow through a circular pipe is given by,

$$f = \frac{16}{R_e}, \quad \text{where} \quad R_e = \text{Reynolds number.}$$

14. Prove that the co-efficient of viscosity by the dash-pot arrangement is given by,

$$\mu = \frac{4Wt^3}{3\pi LD^3V}$$

where W = Weight of the piston, t = Clearance between dash-pot and piston,
L = Length of piston, D = Diameter of piston,
V = Velocity of piston.

15. What are the different methods of determining the co-efficient of viscosity of a liquid ? Describe any two method in details.

16. Prove that the loss of pressure head for the viscous flow through a circular pipe is given by

$$h_f = \frac{32\mu \bar{u} L}{\rho g d^2}$$

where $\bar{u}$ = Average velocity, w = Specific weight.

17. For a laminar steady flow, prove that the pressure gradient in a direction of motion is equal to the shear gradient normal to the direction of motion.

18. Describe Reynolds experiments to demonstrate the two types of flow.

19. For the laminar flow through a circular pipe, prove that :
 (*i*) the shear stress variation across the section of the pipe is linear and
 (*ii*) the velocity variation is parabolic.

(B) NUMERICAL PROBLEMS

1. A crude oil of viscosity 0.9 poise and sp. gr. 0.8 is flowing through a horizontal circular pipe of diameter 80 mm and of length 15 m. Calculate the difference of pressure at the two ends of the pipe, if 50 kg of the oil is collected in a tank in 15 seconds. [**Ans.** 0.559 N/cm^2]

2. A viscous flow is taking place in a pipe of diameter 100 mm. The maximum velocity is 2 m/s. Find the mean velocity and the radius at which this occurs. Also calculate the velocity at 30 mm from the wall of the pipe. [**Ans.** 1 m/s, r = 35.35 mm, u = 1.68 m/s]

3. A fluid of viscosity 0.5 poise and specific gravity 1.20 is flowing through a circular pipe of diameter 100 mm. The maximum shear stress at the pipe wall is given as 147.15 N/m^2, find : (*a*) the pressure gradient, (*b*) the average velocity, and (*c*) the Reynolds number of the flow.
[**Ans.** (*a*) – 64746 N/m^2 per m, (*b*) 3.678 m/s, (*c*) 882.72]

4. Determine (*a*) the pressure gradient, (*b*) the shear stress at the two horizontal parallel plates and (*c*) the discharge per metre width for the laminar flow of oil with a maximum velocity of 1.5 m/s between two horizontal parallel fixed plates which are 80 mm apart. Take viscosity of oil as $\frac{1.962 \text{ Ns}}{\text{m}^2}$.
[**Ans.** (*a*) – 3678.7 N/m^2 per m, (*b*) 147.15 N/m^2, (*c*) .08 m^3/s]

5. Water is flowing between two large parallel plates which are 2.0 mm apart. Determine : (*a*) maximum velocity, (*b*) the pressure drop per unit length and (*c*) the shear stress at walls of the plate if the average velocity is 0.4 m/s. Take viscosity of water as 0.01 poise.
[**Ans.** (*a*) 0.6 m/s, (*b*) 1199.7 N/m^2 per m, (*c*) 1.199 N/m^3]

6. There is a horizontal crack 50 mm wide and 3 mm deep in a wall of thickness 150 mm. Water leaks through the crack. Find the rate of leakage of water through the crack if the difference of pressure between the two ends of the crack is 245.25 N/m^2. Take the viscosity of water as 0.01 poise. [**Ans.** 183.9 cm^3/s]

7. A shaft having a diameter of 10 cm rotates centrally in a journal bearing having a diameter of 10.02 cm and length 20 cm. The annular space between the shaft and the bearing is filled with oil having viscosity of 0.8 poise. Determine the power absorbed in the bearing when the speed of rotation is 500 r.p.m. [**Ans.** 343.6 W]

8. A shaft 150 mm diameter runs in a bearing of length 300 mm, with a radial clearance of 0.04 mm at 40 r.p.m. Find the viscosity of the oil, if the power required to overcome the viscous resistance is 220.725 W. [**Ans.** 6.32 poise]

9. Find the torque required to rotate a vertical shaft of diameter 8 cm at 800 r.p.m. The lower end of the shaft rests in a foot-step bearing. The end of the shaft and surface of the bearing are both flat and are separated by an oil film of thickness 0.075 cm. The viscosity of the oil is given as 1.2 poise. [**Ans.** 0.0538 Nm]

10. A collar bearing having external and internal diameters 20 cm and 10 cm respectively is used to take the thrust of a shaft. An oil film of thickness 0.03 cm is maintained between the collar surface and the bearing. Find the power lost in overcoming the viscous resistance when the shaft rotates at 250 r.p.m. Take $\mu = 0.9$ poise. [**Ans.** 30.165 W]

11. Water is flowing through a 150 mm diameter pipe with a co-efficient of friction $f = .05$. The shear stress at a point 40 mm from the pipe wall is 0.01962 N/cm^2. Calculate the shear stress at the pipe wall. [**Ans.** 0.04198 N/cm^2]

12. An oil dash-pot consists of a piston moving in a cylinder having oil. The piston falls with uniform speed and covers 4.5 cm in 80 seconds. If an additional weight of 1.5 N is placed on the top of the piston, it falls through 4.5 cm in 70 seconds with uniform speed. The diameter of the piston is 10 cm and its length is 15 cm. The clearance between the piston and the cylinder is 0.15 cm, which is uniform throughout. Find the viscosity of oil. [**Ans.** 0.177 poise]

13. The viscosity of oil of sp. gr. 0.8 is measured by a capillary tube of diameter 40 mm. The difference of pressure head between two points 1.5 m apart is 0.3 m of water. The mass of oil collected in a measuring tank is 40 kg in 120 seconds. Find the viscosity of the oil. [**Ans.** 2.36 poise]

14. A capillary tube of diameter 4 mm and length 150 mm is used for measuring viscosity of a liquid. The difference of pressure between the two ends of the tube is 0.7848 N/cm^2 and the viscosity of the liquid is 0.2 poise. Find the rate of flow of liquid through the tube. [**Ans.** 16.43 cm^3/s]

15. A sphere of diameter 3 mm falls 100 mm in 1.5 seconds in a viscous liquid. The density of the sphere is 7000 kg/m^3 and of liquid is 800 kg/m^3. Find the co-efficient of viscosity of the liquid. [**Ans.** 45.61 poise]

16. The viscosity of a liquid is determined by rotating cylinder method, in which case the inner cylinder of diameter 25 cm is stationary. The outer cylinder of diameter 25.5 cm contains the liquid upto a height of 40 cm. The clearance at the bottom of the two cylinders is 0.6 cm. The outer cylinder is rotated at 300 r.p.m. The torque registered on the torsion metre attached to the inner cylinder is 4.905 Nm. Find the viscosity of liquid. [**Ans.** .77 poise]

17. Calculate : (*a*) the pressure gradient along the flow, (*b*) the average velocity, and (*c*) the discharge for an oil of viscosity 0.02 Ns/m^2 flowing between two stationary parallel plates 1 m wide maintained 10 mm apart. The velocity midway between the plates is 2.5 m/s.
[**Ans.** (a) – 4000 N/m^2 per m, (b) 1.667 m/s, (c) .01667 m^3/s]

18. Calculate :
    (*i*) the pressure gradient along the flow,
    (*ii*) the average velocity, and
    (*iii*) the discharge for an oil of viscosity 0.03 N s/m^2 flowing between two stationary plates which are parallel and are at 10 mm apart. Width of plates is 2 m. The velocity midway between the plates is 2.0 m/s.

19. A cylinder of 100 mm diameter, 0.15 m length and weighing 10 N slides axially in a vertical pipe of 104 mm dia. If the space between cylinder surface and pipe wall is filled with liquid of viscosity μ and the cylinder slides downwards at a velocity of 0.45 m/s, determine μ.
[**Hint.** D = 100 mm = 0.1, L = 0.15 m, W = 10 N, D_p = 1.4 mm = 0.104 m,
V = 0.45 m/s. Hence t = (0.104 – 0.1)/2 = 0.002 m.

$$\mu = \frac{4Wt^3}{3\pi D^3 LV} = \frac{4 \times 10 \times 0.002^3}{3\pi \times 0.1^3 \times 0.15 \times .45} = \mathbf{503 \times 10^{-6}\ N\ s/m^2}]$$

20. A liquid is pumped through a 15 cm diameter and 300 m long pipe at the rate of 20 tonnes per hour. The density of liquid is 910 kg/m^3 and kinematic viscosity = 0.002 m^2/s. Determine the power required and show that the flow is viscous.

[**Hint.** D = 15 cm = 0.15 m, L = 300 m, W = 20 tonnes/hr

= 20 × 1000 kgf/60 × 60 sec = 5.555 kgf/sec = 5.555 × 9.81 N/s.

$$Q = \frac{W}{\rho g} = \frac{5.555 \times 9.81}{910 \times 9.81} = 0.0061 \text{ m}^3/\text{s}. \quad V = \frac{Q}{A} = \frac{0.0061}{\frac{\pi}{4}(.15^2)}$$

= 0.345 m/s, v = 0.002 m^2/s.

Now
$$R_e = \frac{\rho VD}{\mu} = \frac{V \times D}{\nu} = \frac{0.345 \times 0.15}{0.002} = 25.87$$

which is less than 2000. Hence flow is viscous.

$$h_f = 32\,\mu LV/\rho g D^2, \text{ where } \nu = \frac{\mu}{\rho} \quad \therefore \mu = \nu \times \rho = 0.002 \times 910 = 1.82$$

Hence,
$$h_f = \frac{32 \times 1.82 \times 300 \times 0.345}{\left(910 \times 9.81 \times 0.15^2\right)} = 30$$

$\therefore$
$$P = \rho g.Q.h_f/1000 = 910 \times 9.81 \times 0.0061 \times 30/1000 = \mathbf{1.633\ kW.}\ ]$$

21. An oil of specific gravity 0.9 and viscosity 10 poise is flowing through a pipe of diameter 110 mm. The velocity at the centre is 2 m/s, find : (*i*) pressure gradient in the direction of flow, (*ii*) shear stress at the pipe wall ; (*iii*) Reynolds number, and (*iv*) velocity at a distance of 30 mm from the wall.

[**Hint.** ρ = 900 kg/m^3 ; μ = 10 poise = 1 N s/m^2 ; D = 110 mm = 0.11 m,

$$U_{max} = 2 \text{ m/s} ; \bar{u} = 1 \text{ m/s} ; U_{max} = \frac{1}{4\mu}\left(\frac{-dp}{dx}\right) R^2$$

(*i*) $\left(\frac{-dp}{dx}\right) = \frac{4\mu \times U_{max}}{R^2} = \frac{4 \times 1 \times 2}{0.055^2} = 2644.6 \text{ N/m}^3$;

(*ii*) $\tau_0 = \left(\frac{-dp}{dx}\right) \times \frac{R}{2} = 2644.6 \times \frac{0.055}{2} = 72.72 \text{ N/m}^2$;

(*iii*) $R_e = \frac{\rho \times \bar{u} \times D}{\mu} = \frac{900 \times 1 \times 0.11}{1} = 99$; and

(*iv*) $u = \frac{1}{4\mu}\left(\frac{-dp}{dx}\right)(R^2 - r^2) = \frac{1}{4 \times 1}(2644.6)(0.055^2 - 0.025^2) = 1.586$ m/s.]

22. Determine (*i*) the pressure gradient, (*ii*) the shear stress at the two horizontal plates, (*iii*) the discharge per metre width for laminar flow of oil with a maximum velocity of 2 m/s between two plates which are 150 mm apart. Given : μ = 2.5 N s/m^2. *(Delhi University, December 2002)*

[**Hint.** U_{max} = 2 m/s, t = 150 mm = 0.15 m, μ = 2.5 N s/m^2

(*i*) $U_{max} = -\frac{1}{8\mu}\frac{dp}{dx}t^2 \quad \therefore \quad \frac{dp}{dx} = \frac{-8\mu\, U_{max}}{t^2} = \frac{-8 \times 2.5 \times 2}{0.15^2} = \mathbf{-\,1777.77\ N/m^2.}$

(*ii*) $\tau_0 = -\frac{1}{2}\frac{dp}{dx} \times t = -\frac{1}{2}(-1777.77) \times 0.15 = \mathbf{133.33\ N/m^2.}$

(*iii*) Q = Mean velocity × Area = $\left(\frac{2}{3}U_{max}\right) \times (t \times 1) = \left(\frac{2}{3} \times 2\right) \times (0.15 \times 1) = \mathbf{0.2\ m^3/s.}]$

10

CHAPTER

TURBULENT FLOW

▶ 10.1 INTRODUCTION

The laminar flow has been discussed in chapter 9. In laminar flow the fluid particles move along straight parallel path in layers or laminae, such that the paths of individual fluid particles do not cross those of neighbouring particles. Laminar flow is possible only at low velocities and when the fluid is highly viscous. But when the velocity is increased or fluid is less viscous, the fluid particles do not move in straight paths. The fluid particles move in random manner resulting in general mixing of the particles. This type of flow is called turbulent flow.

A laminar flow changes to turbulent flow when (*i*) velocity is increased or (*ii*) diameter of a pipe is increased or (*iii*) the viscosity of fluid is decreased. O. Reynold was first to demonstrate that the transition from laminar to turbulent depends not only on the mean velocity but on the quantity $\dfrac{\rho VD}{\mu}$. This quantity $\dfrac{\rho VD}{\mu}$ is a dimensionless quantity and is called Reynolds number (R_e). In case of circular pipe if $R_e < 2000$ the flow is said to be laminar and if $R_e > 4000$, the flow is said to be turbulent. If R_e lies between 2000 to 4000, the flow changes from laminar to turbulent.

▶ 10.2 REYNOLDS EXPERIMENT

The type of flow is determined from the Reynolds number *i.e.*, $\dfrac{\rho V \times d}{\mu}$. This was demonstrated by O. Reynold in 1883. His apparatus is shown in Fig. 10.1.

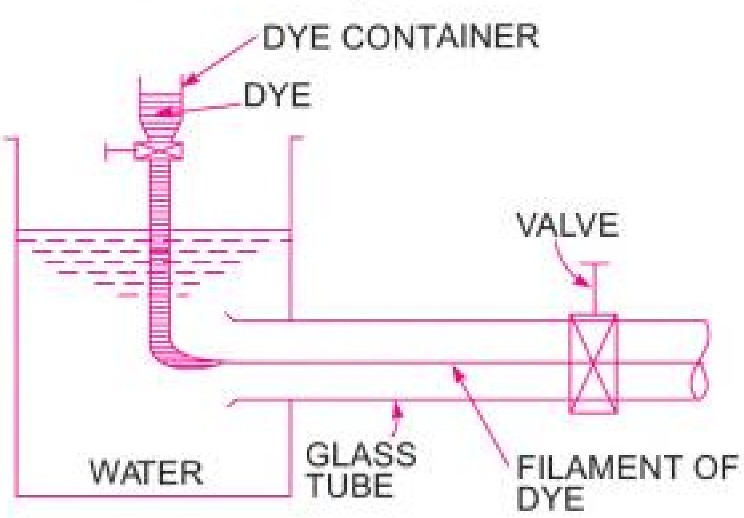


Fig. 10.1 *Reynold apparatus.*

The apparatus consists of :

(*i*) A tank containing water at constant head,

(*ii*) A small tank containing some dye,

(*iii*) A glass tube having a bell-mouthed entrance at one end and a regulating value at other ends.

The water from the tank was allowed to flow through the glass tube. The velocity of flow was varied by the regulating valve. A liquid dye having same specific weight as water was introduced into the glass tube as shown in Fig. 10.1.

The following observations were made by Reynold :

(*i*) When the velocity of flow was low, the dye filament in the glass tube was in the form of a straight line. This straight line of dye filament was parallel to the glass tube, which was the case of laminar flow as shown in Fig. 10.2 (*a*).

(*ii*) With the increase of velocity of flow, the dye-filament was no longer a straight-line but it became a wavy one as shown in Fig. 10.2 (*b*). This shows that flow is no longer laminar.

(*iii*) With further increase of velocity of flow, the wavy dye-filament broke-up and finally diffused in water as shown in Fig. 10.2 (*c*). This means that the fluid particles of the dye at this higher velocity are moving in random fashion, which shows the case of turbulent flow. Thus in case of turbulent flow the mixing of dye-filament and water is intense and flow is irregular, random and disorderly.

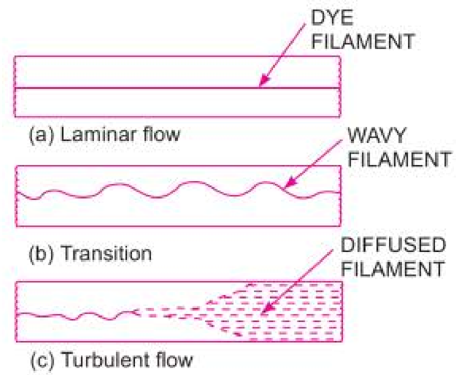


Fig. 10.2 *Different stages of filament.*

In case of laminar flow, the loss of pressure head was found to be proportional to the velocity but in case of turbulent flow, Reynold observed that loss of head is approximately proportional to the square of velocity. More exactly the loss of head, $h_f \propto V^n$, where n varies from 1.75 to 2.0

▶ 10.3 FRICTIONAL LOSS IN PIPE FLOW

When a liquid is flowing through a pipe, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity. This viscous action causes loss of energy which is usually known as frictional loss.

On the basis of his experiments, William Froude gave the following laws of fluid fraction for turbulent flow.

The frictional resistance for turbulent flow is :

(*i*) proportional to V^n, where n varies from 1.5 to 2.0,

(*ii*) proportional to the density of fluid,

(*iii*) proportional to the area of surface in contact,

(*iv*) independent of pressure,

(*v*) dependent on the nature of the surface in contact.

10.3.1 Expression for Loss of Head Due to Friction in Pipes. Consider a uniform horizontal pipe, having steady flow as shown in Fig. 10.3. Let 1-1 and 2-2 are two sections of pipe.

Let p_1 = pressure intensity at section 1-1,

V_1 = velocity of flow at section 1-1,

L = length of the pipe between sections 1-1 and 2-2,

d = diameter of pipe,

f' = frictional resistance per unit wetted area per unit velocity,

h_f = loss of head due to friction,

and p_2, V_2 = are values of pressure intensity and velocity at section 2-2.

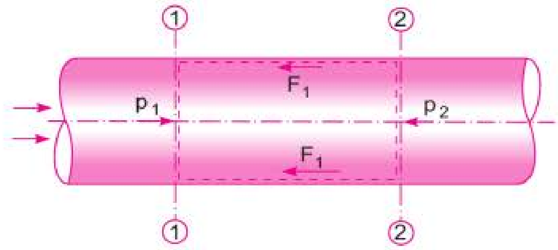


Fig. 10.3 *Uniform horizontal pipe.*

Applying Bernoulli's equations between sections 1-1 and 2-2,

Total head at 1-1 = Total head at 2-2 + loss of head due to friction between 1-1 and 2-2

or
$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f$$

But $z_1 = z_2$ as pipe is horizontal

$V_1 = V_2$ as dia. of pipe is same at 1-1 and 2-2

$\therefore$
$$\frac{p_1}{\rho g} = \frac{p_2}{\rho g} + h_f \text{ or } h_f = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \qquad ...(i)$$

But h_f is the head lost due to friction and hence intensity of pressure will be reduced in the direction of flow by frictional resistance.

Now frictional resistance = frictional resistance per unit wetted area per unit velocity × wetted area × velocity^2

or
$$F_1 = f' \times \pi dL \times V^2 \quad [\because \text{ wetted area} = \pi d \times L, \text{ velocity} = V = V_1 = V_2]$$
$$= f' \times P \times L \times V^2 \qquad [\because \pi d = \text{Perimeter} = P] \quad ...(ii)$$

The forces acting on the fluid between sections 1-1 and 2-2 are :

1. pressure force at section 1-1 = $p_1 \times A$

where A = Area of pipe

2. pressure force at section 2-2 = $p_2 \times A$
3. frictional force F_1 as shown in Fig. 10.3.

Resolving all forces in the horizontal direction, we have

$$p_1A - p_2A - F_1 = 0 \qquad ...(10.1)$$

or
$$(p_1 - p_2)A = F_1 = f' \times P \times L \times V^2 \qquad [\because \text{ From } (ii), F_1 = f'PLV^2]$$

or
$$p_1 - p_2 = \frac{f' \times P \times L \times V^2}{A}$$

But from equation (*i*), $p_1 - p_2 = \rho g h_f$

Equating the value of $(p_1 - p_2)$, we get

$$\rho g h_f = \frac{f' \times P \times L \times V^2}{A}$$

or

$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2 \qquad ...(iii)$$

In equation (*iii*), $\frac{P}{A} = \frac{\text{Wetted perimeter}}{\text{Area}} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$

$\therefore$

$$h_f = \frac{f'}{\rho g} \times \frac{4}{d} \times L \times V^2 = \frac{f'}{\rho g} \times \frac{4LV^2}{d} \qquad ...(iv)$$

Putting $\frac{f'}{\rho} = \frac{f}{2}$, where f is known as co-efficient of friction.

Equation (*iv*), becomes as $h_f = \frac{4 \cdot f}{2g} \cdot \frac{LV^2}{d} = \frac{4f \cdot L \cdot V^2}{d \times 2g}$ ...(10.2)

Equation (10.2) is known as Darcy-Weisbach equation. This equation is commonly used for finding loss of head due to friction in pipes.

Sometimes equation (10.2) is written as

$$h_f = \frac{f \cdot L \cdot V^2}{d \times 2g} \qquad ...(10.2A)$$

Then f is known as friction factor.

10.3.2 Expression for Co-efficient of Friction in Terms of Shear Stress. The equation (10.1) gives the forces acting on a fluid between sections 1-1 and 2-2 of Fig. 10.3 in horizontal direction as

$$p_1A - p_2A - F_1 = 0$$

or

$$\begin{aligned}(p_1 - p_2)A &= F_1 = \text{force due to shear stress } \tau_0 \\ &= \text{shear stress} \times \text{surface area} \\ &= \tau_0 \times \pi d \times L\end{aligned}$$

or

$$(p_1 - p_2)\frac{\pi}{4} d^2 = \tau_0 \times \pi d \times L \qquad \left\{\because A = \frac{\pi}{4} d^2\right\}$$

Cancelling πd from both sides, we have

$$(p_1 - p_2)\frac{d}{4} = \tau_0 \times L$$

or

$$(p_1 - p_2) = \frac{4\tau_0 \times L}{d} \qquad ...(10.3)$$

Equation (10.2) can be written as $h_f = \frac{p_1 - p_2}{\rho g} = \frac{4f \cdot L \cdot V^2}{d \times 2g}$

or $$(p_1 - p_2) = \frac{4f \cdot L \cdot V^2}{d \times 2g} \times \rho g \qquad ...(10.4)$$

Equating the value of $(p_1 - p_2)$ in equations (10.3) and (10.4),

$$\frac{4\tau_0 \times L}{d} = \frac{4f \cdot L \cdot V^2}{d \times 2g} \times \rho g$$

or $$\tau_0 = \frac{fV^2 \times \rho g}{2g} = \frac{fV^2}{2g} \times \rho g$$

or $$\tau_0 = f\,\frac{\rho V^2}{2} \qquad ...(10.5)$$

$\therefore$ $$f = \frac{2\tau_0}{\rho V^2}. \qquad ...(10.6)$$

▶ 10.4 SHEAR STRESS IN TURBULENT FLOW

The shear stress in viscous flow is given by Newton's law of viscosity as

$$\tau_v = \mu\,\frac{du}{dy},$$ where τ_v = shear stress due to viscosity.

Similar to the expression for viscous shear, J. Boussinesq expressed the turbulent shear in mathematical form as

$$\tau_t = \eta\,\frac{d\bar{u}}{dy} \qquad ...(10.7)$$

where τ_t = shear stress due to turbulence

η = eddy viscosity

$\bar{u}$ = average velocity at a distance y from boundary.

The ratio of η (eddy viscosity) and ρ (mass density) is known as kinematic eddy viscosity and is denoted by ε (epsilon). Mathematically it is written as

$$\varepsilon = \frac{\eta}{\rho} \qquad ...(10.8)$$

If the shear stress due to viscous flow is also considered, then the total shear stress becomes as

$$\tau = \tau_v + \tau_t = \mu\,\frac{du}{dy} + \eta\,\frac{d\bar{u}}{dy} \qquad ...(10.9)$$

The value of $\eta = 0$ for laminar flow. For other cases the value of η may be several thousand times the value of μ. To find shear stress in turbulent flow, equation (10.7) given by Boussinesq is used. But as the value of η (eddy viscosity) cannot be predicted, this equation is having limited use.

10.4.1 Reynolds Expression for Turbulent Shear Stress. Reynolds in 1886 developed an expression for turbulent shear stress between two layers of a fluid at a small distance apart, which is given as

$$\tau = \rho u' v' \qquad ...(10.10)$$

where u', v' = fluctuating component of velocity in the direction of x and y due to turbulence.

As u' and v' are varying and hence τ will also vary. Hence to find the shear stress, the time average on both the sides of the equation (10.10) is taken. Then equation (10.10) becomes as

$$\bar{\tau} = \overline{\rho u' v'} \quad ...(10.11)$$

The turbulent shear stress given by equation (10.11) is known as Reynold stress.

10.4.2 Prandtl Mixing Length Theory for Turbulent Shear Stress. In equation (10.11), the turbulent shear stress can only be calculated if the value of $\overline{u' v'}$ is known. But it is very difficult to measure $\overline{u' v'}$. To overcome this difficulty, *L.* Prandtl in 1925, presented a mixing length hypothesis which can be used to express turbulent shear stress in terms of measurable quantities.

According to Prandtl, the mixing length l, is that distance between two layers in the transverse direction such that the lumps of fluid particles from one layer could reach the other layer and the particles are mixed in the other layer in such a way that the momentum of the particles in the direction of x is same. He also assumed that the velocity fluctuation in the x-direction u' is related to the mixing length l as

$$u' = l\,\frac{du}{dy}$$

and v', the fluctuation component of velocity in y-direction is of the same order of magnitude as u' and hence

$$v' = l\,\frac{du}{dy}$$

Now $\overline{u' \times v'}$ becomes as $\overline{u' v'} = \left(l\,\frac{du}{dy}\right) \times \left(l\,\frac{du}{dy}\right) = l^2 \left(\frac{du}{dy}\right)^2$

Substituting the value of $\overline{u' v'}$ in equation (10.11), we get the expression for shear stress in turbulent flow due to Prandtl as

$$\bar{\tau} = \rho l^2 \left(\frac{du}{dy}\right)^2 \quad ...(10.12)$$

Thus the total shear stress at any point in turbulent flow is the sum of shear stress due to viscous shear and turbulent shear and can be written as

$$\bar{\tau} = \mu\,\frac{du}{dy} + \rho l^2 \left(\frac{du}{dy}\right)^2 \quad ...(10.13)$$

But the viscous shear stress is negligible except near the boundary. Equation (10.13) is used for most of turbulent fluid flow problems for determining shear stress in turbulent flow.

▶ 10.5 VELOCITY DISTRIBUTION IN TURBULENT FLOW IN PIPES

In case of turbulent flow, the total shear stress at any point is the sum of viscous shear stress and turbulent shear stress. Also the viscous shear stress is negligible except near the boundary. Hence it can be assumed that the shear stress in turbulent flow is given by equation (10.12). From this equation, the velocity distribution can be obtained if the relation between l, the mixing length and y is known. Prandtl assumed that the mixing length, l is a linear function of the distance y from the pipe wall *i.e.*, $l = ky$, where k is a constant, known as Karman constant and = 0.4.

Substituting the value of l in equation (10.12), we get

$$\bar{\tau} \text{ or } \tau = \rho \times (ky)^2 \times \left(\frac{du}{dy}\right)^2$$

or
$$\tau = \rho k^2 y^2 \left(\frac{du}{dy}\right)^2 \text{ or } \left(\frac{du}{dy}\right)^2 = \tau/\rho k^2 y^2$$

or
$$\frac{du}{dy} = \sqrt{\frac{\tau}{\rho k^2 y^2}} = \frac{1}{ky}\sqrt{\frac{\tau}{\rho}} \qquad \text{...(10.14)}$$

For small values of y that is very close to the boundary of the pipe, Prandtl assumed shear stress τ to be constant and approximately equal to τ_0 which presents the turbulent shear stress at the pipe boundary. Substituting $\tau = \tau_0$ in equation (10.14), we get

$$\frac{du}{dy} = \frac{1}{ky}\sqrt{\frac{\tau_0}{\rho}} \qquad \text{...(10.15)}$$

In equation (10.15), $\sqrt{\frac{\tau_0}{\rho}}$ has the dimensions $\sqrt{\frac{ML^{-1}T^{-2}}{ML^{-3}}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$. But $\frac{L}{T}$ is velocity and hence $\sqrt{\frac{\tau_0}{\rho}}$ has the dimension of velocity, which is known as shear velocity and is denoted by u_*.

Thus $\sqrt{\frac{\tau_0}{\rho}} = u_*$, then equation (10.15) becomes $\frac{du}{dy} = \frac{1}{ky} u_*$.

For a given case of turbulent flow, u_* is constant. Hence integrating above equation, we get

$$u = \frac{u_*}{k} \log_e y + C \qquad \text{...(10.16)}$$

where C = constant of integration.

Equation (10.16) shows that in turbulent flow, the velocity varies directly with the logarithm of the distance from the boundary or in other words the velocity distribution in turbulent flow is logarithmic in nature. To determine the constant of integration, C the boundary condition that at $y = R$ (radius of pipe), $u = u_{max}$ is substituted in equation (10.16).

Hence
$$u_{max} = \frac{u_*}{k} \log_e R + C \quad \therefore \quad C = u_{max} - \frac{u_*}{k} \log_e R$$

Substituting the value of C in equation (10.16), we get

$$u = \frac{u_*}{k} \log_e y + u_{max} - \frac{u_*}{k} \log_e R = u_{max} + \frac{u_*}{k} (\log_e y - \log_e R)$$

$$= u_{max} + \frac{u_*}{0.4} \log_e (y/R) \qquad [\because k = 0.4 = \text{Karman constant}]$$

$$= u_{max} + 2.5\, u_* \log_e (y/R) \qquad \text{...(10.17)}$$

Equation (10.17) is called 'Prandtl's universal velocity distribution equation for turbulent flow in pipes. This equation is applicable to smooth as well as rough pipe boundaries. Equation (10.17) is also written as

$$u_{max} - u = -2.5\, u_* \log_e (y/R) = 2.5\, u_* \log_e (R/y)$$

Dividing by u_*, we get

$$\frac{u_{max} - u}{u_*} = 2.5 \log_e (R/y) = 2.5 \times 2.3 \log_{10} (R/y) \quad [\because \log_e (R/y) = 2.3 \log_{10} (R/y)]$$

or

$$\frac{u_{max} - u}{u_*} = 5.75 \log_{10} (R/y) \qquad \text{...(10.18)}$$

In equation (10.18), the difference between the maximum velocity u_{max}, and local velocity u at any point *i.e.*, $(u_{max} - u)$ is known as '**velocity defect**'.

10.5.1 Hydrodynamically Smooth and Rough Boundaries. Let k is the average height of the irregularities projecting from the surface of a boundary as shown in Fig. 10.4. If the value of k is large for a boundary then the boundary is called rough boundary and if the value of k is less, then boundary is known as smooth boundary, in general. This is the classification of rough and smooth boundary based on boundary characteristics. But for proper classification, the flow and fluid characteristics are also to be considered.

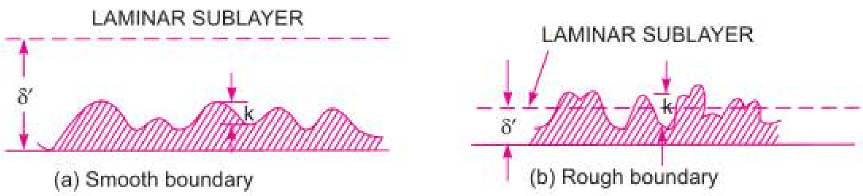


Fig. 10.4 *Smooth and rough boundaries.*

For turbulent flow analysis along a boundary, the flow is divided in two portions. The first portion consists of a thin layer of fluid in the immediate neighbourhood of the boundary, where viscous shear stress predominates while the shear stress due to turbulence is negligible. This portion is known as laminar sub-layer. The height upto which the effect of viscosity predominates in this zone is denoted by δ'. The second portion of flow, where shear stress due to turbulence are large as compared to viscous stress is known as turbulent zone.

If the average height k of the irregularities, projecting from the surface of a boundary is much less than δ', the thickness of laminar sub-layer as shown in Fig. 10.4 (*a*), the boundary is called smooth boundary. This is because, outside the laminar sub-layer the flow is turbulent and eddies of various size present in turbulent flow try to penetrate the laminar sub-layer and reach the surface of the boundary. But due to great thickness of laminar sub-layer the eddies are unable to reach the surface irregularities and hence the boundary behaves as a smooth boundary. This type of boundary is called hydrodynamically smooth boundary.

Now, if the Reynolds number of the flow is increased then the thickness of laminar sub-layer will decrease. If the thickness of laminar sub-layer becomes much smaller than the average height k of irregularities of the surface as shown in Fig. 10.4 (*b*), the boundary will act as rough boundary. This is because the irregularities of the surface are above the laminar sub-layer and the eddies present in turbulent zone will come in contact with the irregularities of the surface and lot of energy will be lost. Such a boundary is called hydrodynamically rough boundary.

From Nikuradse's experiment :

1. If $\frac{k}{\delta'}$ is less than 0.25 or $\frac{k}{\delta'} < 0.25$, the boundary is called smooth boundary.

2. If $\frac{k}{\delta'}$ is greater than 6.0, the boundary is rough,

3. If $0.25 < \left(\frac{k}{\delta'}\right) < 6.0$, the boundary is in transition.

In terms of roughness Reynolds number $\frac{u_* k}{\nu}$:

1. If $\frac{u_* k}{\nu} < 4$, boundary is considered smooth,

2. If $\frac{u_* k}{\nu}$ lies between 4 and 100, boundary is in transition stage, and

3. If $\frac{u_* k}{\nu} > 100$, the boundary is rough.

10.5.2 Velocity Distribution for Turbulent Flow in Smooth Pipes. The velocity distribution for turbulent flow in smooth or rough pipe is given by equation (10.16) as

$$u = \frac{u_*}{k} \log_e y + C$$

It may be seen that at $y = 0$, the velocity u at wall is $-\infty$. This means that velocity u is positive at some distance far away from the wall and $-\infty$ (minus infinity) at the wall. Hence at some finite distance from wall, the velocity will be equal to zero. Let this distance from pipe wall is y'. Now the constant C is determined from the boundary condition *i.e.,* at $y = y'$, $u = 0$. Hence above equation becomes as

$$0 = \frac{u_*}{k} \log_e y' + C \text{ or } C = -\frac{u_*}{k} \log_e y'$$

Substituting the value of C in the above equation, we get

$$u = \frac{u_*}{k} \log_e y - \frac{u_*}{k} \log_e y' = \frac{u_*}{k} \log_e (y/y')$$

Substituting the value of $k = 0.4$, we get

$$u = \frac{u_*}{0.4} \log_e (y/y') = 2.5\, u_* \log_e (y/y')$$

$$\frac{u}{u_*} = 2.5 \times 2.3 \log_{10} (y/y') \qquad [\because \log_e (y/y') = 2.3 \log_{10} (y/y')]$$

or

$$\frac{u}{u_*} = 5.75 \log_{10} (y/y') \qquad \text{...(10.19)}$$

For the smooth boundary, there exists a laminar sub-layer as shown in Fig. 10.4 (*a*). The velocity distribution in the laminar sub-layer is parabolic in nature. Thus in the laminar sub-layer, logarithmic velocity distribution does not hold good. Thus it can be assumed that y' is proportional to δ', where δ' is the thickness of laminar sub-layer. From Nikuradse's experiment the value of y' is given as

$$y' = \frac{\delta'}{107}$$

where $\delta' = \dfrac{11.6\nu}{u_*}$, where ν = kinematic viscosity of fluid.

$$\therefore \qquad y' = \frac{11.6\nu}{u_*} \times \frac{1}{107} = \frac{0.108\nu}{u_*}$$

Substituting this value of y' in equation (10.19), we obtain

$$\frac{u}{u_*} = 5.75 \log_{10} \left(\frac{y}{\frac{.108\nu}{u_*}} \right)$$

$$= 5.75 \log_{10} \left(\frac{y u^*}{.108\, \nu} \right) = 5.75 \log_{10} \left(\frac{u_* y}{\nu} \times 9.259 \right)$$

$$= 5.75 \log_{10} \frac{u_* y}{\nu} + 5.75 \log_{10} 9.259 \qquad \left[\because \frac{1}{0.108} = 9.259 \right]$$

$$= 5.75 \log_{10} \frac{u_* y}{\nu} + 5.55 \qquad \text{...(10.20)}$$

10.5.3 Velocity Distribution for Turbulent Flow in Rough Pipes. In case of rough boundaries, the thickness of laminar sub-layer is very small as shown in Fig. 10.4 (*b*). The surface irregularities are above the laminar sub-layer and hence the laminar sub-layer is completely destroyed. Thus y' can be considered proportional to the height of protrusions k. Nikuradse's experiment shows the value of y' for pipes coated with uniform sand (rough pipes) as $y' = \dfrac{k}{30}$.

Substituting this value of y' in equation (10.19), we get

$$\frac{u}{u_*} = 5.75 \log_{10} \left(\frac{y}{k/30} \right) = 5.75 \left[\log_{10} (y/k) \times 30 \right]$$

$$= 5.75 \log_{10} (y/k) + 5.75 \log_{10} (30.0) = 5.75 \log_{10} (y/k) + 8.5 \qquad \text{...(10.21)}$$

Problem 10.1 *A pipe-line carrying water has average height of irregularities projecting from the surface of the boundary of the pipe as 0.15 mm. What type of boundary is it ? The shear stress developed is 4.9 N/m². The kinematic viscosity of water is .01 stokes.*

Solution. Given :

Average height of irregularities, $k = 0.15 \text{ mm} = 0.15 \times 10^{-3}$ m

Shear stress developed, $\tau_0 = 4.9$ N/m^2

Kinematic viscosity, $\nu = 0.01$ stokes $= .01$ cm^2/s $= .01 \times 10^{-4}$ m^2/s

Density of water, $\rho = 1000$ kg/m^3

Shear velocity, $u_* = \sqrt{\tau_0 / \rho} = \sqrt{\dfrac{4.9}{1000}} = \sqrt{0.0049} = 0.07$ m/s

Roughness Reynold number $= \frac{u_* k}{\nu} = \frac{0.07 \times 0.15 \times 10^{-3}}{.01 \times 10^{-4}} = 10.5.$

Since $\frac{u_* k}{\nu}$ lies between 4 and 100 and hence pipe surface behaves as in transition.

Problem 10.2 *A rough pipe is of diameter 8.0 cm. The velocity at a point 3.0 cm from wall is 30% more than the velocity at a point 1 cm from pipe wall. Determine the average height of the roughness.*

Solution. Given :

Dia. of rough pipe, $D = 8 \text{ cm} = .08 \text{ m}$

Let velocity of flow at 1 cm from pipe wall $= u$

Then velocity of flow at 3 cm from pipe wall $= 1.3\, u$

The velocity distribution for rough pipe is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5, \text{ where } k = \text{height of roughness.}$$

For a point, 1 cm from pipe wall, we have

$$\frac{u}{u_*} = 5.75 \log_{10} (1.0/k) + 8.5 \qquad ...(i)$$

For a point, 3 cm from pipe wall, velocity is 1.3 u and hence

$$\frac{1.3u}{u_*} = 5.75 \log_{10} (3.0/k) + 8.5 \qquad ...(ii)$$

Dividing (*ii*) by (*i*), we get $1.3 = \frac{5.75 \log_{10}(3.0/k) + 8.5}{5.75 \log_{10}(1/k) + 8.5}$

or $1.3[5.75 \log_{10} (1/k) + 8.5] = 5.75 \log_{10} (3.0/k) + 8.5$

or $7.475 \log_{10} (1/k) + 11.05 = 5.75 \log_{10} (3.0/k) + 8.5$

or $7.475 \log_{10} (1/k) - 5.75 \log_{10} (3/k) = 8.5 - 11.05 = -2.55$

or $7.475 [\log_{10} 1.0 - \log_{10} k] - 5.75 [\log_{10} 3.0 - \log_{10} k] = -2.55$

or $7.475 [0 - \log_{10} k] - 5.75 [.4771 - \log_{10} k] = -2.55$

or $-7.475 \log_{10} k - 2.7433 + 5.75 \log_{10} k = -2.55$

or $-1.725 \log_{10} k = 2.7433 - 2.55 = 0.1933$

or $\log_{10} k = \frac{0.1933}{-1.725} = -0.1120 = \bar{1}.888$

$k =$ **.7726 cm. Ans.**

Problem 10.3 *A smooth pipe of diameter 80 mm and 800 m long carries water at the rate of 0.480 m³/minute. Calculate the loss of head, wall shearing stress, centre line velocity, velocity and shear stress at 30 mm from pipe wall. Also calculate the thickness of laminar sub-layer. Take kinematic viscosity of water as 0.015 stokes. Take the value of co-efficient of friction 'f' from the relation given as*

$$f = \frac{.0791}{(R_e)^{1/4}}, \text{ where } R_e = \text{Reynolds number.}$$

Solution. Given :

Dia. of smooth pipe, $d = 80 \text{ mm} = .08 \text{ m}$

Length of pipe, $L = 800 \text{ m}$

Discharge, $Q = 0.048 \text{ m}^3/\text{minute} = \dfrac{0.48}{60} = .008 \text{ m}^3/\text{s}$

Kinematic viscosity, $\nu = .015 \text{ stokes} = .015 \times 10^{-4} \text{ m}^2/\text{s}$ [Stokes = cm^2/s]

Density of water, $\rho = 1000 \text{ kg/m}^3$

Mean velocity, $V = \dfrac{Q}{\text{Area}} = \dfrac{0.008}{\frac{\pi}{4}(.08)^2} = 1.591 \text{ m/s}$

$\therefore$ Reynolds number, $R_e = \dfrac{V \times d}{\nu} = \dfrac{1.591 \times 0.08}{.015 \times 10^{-4}} = 8.485 \times 10^4$

As the Reynolds number is more than 4000, the flow is turbulent.

Now the value of 'f' is given by $f = \dfrac{.0791}{R_e^{1/4}} = \dfrac{.0791}{\left(8.485 \times 10^4\right)^{1/4}} = .004636$

(i) Head lost is given by equation (10.2) as

$$h_f = \frac{4 \,.\, f \,.\, L \,.\, V^2}{d \times 2g} = \frac{4 \times .004636 \times 800 \times 1.591^2}{.08 \times 2 \times 9.81} = \textbf{23.42 m. Ans.}$$

(ii) Wall shearing stress, τ_0 is given by equation (10.5) as

$$\tau_0 = \frac{f \rho V^2}{2} = .004636 \times \frac{1000}{2} \times 1.591^2 = \textbf{5.866 N/m}^2\textbf{. Ans.}$$

(iii) Centre-line velocity, u_{max} for smooth pipe is given by equation (10.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{\nu} + 5.55 \qquad ...(i)$$

where u_* is shear velocity and $= \sqrt{\dfrac{\tau_0}{\rho}} = \sqrt{\dfrac{5.866}{1000}} = 0.0765 \text{ m/s}$

The velocity will be maximum when $y = \dfrac{d}{2} = \dfrac{.08}{2} = .04 \text{ m}$.

Hence at $y = .04$ m, $u = u_{max.}$ Substituting these values in (i), we get

$$\frac{u_{max}}{.0765} = 5.75 \log_{10} \frac{0.0765 \times .04}{.015 \times 10^{-4}} + 5.55$$

$$= 5.75 \log_{10} 2040 + 5.55$$

$$= 5.75 \times 3.309 + 5.55 = 19.03 + 5.55 = 24.58$$

$\therefore$ $u_{max} = .0765 \times 24.58 = \textbf{1.88 m/s. Ans.}$

(iv) The shear stress, τ at any point is given by

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \qquad ...(A)$$

where r = distance from centre of pipe
and hence shear stress at pipe wall where $r = R$ is

$$\tau_0 = -\frac{\partial p}{\partial x}\frac{R}{2} \qquad \text{...}(B)$$

Dividing equation (A) by equation (B), we get

$$\frac{\tau}{\tau_0} = \frac{r}{R}$$

∴ Shear stress $\tau = \frac{\tau_0 r}{R}$

A point 30 mm from pipe wall is having $r = 4 - 3 = 1$ cm = .01 m

∴ τ at (r = .01 m) $= \frac{\tau_0 \times .01}{.04} = \frac{5.866}{4}$ = **1.4665 N/m². Ans.**

Velocity at a point 3 cm from pipe wall means $y = 3$ cm = .03 m

and is given by equation (10.20) as $\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{\nu} + 5.55$, where $u_* = .0765$, $y = .03$

$$\therefore \quad \frac{u}{.0765} = 5.75 \log_{10} \frac{.0765 \times .03}{.015 \times 10^{-4}} + 5.55$$

$$= 5.75 \log_{10} 1530 + 5.55 = 23.86$$

∴ $u = 0.0765 \times 23.86$ = **1.825 m/s. Ans.**

(v) Thickness of laminar sub-layer is given by

$$\delta' = \frac{11.6 \times \nu}{u_*} = \frac{11.6 \times .015 \times 10^{-4}}{.0765} = 2.274 \times 10^{-4} \text{ m}$$

$= 2.274 \times 10^{-2}$ cm = **.02274 cm. Ans.**

Problem 10.4 *Determine the wall shearing stress in a pipe of diameter 100 mm which carries water. The velocities at the pipe centre and 30 mm from the pipe centre are 2 m/s and 1.5 m/s respectively. The flow in pipe is given as turbulent.*

Solution. Given :

Dia. of pipe, D = 100 mm = 0.10 m

∴ Radius, $R = \frac{0.10}{2} = 0.05$ m

Velocity at centre, $u_{max} = 2$ m/s

Velocity at 30 mm or 0.03 m from centre = 1.5 m/s

∴ Velocity (at r = 0.03 m), u = 1.5 m/s

Let the wall shearing stress $= \tau_0$

For turbulent flow, the velocity distribution in terms of centre line velocity (u_{max}) is given by equation (10.18) as

$$\frac{u_{max} - u}{u_*} = 5.75 \log_{10}\left(\frac{R}{y}\right)$$

where u = 1.5 m/s at $y = (R - r) = 0.05 - 0.03 = .02$ m

$\therefore$ $$\frac{2.0 - 1.5}{u_*} = 5.75 \log_{10} \frac{.05}{.02} = 2.288 \text{ or } \frac{0.5}{u_*} = 2.288$$

$\therefore$ $$u_* = \frac{0.5}{2.288} = 0.2185 \text{ m/s}$$

Using the relation $$u_* = \sqrt{\tau_0 / \rho}, \text{ where } \rho \text{ for water} = 1000 \text{ kg/m}^3$$

$\therefore$ $$0.2185 = \sqrt{\frac{\tau_0}{1000}} \text{ or } \frac{\tau_0}{1000} = 0.2185^2 = 0.0477$$

or $$\tau_0 = 0.0477 \times 1000 = \mathbf{47.676 \text{ N/m}^2}. \text{ Ans.}$$

10.5.4 Velocity Distribution for Turbulent Flow in Terms of Average Velocity. The average velocity $\overline{U}$, through the pipe is obtained by first finding the total discharge Q and then dividing the total discharge by the area of the pipe.

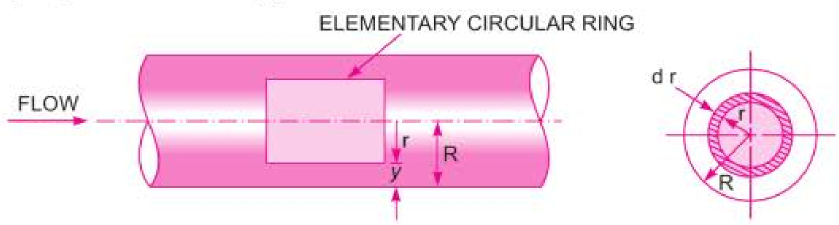


Fig. 10.5 *Average velocity for turbulent flow.*

Consider an elementary circular ring of radius 'r' and thickness dr as shown in Fig. 10.5. The distance of the ring from pipe wall is $y = (R - r)$, where R = radius of pipe.

Then the discharge, dQ, through the ring is given by

$$dQ = \text{area of ring} \times \text{velocity}$$
$$= 2\pi r dr \times u = u \times 2\pi r dr$$

Total discharge, $$Q = \int dQ = \int_0^R u \times 2\pi r dr \qquad \text{...(10.22)}$$

(*a*) **For smooth pipes.** For smooth pipes, the velocity distribution is given by equation (10.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{\nu} + 5.5$$

or $$u = \left[5.75 \log_{10} \frac{u_* y}{\nu} + 5.5\right] \times u_*$$

But $$y = (R - r)$$

$\therefore$ $$u = \left[5.75 \log_{10} \frac{u_*(R - r)}{\nu} + 5.5\right] \times u_*$$

Substituting the value of u in equation (10.22), we get

$$Q = \int_0^R \left[5.75 \log_{10} \frac{u_*(R - r)}{\nu} + 5.5\right] u_* \times 2\pi r dr$$

$\therefore$ Average velocity, $\overline{U} = \dfrac{Q}{\text{Area}} = \dfrac{Q}{\pi R^2}$

$$= \frac{1}{\pi R^2} \int_0^R \left[5.75 \log_{10} \frac{u_*(R-r)}{\nu} + 5.5 \right] u_* \, 2\pi r dr$$

Integration of the above equation and subsequent simplification gives the average velocity for turbulent flow in smooth pipes as

$$\frac{\overline{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{\nu} + 1.75 \qquad \text{...(10.23)}$$

(*b*) **For rough pipes.** For rough pipes, the velocity at any point in turbulent flow is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

But $\qquad y = (R - r)$

$\therefore$
$$\frac{u}{u_*} = 5.75 \log_{10} \left(\frac{R-r}{k} \right) + 8.5$$

or
$$u = u_* \left[5.75 \log_{10} \left(\frac{R-r}{k} \right) + 8.5 \right]$$

Substituting the value of u in equation (10.22), we get

$$Q = \int_0^R u_* \left[5.75 \log_{10} \left(\frac{R-r}{k} \right) + 8.5 \right] 2\pi r dr$$

$\therefore$ Average velocity,
$$\overline{U} = \frac{Q}{\pi R^2} = \frac{\displaystyle\int_0^R u_* \left[5.75 \log_{10} \left(\frac{R-r}{k} \right) + 8.5 \right] 2\pi r dr}{\pi R^2}$$

Integration of the above equation and subsequent simplification will give the following relation for average velocity, $\overline{U}$ for turbulent flow in rough pipe as

$$\frac{\overline{U}}{u_*} = 5.75 \log_{10} \frac{R}{k} + 4.75 \qquad \text{...(10.24)}$$

(*c*) **Difference of the velocity at any point and average velocity for smooth and rough pipes.** The velocity at any point for turbulent flow for smooth pipes is given by equation (10.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_*(R-r)}{\nu} + 5.5 \qquad [\because y = R - r]$$

and the average velocity is given by equation (10.23) as

$$\frac{\overline{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{\nu} + 1.75$$

$\therefore$ Difference of velocity u and $\overline{U}$ for smooth pipe is obtained as

$$\frac{u}{u_*} - \frac{\overline{U}}{u_*} = \left[5.75 \log_{10} \frac{u_*(R-r)}{\nu} + 5.5\right] - \left[5.75 \log_{10} \frac{u_* R}{\nu} + 1.75\right]$$

or
$$\frac{u - \overline{U}}{u_*} = 5.75 \left[\log_{10} \frac{u_*(R-r)}{\nu} - \log_{10} \frac{u_* R}{\nu}\right] + 5.5 - 1.75$$

$$= 5.75 \log_{10} \left[\frac{u_*(R-r)}{\nu} \div \frac{u_* R}{\nu}\right] + 3.75$$

$$= 5.75 \log_{10} \left(\frac{R-r}{\nu}\right) + 3.75$$

$$= 5.75 \log_{10} (y/R) + 3.75 \qquad \text{...(10.25)} \quad [\because R - r = y]$$

Similarly the velocity, u at any point for rough pipe is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

and average velocity is given by equation (10.24) as

$$\frac{\overline{U}}{u_*} = 5.75 \log_{10} (R/k) + 4.75$$

$\therefore$ Difference of velocity u and $\overline{U}$ for rough pipe is given by

$$\frac{u}{u_*} - \frac{\overline{U}}{u_*} = [5.75 \log_{10} (y/k) + 8.5] - [5.75 \log_{10} (R/k) + 4.75]$$

$$= 5.75 \log_{10} [(y/k) \div (R/k)] + 8.5 - 4.75$$

or
$$\frac{u - \overline{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75 \qquad \text{...(10.26)}$$

Equations (10.25) and (10.26) are the same. This shows that the difference of velocity at any point and the average velocity will be the same in case of smooth as well as rough pipes.

Problem 10.5 *Determine the distance from the pipe wall at which the local velocity is equal to the average velocity for turbulent flow in pipes.*

Solution. Given :

Local velocity at a point = average velocity

or $u = \overline{U}$

For a smooth or rough pipe, the difference of velocity at any point and average velocity is given by equation (10.25) or equation (10.26) as

$$\frac{u - \overline{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75$$

Substituting the given condition *i.e.,* $u = \overline{U}$, we get

$$\frac{\overline{U} - \overline{U}}{u_*} = 0 = 5.75 \log_{10} (y/R) + 3.75 \quad \text{or} \quad 5.75 \log_{10} (y/R) = 3.75$$

or $$\log_{10}(y/R) = -\frac{3.75}{5.75} = -0.6521 = -\bar{1}.3479$$

$\therefore$ $$y/R = 0.22279 \simeq 0.2228 \text{ or } y = \mathbf{.2228\ R.}\ \textbf{Ans.}$$

Problem 10.6 *For turbulent flow in a pipe of diameter 300 mm, find the discharge when the centre-line velocity is 2.0 m/s and the velocity at a point 100 mm from the centre as measured by pitot-tube is 1.6 m/s.*

Solution. Given :

Dia. of pipe, $D = 300 \text{ mm} = 0.3 \text{ m}$

$\therefore$ Radius, $R = \frac{0.3}{2} = 0.15 \text{ m}$

Velocity at centre, $u_{max} = 2.0 \text{ m/s}$

Velocity (at $r = 100 \text{ mm} = 0.1 \text{ m}$), $u = 1.6 \text{ m/s}$

Now $y = R - r = 0.15 - 0.10 = 0.05 \text{ m}$

$\therefore$ Velocity (at $r = 0.1$ m or at $y = 0.05$ m), $u = 1.6$ m/s

The velocity in terms of centre-line velocity is given by equation (10.18) as

$$\frac{u_{max} - u}{u_*} = 5.75 \log_{10}(R/y)$$

Substituting the values, we get $$\frac{2.0 - 1.6}{u_*} = 5.75 \log_{10} \frac{.15}{.05} \qquad \left[\because \begin{array}{l} y = .05 \text{ m} \\ R = 0.15 \text{ m} \end{array}\right]$$

$$= 5.75 \log_{10} 3.0 = 2.7434$$

or $$\frac{0.4}{u_*} = 2.7434$$

$\therefore$ $$u_* = \frac{0.4}{2.7434} = 0.1458 \text{ m/s} \qquad ...(i)$$

Using equation (10.26) which gives relation between velocity at any point and average velocity, we have

$$\frac{u - \bar{U}}{u_*} = 5.75 \log_{10}(y/R) + 3.75$$

at $y = R$, velocity u becomes $= u_{max}$

$\therefore$ $$\frac{u_{max} - \bar{U}}{u_*} = 5.75 \log_{10}(R/R) + 3.75 = 5.75 \times 0 + 3.75 = 3.75$$

But $u_{max} = 2.0$ and u_* from $(i) = 0.1458$

$\therefore$ $$\frac{2.0 - \bar{U}}{0.1458} = 3.75$$

or $$\bar{U} = 2.0 - .1458 \times 3.75 = 2.0 - 0.5467 = 1.4533 \text{ m/s}$$

$\therefore$ Discharge, $Q = \text{Area} \times \text{average velocity}$

$$= \frac{\pi}{4} D^2 \times \overline{U} = \frac{\pi}{4} (0.3)^2 \times 1.4533 = \mathbf{0.1027\ m^3/s.}\ \textbf{Ans.}$$

10.5.5 Velocity Distribution for Turbulent Flow in Smooth Pipes by Power Law. The velocity distribution for turbulent flow as given by equations (10.18), (10.20) and (10.21) are logarithmic in nature. These equations are not convenient to use. Nikuradse carried out experiments for different Reynolds number to determine the velocity distribution law in smooth pipes. He expressed the velocity distribution in exponential form as

$$\frac{u}{u_{max}} = (y/R)^{1/n} \qquad \text{...(10.27)}$$

where exponent $\frac{1}{n}$ depends on Reynolds number

The value of $\left(\frac{1}{n}\right)$ decreases, with increasing Reynolds number.

For $R_e = 4 \times 10^3$, $\frac{1}{n} = \frac{1}{6}$

$R_e = 1.1 \times 10^5$, $\frac{1}{n} = \frac{1}{7}$

$R_e \geq 2 \times 10^6$, $\frac{1}{n} = \frac{1}{10}$

Thus if $\frac{1}{n} = \frac{1}{7}$, the velocity distribution law becomes as

$$\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^{1/7} \qquad \text{...(10.28)}$$

Equation (10.28) is known as 1/7th power law of velocity distribution for smooth pipes.

10.6 RESISTANCE OF SMOOTH AND ROUGH PIPES

The loss of head, due to friction in pipes is given by equation (10.2) as

$$h_f = \frac{4 \,.\, f \,.\, L \,.\, V^2}{d \times 2g}$$

In this equation, the value of co-efficient of friction, f should be known accurately for predicting the loss of head due to friction in pipes. On the basis of dimensional analysis, it can be shown that the pressure loss in a straight pipe of diameter D, length L, roughness k, average velocity of flow $\overline{U}$, viscosity and density of fluid μ and ρ is

$$\Delta p = \frac{\rho \overline{U}^2}{2} \phi \left[R_e, \frac{k}{D}, \frac{L}{D}\right] \quad \text{or} \quad \frac{\Delta p}{\frac{\rho \overline{U}^2}{2}} = \phi \left[R_e, \frac{k}{D}, \frac{L}{D}\right]$$

Experimentally it was found that pressure drop is a function of $\frac{L}{D}$ to the first power and hence

$$\frac{\Delta p}{\frac{\rho \bar{U}^2}{2}} = \frac{L}{D} \phi \left[R_e, \frac{k}{D}\right] \quad \text{or} \quad \frac{\Delta p \times D}{L \frac{\rho \bar{U}^2}{2}} = \phi \left[R_e, \frac{k}{D}\right]$$

The term of the right hand side is called co-efficient of friction f. Thus $f = \phi \left[R_e, \frac{k}{D}\right]$

This equation shows that friction co-efficient is a function of Reynolds number and k/D ratio, where k is the average height of pipe wall roughness protrusions.

(*a*) **Variation of 'f' for Laminar Flow.** In viscous flow chapter, it is shown that co-efficient of friction 'f' for laminar flow in pipes is given by

$$f = \frac{16}{R_e} \quad \text{...(10.29)}$$

Thus friction co-efficient is only a function of Reynolds number in case of laminar flow. It is independent of (k/D) ratio.

(*b*) **Variation of 'f' for Turbulent Flow.** For turbulent flow, the co-efficient of friction is a function of R_e and k/D ratio. For relative roughness (k/D), in the turbulent flow the boundary may be smooth or rough and hence the value of 'f' will be different for these boundaries.

(*i*) **'f' for smooth pipes.** For turbulent flow in smooth pipes, co-efficient of friction is a function of Reynolds number only. The value of laminar sub-layer in case of smooth pipe is large as compared to the average height of surface roughness k. The value of 'f' for smooth pipe for Reynolds number varying from 4000 to 100000 is given by the relation

$$f = \frac{.0791}{(R_e)^{1/4}} \quad \text{...(10.30)}$$

The equation (10.30) is given by Blasius.

The value of 'f' for $R_e > 10^5$ is obtained from equation (10.23) which gives the velocity distribution for smooth pipe in terms of average velocity $(\bar{U})$ as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{(u_* R)}{\nu} + 1.75 \quad \text{...(10.31)}$$

From equation (10.6), we have $f = \frac{2\tau_0}{\rho V^2}$, where V = average velocity

$$\therefore \quad f = \frac{2\tau_0}{\rho \bar{U}^2} = \frac{2}{\bar{U}^2} \left(\sqrt{\frac{\tau_0}{\rho}}\right)^2 = \frac{2}{\bar{U}^2} \times u_*^2 \qquad \left[\because \sqrt{\frac{\tau_0}{\rho}} = u_*\right]$$

$$\therefore \quad u_*^2 = \frac{f \bar{U}^2}{2}$$

or

$$u_* = \bar{U} \sqrt{\frac{f}{2}} \quad \text{...(10.31A)}$$

Substituting the value of u_* in equation (10.31), we get

$$\frac{\bar{U}}{\bar{U}\sqrt{f/2}} = 5.75 \log_{10} \left(\frac{\sqrt{f/2}}{\nu}\right) R + 1.75$$

or
$$\frac{1}{\sqrt{f/2}} = 5.75 \log_{10}\left(\frac{\overline{U}R}{\nu}\sqrt{f/2}\right) + 1.75$$

Taking $R = D/2$ and simplifying, the above equation is written as

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}\left(\frac{\overline{U}D}{\nu}\sqrt{4f}\right) - 0.91$$

But $\frac{\overline{U}D}{\nu} = R_e$ and hence above equation is written as

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}(R_e\sqrt{4f}) - 0.91 \quad \text{...(10.32)}$$

Equation (10.32) is valid upto $R_e = 4 \times 10^6$

Nikuradse's experimental result for turbulent flow in smooth pipe for 'f' is

$$\frac{1}{\sqrt{4f}} = 2.0 \log_{10}(R_e\sqrt{4f}) - 0.8 \quad \text{...(10.33)}$$

This is applicable upto $R_e = 4 \times 10^7$. But the equation (10.33) is solved by hit and trial method. The value of 'f' (*i.e.*, co-efficient of friction) can alternately be obtained as

$$f = .0008 + \frac{.05525}{(R_e)^{0.237}} \quad \text{...(10.34)}$$

The value of 'f'[*i.e.*, friction factor which is used in equation (10.2*A*)] is given by

$$f = 0.0032 + \frac{0.221}{(R_e)^{0.237}} \quad \text{...(10.34A)}$$

(*ii*) **Value of 'f' for rough pipes.** For turbulent flow in rough pipes, the co-efficient of friction is a function of relative roughness (k/D) and it is independent of Reynolds number. This is because the value of laminar sub-layer for rough pipes is very small as compared to the height of surface roughness. The average velocity for rough pipes is given by (10.24) as

$$\frac{\overline{U}}{u_*} = 5.75 \log_{10}(R/k) + 4.75$$

But
$$u_* = \overline{U}\sqrt{f/2}$$

Substituting the value of u_* in the above equation, we get

$$\frac{\overline{U}}{\overline{U}\sqrt{f/2}} = 5.75 \log_{10}(R/k) + 4.75$$

which is simplified to the form as $\frac{1}{\sqrt{4f}} = 2.03 \log_{10}(R/k) + 1.68$ ...(10.35)

But Nikuradse's experimental result gave for rough pipe the following relation for 'f' as

$$\frac{1}{\sqrt{4f}} = 2 \log_{10}(R/k) + 1.74 \quad \text{...(10.36)}$$

(*c*) **Value of 'f' for commercial pipes.** The value of 'f' for commercial pipes such as pipes made of metal, concrete and wood is obtained from Nikuradse's experimental data for smooth and rough pipes. According to Colebrook, by subtracting 2 $\log_{10}(R/k)$ from both sides of equations (10.33) and (10.36), the value of 'f' is obtained for commercial smooth and rough pipes as :

1. Smooth pipes

$$\frac{1}{\sqrt{4f}} - 2\log_{10}(R/k) = 2\log_{10}(R_e\sqrt{4f}) - 0.8 - 2\log_{10}(R/k)$$

$$= 2\log_{10}\left(\frac{R_e\sqrt{4f}}{R/k}\right) - 0.8 \qquad ...(10.37)$$

2. Rough pipes

$$\frac{1}{\sqrt{4f}} - 2\log_{10}(R/k) = 2\log_{10}(R/k) + 1.74 - 2\log_{10}(R/k)$$

$$= 1.74. \qquad ...(10.38)$$

Problem 10.7 *For the problem 10.6, find the co-efficient of friction and the average height of roughness projections.*

Solution. From the solution of problem 10.6, we have

$$R = 0.15 \text{ m}$$
$$u_* = 0.1458 \text{ m/s}$$
$$\overline{U} = 1.4533 \text{ m/s}$$

For co-efficient of friction, we know that

$$u_* = \overline{U}\sqrt{f/2}$$

or
$$0.1458 = 1.4533\sqrt{f/2}$$

or
$$\sqrt{f/2} = \frac{0.1458}{1.4533} = 0.1$$

$\therefore$
$$f = 2.0 \times (.1)^2 = \mathbf{.02.} \text{ Ans.}$$

Height of roughness projection is obtained from equation (10.36) as

$$\frac{1}{\sqrt{4f}} = 2\log_{10}(R/k) + 1.74$$

Substituting the values of R and f, we get

$$\frac{1}{\sqrt{4 \times 0.2}} = 2\log_{10}\left(\frac{0.15}{k}\right) + 1.74 \text{ or } 3.5355 = 2\log_{10}\left(\frac{.15}{k}\right) + 1.74$$

or
$$\log_{10}\left(\frac{.15}{k}\right) = \frac{3.5355 - 1.74}{2} = 0.8977 = \log_{10} 7.90$$

$\therefore$
$$\frac{0.15}{k} = 7.90$$

$\therefore$
$$k = \frac{0.15}{7.90} = 0.01898 \text{ m} = \mathbf{18.98 \text{ mm.}} \text{ Ans.}$$

Problem 10.8 *Water is flowing through a rough pipe of diameter 500 mm and length 4000 m at the rate of 0.5 m^3/s. Find the power required to maintain this flow. Take the average height of roughness as k = 0.40 mm.*

Solution. Given :

Dia. of rough pipe, $D = 500$ mm $= 0.50$ m

$\therefore$ Radius, $R = \dfrac{D}{2} = 0.25$ m

Length of pipe, $L = 4000$ m

Discharge, $Q = 0.5\ m^3/s$

Average height of roughness, $k = 0.40$ mm $= 0.4 \times 10^{-3}$ m

First find the value of co-efficient of friction. Then calculate the head lost due to friction and then power required.

For a rough pipe, the value of 'f' is given by the equation (10.36) as

$$\frac{1}{\sqrt{4f}} = 2\log_{10}(R/k) + 1.74 = 2\log_{10}\left(\frac{.25}{.4\times10^{-3}}\right) + 1.74$$

$$= 2\log_{10}(625.0) + 1.74 = 5.591 + 1.74 = 7.331$$

or $$\sqrt{4f} = \frac{1}{7.331} = 0.1364 \text{ or } f = (0.1364)^2/4 = .00465$$

Also the average velocity, $$\overline{U} = \frac{\text{Discharge}}{\text{Area}} = \frac{0.5}{\frac{\pi}{4}D^2} = \frac{0.5}{\frac{\pi}{4}(.5)^2} = 2.546$$

$\therefore$ Head lost due to friction, $$h_f = \frac{4.f.L.V^2}{d\times 2g} = \frac{4\times.00465\times4000\times2.546^2}{0.5\times2\times9.81}$$

$$= 49.16 \text{ m} \qquad [\because V = \overline{U} = 2.546,\ d = D = 0.5]$$

$\therefore$ Power required, $$P = \frac{W\times h_f}{1000} = \frac{w.Q.h_f}{1000} = \frac{\rho\times g\times Q\times h_f}{1000} \text{ kW}$$

$$= \frac{1000\times9.81\times0.5\times49.16}{1000} = \mathbf{241.13\ kW.}$$ **Ans.**

Problem 10.9 *A smooth pipe of diameter 400 mm and length 800 m carries water at the rate of 0.04 m^3/s. Determine the head lost due to friction, wall shear stress, centre-line velocity and thickness of laminar sub-layer. Take the kinematic viscosity of water as 0.018 stokes.*

Solution. Given :

Dia. of pipe, $D = 400$ mm $= 0.40$ m

$\therefore$ Radius, $R = \dfrac{D}{2} = 0.20$ m

Length of pipe, $L = 800$ m

Discharge, $Q = 0.04\ m^3/s$

Kinematic viscosity, $\nu = 0.018$ stokes $= 0.018\ cm^2/s = 0.018 \times 10^{-4}\ m^2/s$

Average velocity, $$\overline{U} = \frac{Q}{\text{Area}} = \frac{0.04}{\frac{\pi}{4}(0.4)^2} = 0.3183 \text{ m/s}$$

$\therefore$ Reynolds number, $$R_e = \frac{V\times D}{\nu} = \frac{\overline{U}\times D}{\nu} = \frac{0.3183\times0.4}{.018\times10^{-4}} = 7.073\times10^4$$

The flow is turbulent.

The co-efficient of friction 'f' is obtained from equation (10.30) as

$$f = \frac{.0791}{(R_e)^{1/4}} = \frac{0.0791}{\left(7.073 \times 10^4\right)^{1/4}} = \frac{.0791}{16.30} = .00485$$

(*i*) Head lost due to friction, $h_f = \dfrac{4\,.\,f\,.\,L\,.\,V^2}{D \times 2g} = \dfrac{4\,.\,f\,.\,L\,.\,\overline{U}^2}{D \times 2g}$

$$= \frac{4 \times .00485 \times 800 \times (.3183)^2}{0.40 \times 2 \times 9.81} = \mathbf{0.20\ m.}\ \textbf{Ans.}$$

(*ii*) Wall shear stress (τ_0) is given by equation (10.5) as

$$\tau_0 = \frac{f\,.\,\rho\,.\,V^2}{2} = \frac{f\,.\,\rho\,.\,\overline{U}^2}{2} \qquad [\because V = \overline{U}\,]$$

$$= 0.00485 \times 1000 \times \frac{(.3184)^2}{2.0}\ \text{N/m}^2 = \mathbf{0.245\ N/m^2.}\ \textbf{Ans.}$$

(*iii*) The centre-line velocity (u_{max}) for smooth pipe is given by equation (10.20) as in which $u = u_{max}$ at $y = R$

$$\therefore \qquad \frac{u_{max}}{u_*} = 5.75 \log_{10} \frac{u_* R}{\nu} + 5.55 \text{ [Put in equation (10.20), } u = u_{max} \text{ at } y = R]$$

where the shear velocity $\quad u_* = \sqrt{\dfrac{\tau_0}{\rho}} = \sqrt{\dfrac{0.245}{1000}} = \sqrt{0.000245} = 0.0156$ m/s

Substituting the values of u_*, R and ν in the above equation, we get

$$\frac{u_{max}}{0.0156} = 5.75 \log_{10} \frac{0.0156 \times 0.20}{.018 \times 10^{-4}} + 5.55 = 24.173$$

or $\quad u_{max} = 24.173 \times .0156 = \mathbf{0.377\ m/s.}$ **Ans.**

(*iv*) The thickness of laminar sub-layer (δ') is given by

$$\delta' = \frac{11.6 \times \nu}{u_*} = \frac{11.6 \times .018 \times 10^{-4}}{.0156} = .001338 \text{ m} = \mathbf{1.338\ mm.}\ \textbf{Ans.}$$

Problem 10.10 *A rough pipe of diameter 400 mm and length 1000 m carries water at the rate of 0.4 m³/s. The wall roughness is 0.012 mm. Determine the co-efficient of friction, wall shear stress, centre-line velocity and velocity at a distance of 150 mm from the pipe wall.*

Solution. Given :

Dia. of rough pipe, $\quad D = 400$ mm $= 0.4$ m

∴ Radius, $\quad R = \dfrac{D}{2} = \dfrac{0.4}{2} = 0.20$ m

Length of pipe, $\quad L = 1000$ m

Discharge, $\quad Q = 0.4\ \text{m}^3/\text{s}$

Wall roughness, $\quad k = 0.012$ mm $= 0.012 \times 10^{-3}$ m (*i*) The value of co-efficient of friction '*f*' for rough pipe is given by the equation (10.36) as

$$\frac{1.0}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74$$

or

$$\frac{1.0}{\sqrt{4f}} = 2 \log_{10} \left(\frac{0.20}{.012 \times 10^{-3}}\right) + 1.74$$

$$= 2 \log_{10} (16666.67) + 1.74 = 10.183$$

$$\therefore \quad 4f = \left(\frac{1}{10.183}\right)^2 = .00964$$

$$\therefore \quad f = \frac{.00964}{4.0} = \mathbf{.00241.}\ \textbf{Ans.}$$

(*ii*) Centre-line velocity (u_{max}) for rough pipe is given by equation (10.21) in which u is made $= u_{max}$ at $y = R$ and hence

$$\frac{u_{max}}{u_*} = 5.75 \log_{10} (R/k) + 8.5 \qquad ...(i)$$

where shear velocity, $u_* = \sqrt{\dfrac{\tau_0}{\rho}}$

and τ_0 = wall shear stress $= \dfrac{f \cdot \rho \cdot V^2}{2}$

where $V = \dfrac{\text{Discharge}}{\text{Area}} = \dfrac{Q}{\dfrac{\pi}{4} D^2} = \dfrac{Q}{\dfrac{\pi}{4} (.4)^2}$ = **3.183 m/s. Ans.**

(*iii*) $\therefore$

$$\tau_0 = \frac{f \cdot \rho \cdot V^2}{2} = .00241 \times 1000 \times \frac{3.183^2}{2.0} = \mathbf{12.2\ N/m^2.}\ \textbf{Ans.}$$

$$\therefore \quad u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{12.2}{1000}} = 0.11 \text{ m/s}$$

Substituting the value of u_*, R, k in equation (*i*), we get

$$\frac{u_{max}}{0.11} = 5.75 \log_{10} \left(\frac{0.2}{.012 \times 10^{-3}}\right) + 8.5 = 32.77$$

$$\therefore \quad u_{max} = 32.77 \times 0.11 = \mathbf{3.60\ m/s.}\ \textbf{Ans.}$$

(*iv*) Velocity (u) at a distance y = 150 mm = 0.15 m

The velocity (u) at any point for rough pipe is given by equation (10.21) as

$$\frac{u}{u^*} = 5.75 \log_{10} (y/k) + 8.5$$

where $u_* = 0.11$ m/s and $y = 0.15$ m, $k = 0.012 \times 10^{-3}$ m

$\because$ $$\frac{u}{0.11} = 5.75 \log_{10}\left(\frac{0.15}{.012 \times 10^{-3}}\right) + 8.5 = 32.05$$

$\therefore$ $$u = 32.05 \times 0.11 = \mathbf{3.52\ m/s.}\ \textbf{Ans.}$$

Problem 10.11 *A smooth pipe line of 100 mm diameter carries 2.27 m^3 per minute of water at 20°C with kinematic viscosity of 0.0098 stokes. Calculate the friction factor, maximum velocity as well as shear stress at the boundary.*

Solution. Given :

Dia. of pipe, $D = 100$ mm $= 0.1$ m

$\therefore$ Radius of pipe, $R = 0.05$ m

Discharge, $$Q = 2.27\ m^3/min = \frac{2.27}{60}\ m^3/s = 0.0378\ m^3/s$$

Kinematic viscosity, $\nu = 0.0098$ stokes $= 0.0098\ cm^2/s = 0.0098 \times 10^{-4}\ m^2/s$

Now average velocity is given by $$\overline{U} = \frac{Q}{\text{Area}} = \frac{0.0378}{\frac{\pi}{4}(0.1)^2} = \frac{0.0378 \times 4}{\pi \times 0.01} = 4.817\ m/s$$

$\therefore$ Reynolds number is given by, $$R_e = \frac{\overline{U} \times D}{\nu} = \frac{4.817 \times 0.1}{0.0098 \times 10^{-4}} = 4.9154 \times 10^5.$$

The flow is turbulent and R_e is more than 10^5. Hence for smooth pipe, the co-efficient of friction 'f' is obtained from equation (10.33) as

$$\frac{1}{\sqrt{4f}} = 2.0 \log_{10}(R_e\sqrt{4f}) - 0.8$$

or $$\frac{1}{\sqrt{4f}} = 2.0 \log_{10}(4.9154 \times 10^5 \times \sqrt{4f}) - 0.8$$

$$= 2.0\,[\log_{10} 4.9154 \times 10^5 + \log_{10}\sqrt{4f}\,] - 0.8$$

$$= 2.0\,[5.6915 + \log_{10}\sqrt{4f}\,] - 0.8 = 2 \times 5.6915 + 2\log_{10}\sqrt{4f} - 0.8$$

$$= 11.3830 + \log_{10}\left(\sqrt{4f}\right)^2 - 0.8 = 11.383 + \log_{10}(4f) - 0.8$$

or $$\frac{1}{\sqrt{4f}} - \log_{10}(4f) = 11.383 - 0.8 = 10.583 \qquad ...(i)$$

(*i*) *Friction factor*

Now, friction factor (f^*) $= 4 \times$ co-efficient of friction $= 4f$

Substituting the value of '$4f$' in equation (*i*), we get

$$\frac{1}{\sqrt{f^*}} - \log_{10} f^* = 10.583 \qquad ...(ii)$$

The above equation is solved by hit and trial method.

Let $f^* = 0.1$, then L.H.S. of equation (*ii*), becomes as

$$\text{L.H.S.} = \frac{1}{\sqrt{0.1}} - \log_{10} 0.1 = 3.16 - (-1.0) = 4.16$$

Let $f^* = 0.01$, then L.H.S. of equation (*ii*), becomes as

$$\text{L.H.S.} = \frac{1}{\sqrt{0.01}} - \log_{10} 0.01 = 10 - (-2) = 12$$

But for exact solution, L.H.S. should be 10.583. Hence value of f^* lies between 0.1 and 0.01.

Let $f^* = 0.013$ then L.H.S. of equation (*ii*), becomes as

$$\text{L.H.S.} = \frac{1}{\sqrt{0.013}} - \log_{10} 0.013 = 8.77 - (-1.886) = 8.77 + 1.886 = 10.656$$

which is approximately equal to 10.583.

Hence the value of f^* is equal to 0.013.

$\therefore$ Friction factor, $f^* =$ **0.013. Ans.**

(*ii*) *Maximum velocity* (u_{max})

Now we know that $f^* = 4f$

$$\therefore \quad \text{Co-efficient of friction, } f = \frac{f^*}{4} = \frac{0.013}{4} = 0.00325$$

Now the shear velocity (u_*) in terms of co-efficient of friction and average velocity is given by equation (10.31*A*) as

$$u_* = \overline{U}\sqrt{\frac{f}{2}} = 4.817 \times \sqrt{\frac{0.00325}{2}} = 4.817 \times 0.0403 = 0.194$$

For smooth pipe, the velocity at any point is given by equation (10.20)

$$u = u_* \left[5.75 \log_{10} \frac{u_* \times y}{\nu} + 5.55\right]$$

The velocity will be maximum at the centre of the pipe,

where $y = R = 0.05$

i.e., radius of pipe. Hence the above equation becomes as

$$U_{\max} = u_* \left[5.75 \log_{10} \frac{u_* \times R}{\nu} + 5.55\right]$$

$$= 0.194 \left[5.75 \log_{10} \frac{0.194 \times 0.05}{0.0098 \times 10^{-4}} + 5.55\right]$$

$$= 0.194\,[22.974 + 5.55] = \mathbf{5.528\ m/s.}\ \textbf{Ans.}$$

(*iii*) *Shear stress at the boundary* (τ_0)

We know that $u_* = \sqrt{\frac{\tau_0}{\rho}}$ or $u_*^2 = \frac{\tau_0}{\rho}$

$$\therefore \quad \tau_0 = \rho\, u_*^2 = 1000 \times 0.194^2 = \mathbf{37.63\ N/m^2.}\ \textbf{Ans.}$$

Problem 10.12 *Hydrodynamically smooth pipe carries water at the rate of 300 l/s at 20°C ($\rho = 1000$ kg/m^3, $\nu = 10^{-6}$ m^2/s) with a head loss of 3 m in 100 m length of pipe. Determine the pipe diameter. Use $f = 0.0032 + \frac{0.221}{(R_e)^{0.237}}$ equation for f, where $h_f = \frac{f \times L \times V^2}{D \times 2g}$ and $R_e = \frac{\rho VD}{\mu}$.*

Solution. Given :

Discharge, $Q = 300 \text{ l/s} = 0.3 \text{ m}^3/\text{s}$

Density, $\rho = 1000 \text{ kg/m}^3$

Kinematic viscosity, $\nu = 10^{-6} \text{ m}^2/\text{s}$

Head loss, $h_f = 3 \text{ m}$

Length of pipe, $L = 100 \text{ m}$

Value of friction factor, $f = 0.0032 + \dfrac{0.221}{(R_e)^{0.237}}$

Reynolds number, $$R_e = \frac{\rho VD}{\mu} = \frac{V \times D}{\nu} \qquad \left(\because \frac{\mu}{\rho} = \nu\right)$$

$$= \frac{V \times D}{10^{-6}} = V \times D \times 10^6$$

Find : Diameter of pipe.

Let D = Diameter of pipe

Head loss in terms of friction factor is given as

$$h_f = \frac{f \times L \times V^2}{D \times 2g}$$

or $$3 = \frac{f \times 100 \times V^2}{D \times 2 \times 9.81} \qquad (\because h_f = 3,\ L = 100 \text{ m})$$

or $$f = \frac{3 \times D \times 2 \times 9.81}{100\, V^2} \text{ or } f = \frac{0.5886\, D}{V^2} \qquad ...(i)$$

Now $$Q = A \times V$$

or $$0.3 = \frac{\pi}{4} D^2 \times V \text{ or } D^2 \times V = \frac{4 \times 0.3}{\pi} = 0.382$$

$\therefore$ $$V = \frac{0.382}{D^2} \qquad ...(ii)$$

Also $$f = 0.0032 + \frac{0.221}{(R_e)^{0.237}}$$

or $$\frac{0.5886\, D}{V^2} = 0.0032 + \frac{0.221}{\left(V \times D \times 10^6\right)^{0.237}}$$

$$\left(\because \text{From equation } (i),\ f = \frac{0.5886\, D}{V^2} \text{ and } R_e = V \times D \times 10^6\right)$$

or $$\frac{0.5886\, D}{\left(\dfrac{0.382}{D^2}\right)^2} = 0.0032 + \frac{0.221}{\left(\dfrac{0.382}{D^2} \times D \times 10^6\right)^{0.237}}$$

$$\left(\because \text{From equation}(ii),\ V = \frac{0.382}{D^2}\right)$$

or $$\frac{0.5886 \times D^5}{0.382^2} = 0.0032 + \frac{0.221}{\frac{\left(0.382 \times 10^6\right)^{0.237}}{D^{0.237}}}$$

or $$4.033\ D^5 = 0.0032 + 0.0105 \times D^{0.237}$$

or $$4.033\ D^5 - 0.0105\ D^{0.237} - 0.0032 = 0 \qquad \text{...}(iii)$$

The above equation (*iii*) will be solved by hit and trial method.

(*i*) Assume $D = 1$ m, then L.H.S. of equation (*iii*), becomes as

$$\text{L.H.S.} = 4.033 \times 1^5 - 0.0105 \times 1^{0.237} - 0.0032$$
$$= 4.033 - 0.0105 - 0.0032 = 4.0193$$

By increasing the value of D more than 1 m, the L.H.S. will go on increasing. Hence decrease the value of D.

(*ii*) Assume $D = 0.3$ m, then L.H.S. of equation (*iii*),

becomes as $\text{L.H.S.} = 4.033 \times 0.3^5 - 0.0105 \times 0.3^{0.237} - 0.0032$
$$= 0.0098 - 0.00789 - 0.0032 = -\ 0.00129$$

As this value is negative, the value of D will be slightly more than 0.3.

(*iii*) Assume $D = 0.306$ m, then L.H.S. of equation (*iii*), becomes as

$$\text{L.H.S.} = 4.033 \times 0.306^5 - 0.0105 \times 0.306^{0.237} - 0.0032$$
$$= 0.0108 - 0.00793 - 0.0032 = -\ 0.00033$$

This value of L.H.S. is approximately equal to zero. Actually the value of D will be slightly more than 0.306 m say **0.308 m. Ans.**

Problem 10.13 *Water is flowing through a rough pipe of diameter 600 mm at the rate of 600 litres/second. The wall roughness is 3 mm. Find the power lost for 1 km length of pipe.*

Solution. Given :

Dia. of pipe, $D = 600 \text{ mm} = 0.6 \text{ m}$

∴ Radius of pipe, $R = \frac{0.6}{2} = 0.3 \text{ m}$

Discharge, $Q = 600 \text{ litre/s} = 0.6 \text{ m}^3/\text{s}$

Wall roughness, $k = 3 \text{ mm} = 3 \times 10^{-3} \text{ m} = 0.003 \text{ m}$

Length of pipe, $L = 1 \text{ km} = 1000 \text{ m}$

For rough pipes, the co-efficient of friction in terms of wall roughness, k is given by equation (10.36) as

$$\frac{1}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74 = 2 \log_{10} \left(\frac{0.3}{0.003}\right) + 1.74 = 5.74$$

or $$\sqrt{4f} = \frac{1}{5.76} = 0.1742 \text{ or } 4f = (0.1742)^2 = 0.03035$$

The head loss due to friction is given by, $h_f = \dfrac{4f \times L \times V^2}{D \times 2g}$

where $V = \dfrac{Q}{A} = \dfrac{0.6}{\frac{\pi}{4}(0.6^2)} = 2.122$ m/s

$$h_f = \frac{0.03035 \times 1000 \times 2.122^2}{0.6 \times 2 \times 9.81} = 11.6 \text{ m}$$

The power* lost is given by, $P = \dfrac{\rho g \times Q \times h_f}{1000} = \dfrac{1000 \times 9.81 \times 0.6 \times 11.6}{1000}$ kW = **68.27 kW. Ans.**

HIGHLIGHTS

1. If the Reynold number is less than 2000 in a pipe, the flow is laminar while if the Reynold number is more than 4000, the flow is turbulent in pipes.
2. Loss of pressure head in a laminar flow is proportional to the mean velocity of flow, while in case of turbulent flow it is approximately proportional to the square of velocity.
3. Expression for head loss due to friction in pipes is given by Darcy-Weisbach equation,

$$h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}, \text{ where } f = \text{co-efficient of friction}$$

$$= \frac{f \times L \times V^2}{d \times 2g}, \text{ where } f = \text{friction factor}$$

4. Co-efficient of friction is expressed in terms of shear stress as $= \dfrac{2\tau_0}{\rho V^2}$

   where V = mean velocity of flow, ρ = mass density of fluid.
5. Shear stress in turbulent flow is sum of shear stress due to viscosity and shear stress due to turbulence, *i.e.*,

$$\tau = \tau_v + \tau_t, \quad \text{where} \quad \tau_v = \text{shear stress due to viscosity}, \quad \tau_t = \text{shear stress due to turbulence}$$

$$= \mu \frac{d\bar{u}}{dy} + \eta \frac{d\bar{u}}{dy}$$

6. Turbulent shear stress by Reynolds is given as $\tau = \rho\, u'v'$
   where u' and v' = fluctuating component of velocity.
7. The expression for shear stress in turbulent flow due to Prandtl is $\bar{\tau} = \rho l^2 \left(\dfrac{du}{dy}\right)^2$, where l = mixing length.
8. The velocity distribution in the turbulent flow for pipes is given by the expression

$$u = u_{max} + 2.5\, u^* \log_e (y/R)$$

   where u_{max} = is the centre-line velocity,
   y = distance from the pipe wall,
   R = radius of the pipe,

   and u_* = shear velocity which is equal to $\sqrt{\dfrac{\tau_0}{\rho}}$.

---

* Power $= \rho g \times Q \times h_f$ watt $= \dfrac{\rho g \times Q \times h_f}{1000}$ kW.

9. Velocity defect is the difference between the maximum velocity (u_{max}) and local velocity (u) at any point and is given by $(u_{max} - u) = 5.75 \times u_* \log_{10} (R/y)$.

10. The boundary is known as hydrodynamically smooth if k, the average height of the irregularities projecting from the surface of the boundary is small compared to the thickness of the laminar sub-layer (δ') and boundary is rough if k is large in comparison with the thickness of the sub-layer.

or if $\dfrac{k}{\delta'} < 0.25$, the boundary is smooth ; if $\dfrac{k}{\delta'} > 6.0$, the boundary is rough

and if $\dfrac{k}{\delta'}$ lie between 0.25 to 6.0, the boundary is in transition.

11. Velocity distribution for turbulent flow is

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{\nu} + 5.55 \text{ for smooth pipes}$$

$$= 5.75 \log_{10} (y/k) + 8.5 \text{ for rough pipes}$$

where u = velocity at any point in the turbulent flow,

u_* = shear velocity and $= \sqrt{\dfrac{\tau_0}{\rho}}$, ν = kinematic viscosity of fluid,

y = distance from pipe wall, and k = roughness factor.

12. Velocity distribution in terms of average velocity is

$$\frac{\overline{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{\nu} + 1.75 \text{ for smooth pipes,}$$

$$= 5.75 \log_{10} R/k + 4.75 \text{ for rough pipes.}$$

13. Difference of local velocity and average velocity for smooth and rough pipes is

$$\frac{u - \overline{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75.$$

14. The co-efficient of friction is given by

$$f = \frac{16}{R_e} \text{ ...... for laminar flow,}$$

$$= \frac{0.0791}{(R_e)^{1/4}} \text{ for turbulent flow in smooth pipes for } R_e \geq 4000 \text{ by} \leq 10^5$$

$$= .0008 + \frac{.05525}{(R_e)^{.257}} \text{ for } R_e \leq 10^5 \text{ but} \geq 4 \times 10^7$$

$$\frac{1}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74 \text{ for rough pipes where } R_e = \text{Reynolds number.}$$

EXERCISE

(A) THEORETICAL PROBLEMS

1. What do you understand by turbulent flow ? What factor decides the type of flow in pipes ?
2. (*a*) Derive an expression for the loss of head due to friction in pipes.
   (*b*) Derive Darcy-Weisbach equation. (*J.N.T.U., Hyderabad, S 2002*)
3. Explain the term co-efficient of friction. On what factors does this co-efficient depend ?

4. Obtain an expression for the co-efficient of friction in the terms of shear stress.
5. What do you mean by Prandtl mixing Length Theory ? Find an expression for shear stress due to Prandtl.
6. Derive an expression for Prandtl's universal velocity distribution for turbulent flow in pipes. Why this velocity distribution is called universal ?
7. What is a velocity defect ? Derive an expression for velocity defect in pipes.
8. How would you distinguish between hydrodynamically smooth and rough boundaries ?
9. Obtain an expression for the velocity distribution for turbulent flow in smooth pipes.
10. Show that velocity distribution for turbulent flow through rough pipe is given by

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

where u_* = shear velocity, y = distance from pipe wall, k = roughness factor.
11. Obtain an expression for velocity distribution in terms of average velocity for (*a*) smooth pipes and (*b*) rough pipes.
12. Prove that the difference of local velocity and average velocity for turbulent flow through rough or smooth pipes is given by

$$\frac{u - \overline{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75.$$

13. Obtain an expression for velocity distribution in turbulent flow for (*i*) smooth pipes and (*ii*) rough pipes. (*Delhi University, December, 2002*)

(B) NUMERICAL PROBLEMS

1. A pipe-line carrying water has average height of irregularities projecting from the surface of the boundary of the pipe as 0.20 mm. What type of the boundary is it ? The shear stress development is 7.848 N/m^2. Take value of kinematic viscosity for water as 0.01 stokes. [**Ans.** Boundary is in transition]
2. Determine the average height of the roughness for a rough pipe of diameter 10.0 cm when the velocity at a point 4 cm from wall is 40% more than the velocity at a point 1 cm from pipe wall. [**Ans.** 0.94 cm]
3. A smooth pipe of diameter 10 cm and 1000 m long carries water at the rate of 0.70 m^2/minute. Calculate the loss of head, wall shearing stress, centre line velocity, velocity and shear stress at 3 cm from pipe wall. Also calculate the thickness of the laminar sub-layer. Take kinematic viscosity of water as 0.015 stokes and value of co-efficient of friction 'f' as

$$f = \frac{.0791}{(R_e)^{1/4}}, \text{ where } R_e = \text{Reynolds number.}$$

[**Ans.** 20.05 m, 4.9 N/m^2 ; 1.774 m/s ; 1.65 m/s ; 19.62 N/m^2 ; 0.248 mm]
4. The velocities of water through a pipe of diameter 10 cm, are 4 m/s and 3.5 m/s at the centre of the pipe and 2 cm from the pipe centre respectively. Determine the wall shearing stress in the pipe for turbulent flow. [**Ans.** 15.66 kgf/m^2]
5. For turbulent flow in a pipe of diameter 200 mm, find the discharge when the centre-line velocity is 30 m/s and velocity at a point 80 mm from the centre as measured by pitot-tube is 2.0 m/s. [**Ans.** 64.9 litres/s]
6. For problem 5, find the co-efficient of friction and the average height of roughness projections. [**Ans.** 0.029, 25.2 mm]
7. Water is flowing through a rough pipe of diameter 40 cm and length 3000 m at the rate of 0.4 m^3/s. Find the power required to maintain this flow. Take the average height of roughness as K = 0.3 mm. [**Ans.** 278.5 kN]

8. A smooth pipe of diameter 300 mm and length 600 m carries water at rate of 0.04 m^2/s. Determine the head lost due to friction, wall shear stress, centre-line velocity and thickness of laminar sub-layer. Take the kinematic viscosity of water as 0.018 stokes. [**Ans.** 0.588 m, 0.72 N/cm^2, 0.665 m/s, 0.779 mm]
9. A rough pipe of diameter 300 mm and length 800 m carries water at the rate of 0.4 m^2/s. The wall roughness is 0.015 mm. Determine the co-efficient of friction, wall shear stress, centre line velocity and velocity at a distance of 100 mm from the pipe wall.
[**Ans.** f = .00263, τ_0 = 42.08 N/cm^2, u_{max} = 6.457 m/s, u = 6.249 m/s]
10. Determine the distance from the centre of the pipe, at which the local velocity is equal to the average velocity for turbulent flow in pipes. [**Ans.** 0.7772 R]

11 CHAPTER

FLOW THROUGH PIPES

▶ 11.1 INTRODUCTION

In chapters 9 and 10, laminar flow and turbulent flow have been discussed. We have seen that when the Reynolds number is less than 2000 for pipe flow, the flow is known as laminar flow whereas when the Reynolds number is more than 4000, the flow is known as turbulent flow. In this chapter, the turbulent flow of fluids through pipes running full will be considered. If the pipes are partially full as in the case of sewer lines, the pressure inside the pipe is same and equal to atmospheric pressure. Then the flow of fluid in the pipe is not under pressure. This case will be taken in the chapter of flow of water through open channels. Here we will consider flow of fluids through pipes under pressure only.

▶ 11.2 LOSS OF ENERGY IN PIPES

When a fluid is flowing through a pipe, the fluid experiences some resistance due to which some of the energy of fluid is lost. This loss of energy is classified as :

Energy Losses

1. Major Energy Losses

   This is due to friction and it is calculated by the following formulae :
   (*a*) Darcy-Weisbach Formula
   (*b*) Chezy's Formula

2. Minor Energy Losses

   This is due to
   (*a*) Sudden expansion of pipe
   (*b*) Sudden contraction of pipe
   (*c*) Bend in pipe
   (*d*) Pipe fittings etc.
   (*e*) An obstruction in pipe.

▶ 11.3 LOSS OF ENERGY (OR HEAD) DUE TO FRICTION

(*a*) **Darcy-Weisbach Formula.** The loss of head (or energy) in pipes due to friction is calculated from Darcy-Weisbach equation which has been derived in chapter 10 and is given by

$$h_f = \frac{4 \, . \, f \, . \, L \, . \, V^2}{d \times 2g} \qquad \text{...(11.1)}$$

where h_f = loss of head due to friction

f = co-efficient of friction which is a function of Reynolds number

$= \dfrac{16}{R_e}$ for $R_e < 2000$ (viscous flow)

$= \dfrac{0.079}{R_e^{1/4}}$ for R_e varying from 4000 to 10^6

L = length of pipe,

V = mean velocity of flow,

d = diameter of pipe.

(*b*) **Chezy's Formula for loss of head due to friction in pipes.** Refer to chapter 10 article 10.3.1 in which expression for loss of head due to friction in pipes is derived. Equation (*iii*) of article 10.3.1, is

$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2 \qquad \text{...(11.2)}$$

where h_f = loss of head due to friction, P = wetted perimeter of pipe,
A = area of cross-section of pipe, L = length of pipe,
and V = mean velocity of flow.

Now the ratio of $\dfrac{A}{P}\left(= \dfrac{\text{Area of flow}}{\text{Perimeter (wetted)}}\right)$ is called hydraulic mean depth or hydraulic radius and is denoted by m.

$\therefore$ Hydraulic mean depth, $m = \dfrac{A}{P} = \dfrac{\frac{\pi}{4}d^2}{\pi d} = \dfrac{d}{4}$

Substituting $\dfrac{A}{P} = m$ or $\dfrac{P}{A} = \dfrac{1}{m}$ in equation (11.2), we get

$$h_f = \frac{f'}{\rho g} \times L \times V^2 \times \frac{1}{m} \text{ or } V^2 = h_f \times \frac{\rho g}{f'} \times m \times \frac{1}{L} = \frac{\rho g}{f'} \times m \times \frac{h_f}{L}$$

$$\therefore \qquad V = \sqrt{\frac{\rho g}{f'} \times m \times \frac{h_f}{L}} = \sqrt{\frac{\rho g}{f'}}\sqrt{m\frac{h_f}{L}} \qquad \text{...(11.3)}$$

Let $\sqrt{\dfrac{\rho g}{f'}} = C$, where C is a constant known as Chezy's constant and $\dfrac{h_f}{L} = i$, where i is loss of head per unit length of pipe.

Substituting the values of $\sqrt{\dfrac{\rho g}{f'}}$ and $\sqrt{\dfrac{h_f}{L}}$ in equation (11.3), we get

$$V = C\sqrt{mi} \qquad \text{...(11.4)}$$

Equation (11.4) is known as Chezy's formula. Thus the loss of head due to friction in pipe from Chezy's formula can be obtained if the velocity of flow through pipe and also the value of C is known. The value of m for pipe is always equal to $d/4$.

Problem 11.1 *Find the head lost due to friction in a pipe of diameter 300 mm and length 50 m, through which water is flowing at a velocity of 3 m/s using (i) Darcy formula, (ii) Chezy's formula for which C = 60.*

Take ν for water = 0.01 stoke.

Solution. Given :

Dia. of pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

Length of pipe, $L = 50 \text{ m}$

Velocity of flow, $V = 3 \text{ m/s}$

Chezy's constant, $C = 60$

Kinematic viscosity, $\nu = 0.01 \text{ stoke} = 0.01 \text{ cm}^2/\text{s}$
$= 0.01 \times 10^{-4} \text{ m}^2/\text{s}.$

(*i*) **Darcy Formula** is given by equation (11.1) as

$$h_f = \frac{4\,.\,f\,.\,L\,.\,V^2}{d \times 2g}$$

where 'f' = co-efficient of friction is a function of Reynolds number, R_e

But R_e is given by $$R_e = \frac{V \times d}{\nu} = \frac{3.0 \times 0.30}{.01 \times 10^{-4}} = 9 \times 10^5$$

$\therefore$ Value of $$f = \frac{0.079}{R_e^{1/4}} = \frac{0.079}{\left(9 \times 10^5\right)^{1/4}} = .00256$$

$\therefore$ Head lost, $$h_f = \frac{4 \times .00256 \times 50 \times 3^2}{0.3 \times 2.0 \times 9.81} = \mathbf{.7828\ m.}\ \textbf{Ans.}$$

(*ii*) **Chezy's Formula.** Using equation (11.4)

$$V = C\sqrt{mi}$$

where $C = 60$, $m = \dfrac{d}{4} = \dfrac{0.30}{4} = 0.075 \text{ m}$

$\therefore$ $$3 = 60\sqrt{.075 \times i} \text{ or } i = \left(\frac{3}{60}\right)^2 \times \frac{1}{.075} = 0.0333$$

But $$i = \frac{h_f}{L} = \frac{h_f}{50}$$

Equating the two values of i, we have $\dfrac{h_f}{50} = .0333$

$\therefore$ $h_f = 50 \times .0333 = \mathbf{1.665\ m.}$ **Ans.**

Problem 11.2 *Find the diameter of a pipe of length 2000 m when the rate of flow of water through the pipe is 200 litres/s and the head lost due to friction is 4 m. Take the value of C = 50 in Chezy's formulae.*

Solution. Given :

Length of pipe, $L = 2000$ m

Discharge, $Q = 200$ litre/s $= 0.2\ m^3/s$

Head lost due to friction, $h_f = 4$ m

Value of Chezy's constant, $C = 50$

Let the diameter of pipe $= d$

Velocity of flow,
$$V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{\frac{\pi}{4}d^2} = \frac{0.2}{\frac{\pi}{4}d^2} = \frac{0.2 \times 4}{\pi d^2}$$

Hydraulic mean depth,
$$m = \frac{d}{4}$$

Loss of head per unit length, $i = \frac{h_f}{L} = \frac{4}{2000} = .002$

Chezy's formula is given by equation (11.4) as $V = C\sqrt{mi}$

Substituting the values of V, m, i and C, we get

$$\frac{0.2 \times 4}{\pi d^2} = 50\sqrt{\frac{d}{4} \times .002} \text{ or } \sqrt{\frac{d}{4} \times .002} = \frac{0.2 \times 4}{\pi d^2 \times 50} = \frac{.00509}{d^2}$$

Squaring both sides, $\frac{d}{4} \times .002 = \frac{.00509^2}{d^4} = \frac{.0000259}{d^4}$ or $d^5 = \frac{4 \times .0000259}{.002} = 0.0518$

$\therefore$ $d = \sqrt[5]{0.0518} = (.0518)^{1/5} = 0.553$ m $=$ **553 mm. Ans.**

Problem 11.3 *A crude oil of kinematic viscosity 0.4 stoke is flowing through a pipe of diameter 300 mm at the rate of 300 litres per sec. Find the head lost due to friction for a length of 50 m of the pipe.*

Solution. Given :

Kinematic viscosity, $\nu = 0.4$ stoke $= 0.4\ cm^2/s = .4 \times 10^{-4}\ m^2/s$

Dia. of pipe, $d = 300$ mm $= 0.30$ m

Discharge, $Q = 300$ litres/s $= 0.3\ m^3/s$

Length of pipe, $L = 50$ m

Velocity of flow,
$$V = \frac{Q}{\text{Area}} = \frac{0.3}{\frac{\pi}{4}(0.3)^2} = 4.24 \text{ m/s}$$

$\therefore$ Reynolds number,
$$R_e = \frac{V \times d}{\nu} = \frac{4.24 \times 0.30}{0.4 \times 10^{-4}} = 3.18 \times 10^4$$

As R_e lies between 4000 and 100000, the value of f is given by

$$f = \frac{.079}{(R_e)^{1/4}} = \frac{.079}{\left(3.18 \times 10^4\right)^{1/4}} = .00591$$

$\therefore$ Head lost due to friction, $h_f = \dfrac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} = \dfrac{4 \times .00591 \times 50 \times 4.24^2}{0.3 \times 2 \times 9.81}$ = **3.61 m. Ans.**

Problem 11.4 *An oil of sp. gr. 0.7 is flowing through a pipe of diameter 300 mm at the rate of 500 litres/s. Find the head lost due to friction and power required to maintain the flow for a length of 1000 m. Take* ν *= .29 stokes.*

Solution. Given :

Sp. gr. of oil, $S = 0.7$

Dia. of pipe, $d = 300$ mm $= 0.3$ m

Discharge, $Q = 500$ litres/s $= 0.5$ m^3/s

Length of pipe, $L = 1000$ m

Velocity, $V = \dfrac{Q}{\text{Area}} = \dfrac{0.5}{\dfrac{\pi}{4} d^2} = \dfrac{0.5 \times 4}{\pi \times 0.3^2} = 7.073$ m/s

$\therefore$ Reynolds number, $R_e = \dfrac{V \times d}{\nu} = \dfrac{7.073 \times 0.3}{0.29 \times 10^{-4}} = 7.316 \times (10)^4$

$\therefore$ Co-efficient of friction, $f = \dfrac{.079}{R_e^{1/4}} = \dfrac{0.79}{\left(7.316 \times 10^4\right)^{1/4}} = .0048$

$\therefore$ Head lost due to friction, $h_f = \dfrac{4 \times f \times L \times V^2}{d \times 2g} = \dfrac{4 \times .0048 \times 1000 \times 7.073^2}{0.3 \times 2 \times 9.81} = 163.18$ m

Power required $= \dfrac{\rho g \cdot Q \cdot h_f}{1000}$ kW

where ρ = density of oil $= 0.7 \times 1000 = 700$ kg/m^3

$\therefore$ Power required $= \dfrac{700 \times 9.81 \times 0.5 \times 163.18}{1000}$ = **560.28 kW. Ans.**

Problem 11.5 *Calculate the discharge through a pipe of diameter 200 mm when the difference of pressure head between the two ends of a pipe 500 m apart is 4 m of water. Take the value of 'f' = 0.009 in the formula* $h_f = \dfrac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$.

Solution. Given :

Dia. of pipe, $d = 200$ mm $= 0.20$ m

Length of pipe, $L = 500$ m

Difference of pressure head, $h_f = 4$ m of water

$f = .009$

Using equation (11.1), we have $h_f = \dfrac{4 \times f \times L \times V^2}{d \times 2g}$

or $4.0 = \dfrac{4 \times .009 \times 500 \times V^2}{0.2 \times 2 \times 9.81}$ or $V^2 = \dfrac{4.0 \times 0.2 \times 2 \times 9.81}{4.0 \times .009 \times 500} = 0.872$

$\therefore$ $V = \sqrt{0.872} = 0.9338 \simeq 0.934$ m/s

$\therefore$ Discharge, Q = velocity × area

$= 0.934 \times \frac{\pi}{4} d^2 = 0.934 \times \frac{\pi}{4} (0.2)^2$

$= 0.0293$ m^3/s = **29.3 litres/s. Ans.**

Problem 11.6 *Water is flowing through a pipe of diameter 200 mm with a velocity of 3 m/s. Find the head lost due to friction for a length of 5 m if the co-efficient of friction is given by* $f = 0.02 + \frac{.09}{R_e^{0.3}}$, *where* R_e *is Reynolds number. The kinematic viscosity of water = .01 stoke.*

Solution. Given :

Dia. of pipe, $d = 200$ mm $= 0.20$ m

Velocity, $V = 3$ m/s

Length, $L = 5$ m

Kinematic viscosity, $\nu = 0.01$ stoke $= .01 \times 10^{-4}$ m^2/s

$\therefore$ Reynolds number, $R_e = \frac{V \times d}{\nu} = \frac{3 \times 0.20}{.01 \times 10^{-4}} = 6 \times 10^5$

Value of $f = .02 + \frac{0.9}{R_e^{0.3}} = .02 + \frac{.09}{\left(6 \times 10^5\right)^{.3}} = .02 + \frac{0.09}{54.13}$

$= .02 + .00166 = 0.02166$

$\therefore$ Head lost due to friction, $h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4.0 \times .02166 \times 5.0 \times 3^2}{0.20 \times 2.0 \times 9.81}$

= **0.993 m of water. Ans.**

Problem 11.7 *An oil of sp. gr. 0.9 and viscosity 0.06 poise is flowing through a pipe of diameter 200 mm at the rate of 60 litres/s. Find the head lost due to friction for a 500 m length of pipe. Find the power required to maintain this flow.*

Solution. Given :

Sp. gr. of oil $= 0.9$

Viscosity, $\mu = 0.06$ poise $= \frac{0.06}{10}$ Ns/m^2

Dia. of pipe, $d = 200$ mm $= 0.2$ m

Discharge, $Q = 60$ litres/s $= 0.06$ m^3/s

Length, $L = 500$ m

Density $\rho = 0.9 \times 1000 = 900$ kg/m^3

$\therefore$ Reynolds number, $R_e = \frac{\rho V d}{\mu} = 900 \times \frac{V \times 0.2}{\frac{0.06}{10}}$

where $V = \frac{Q}{\text{Area}} = \frac{0.06}{\frac{\pi}{4} d^2} = \frac{0.06}{\frac{\pi}{4}(.2)^2} = 1.909$ m/s $\simeq 1.91$ m/s

$$\therefore \qquad R_e = 900 \times \frac{1.91 \times 0.2 \times 10}{0.06} = 57300$$

As R_e lies between 4000 and 10^5, the value of co-efficient of friction, f is given by

$$f = \frac{0.079}{R_e^{\,0.25}} = \frac{0.079}{(57300)^{0.25}} = .0051$$

$$\therefore \text{ Head lost due to friction, } h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .0051 \times 500 \times 1.91^2}{0.2 \times 2 \times 9.81}$$

$$= \textbf{9.48 m of water. Ans.}$$

$$\therefore \text{ Power required } = \frac{\rho g \cdot Q \cdot h_f}{1000} = \frac{900 \times 9.81 \times 0.06 \times 9.48}{1000} = \textbf{5.02 kW. Ans.}$$

▶ 11.4 MINOR ENERGY (HEAD) LOSSES

The loss of head or energy due to friction in a pipe is known as major loss while the loss of energy due to change of velocity of the following fluid in magnitude or direction is called minor loss of energy. The minor loss of energy (or head) includes the following cases :

1. Loss of head due to sudden enlargement,
2. Loss of head due to sudden contraction,
3. Loss of head at the entrance of a pipe,
4. Loss of head at the exit of a pipe,
5. Loss of head due to an obstruction in a pipe,
6. Loss of head due to bend in the pipe,
7. Loss of head in various pipe fittings.

In case of long pipe the above losses are small as compared with the loss of head due to friction and hence they are called minor losses and even may be neglected without serious error. But in case of a short pipe, these losses are comparable with the loss of head due to friction.

11.4.1 Loss of Head Due to Sudden Enlargement. Consider a liquid flowing through a pipe which has sudden enlargement as shown in Fig. 11.1. Consider two sections (1)-(1) and (2)-(2) before and after the enlargement.

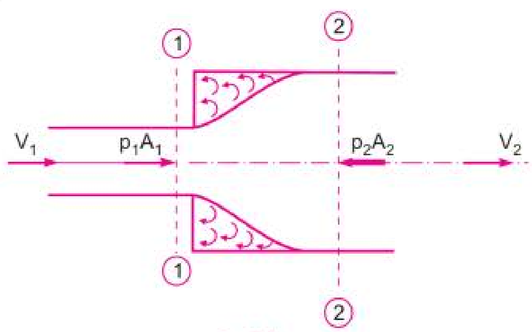


Fig. 11.1 *Sudden enlargement.*

Let p_1 = pressure intensity at section 1-1,

V_1 = velocity of flow at section 1-1,

A_1 = area of pipe at section 1-1,

p_2, V_2 and A_2 = corresponding values at section 2-2.

Due to sudden change of diameter of the pipe from D_1 to D_2, the liquid flowing from the smaller pipe is not able to follow the abrupt change of the boundary. Thus the flow separates from the boundary and turbulent eddies are formed as shown in Fig. 11.1. The loss of head (or energy) takes place due to the formation of these eddies.

Let p' = pressure intensity of the liquid eddies on the area $(A_2 - A_1)$

h_e = loss of head due to sudden enlargement

Applying Bernoulli's equation at sections 1-1 and 2-2,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{loss of head due to sudden enlargement}$$

But $z_1 = z_2$ as pipe is horizontal

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_e$$

or
$$h_e = \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) + \left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right) \quad \text{...}(i)$$

Consider the control volume of liquid between sections 1-1 and 2-2. Then the force acting on the liquid in the control volume in the direction of flow is given by

$$F_x = p_1 A_1 + p'(A_2 - A_1) - p_2 A_2$$

But experimentally it is found that $p' = p_1$

$$\therefore \quad F_x = p_1 A_1 + p_1(A_2 - A_1) - p_2 A_2 = p_1 A_2 - p_2 A_2$$
$$= (p_1 - p_2)A_2 \quad \text{...}(ii)$$

Momentum of liquid/sec at section 1-1 = mass × velocity

$$= \rho A_1 V_1 \times V_1 = \rho A_1 V_1^2$$

Momentum of liquid/sec at section 2-2 = $\rho A_2 V_2 \times V_2 = \rho A_2 V_2^2$

$\therefore$ Change of momentum/sec = $\rho A_2 V_2^2 - \rho A_1 V_1^2$

But from continuity equation, we have

$$A_1 V_1 = A_2 V_2 \text{ or } A_1 = \frac{A_2 V_2}{V_1}$$

$$\therefore \quad \text{Change of momentum/sec} = \rho A_2 V_2^2 - \rho \times \frac{A_2 V_2}{V_1} \times V_1^2 = \rho A_2 V_2^2 - \rho A_2 V_1 V_2$$
$$= \rho A_2 [V_2^2 - V_1 V_2] \quad \text{...}(iii)$$

Now net force acting on the control volume in the direction of flow must be equal to the rate of change of momentum or change of momentum per second. Hence equating (*ii*) and (*iii*)

$$(p_1 - p_2)A_2 = \rho A_2 [V_2^2 - V_1 V_2]$$

or
$$\frac{p_1 - p_2}{\rho} = V_2^2 - V_1 V_2$$

Dividing by g on both sides, we have $\dfrac{p_1 - p_2}{\rho g} = \dfrac{V_2^2 - V_1 V_2}{g}$ or $\dfrac{p_1}{\rho g} - \dfrac{p_2}{\rho g} = \dfrac{V_2^2 - V_1 V_2}{g}$

Substituting the value of $\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right)$ in equation (*i*), we get

$$h_e = \frac{V_2^2 - V_1V_2}{g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = \frac{2V_2^2 - 2V_1V_2 + V_1^2 - V_2^2}{2g}$$

$$= \frac{V_2^2 + V_1^2 - 2V_1V_2}{2g} = \left(\frac{V_1 - V_2}{2g}\right)^2$$

$$\therefore \qquad h_e = \frac{(V_1 - V_2)^2}{2g}. \qquad \text{...(11.5)}$$

11.4.2 Loss of Head due to Sudden Contraction.

Consider a liquid flowing in a pipe which has a sudden contraction in area as shown in Fig. 11.2. Consider two sections 1-1 and 2-2 before and after contraction. As the liquid flows from large pipe to smaller pipe, the area of flow goes on decreasing and becomes minimum at a section *C-C* as shown in Fig. 11.2. This section *C-C* is called Vena-contracta. After section *C-C,* a sudden enlargement of the area takes place. The loss of head due to sudden contraction is actually due to sudden enlargement from Vena-contracta to smaller pipe.

Let A_c = Area of flow at section *C-C*

V_c = Velocity of flow at section *C-C*

A_2 = Area of flow at section 2-2

V_2 = Velocity of flow at section 2-2

h_c = Loss of head due to sudden contraction.

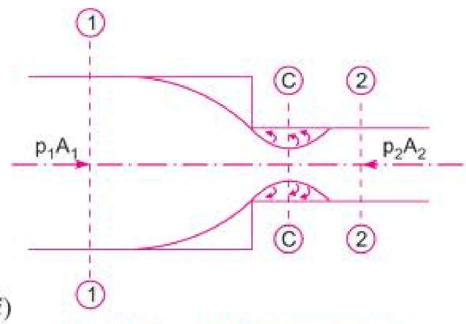


Fig. 11.2 *Sudden contraction.*

Now h_c = actual loss of head due to enlargement from section *C-C* to section 2-2 and is given by equation (11.5) as

$$= \frac{(V_c - V_2)^2}{2g} = \frac{V_2^2}{2g}\left[\frac{V_c}{V_2} - 1\right]^2 \qquad \text{...}(i)$$

From continuity equation, we have

$$A_cV_c = A_2V_2 \quad \text{or} \quad \frac{V_c}{V_2} = \frac{A_2}{A_c} = \frac{1}{(A_c / A_2)} = \frac{1}{C_c} \qquad \left[\because C_c = \frac{A_c}{A_2}\right]$$

Substituting the value of $\frac{V_c}{V_2}$ in (*i*), we get

$$h_c = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2 \qquad \text{...(11.6)}$$

$$= \frac{kV_2^2}{2g}, \text{ where } k = \left[\frac{1}{C_c} - 1\right]^2$$

If the value of C_c is assumed to be equal to 0.62, then

$$k = \left[\frac{1}{0.62} - 1\right]^2 = 0.375$$

Then h_c becomes as $$h_c = \frac{kV_2^2}{2g} = 0.375\,\frac{V_2^2}{2g}$$

If the value of C_c is not given then the head loss due to contraction is taken as

$$= 0.5\,\frac{V_2^2}{2g} \text{ or } h_c = 0.5\,\frac{V_2^2}{2g}. \qquad \text{...(11.7)}$$

Problem 11.8 *Find the loss of head when a pipe of diameter 200 mm is suddenly enlarged to a diameter of 400 mm. The rate of flow of water through the pipe is 250 litres/s.*

Solution. Given :

Dia. of smaller pipe, $D_1 = 200$ mm $= 0.20$ m

$\therefore$ Area, $$A_1 = \frac{\pi}{4} D_1^{\,2} = \frac{\pi}{4} (.2)^2 = 0.03141 \text{ m}^2$$

Dia. of large pipe, $$D_2 = 400 \text{ mm} = 0.4 \text{ m}$$

$\therefore$ Area, $$A_2 = \frac{\pi}{4} \times (0.4)^2 = 0.12564 \text{ m}^2$$

Discharge, $$Q = 250 \text{ litres/s} = 0.25 \text{ m}^3\text{/s}$$

Velocity, $$V_1 = \frac{Q}{A_1} = \frac{0.25}{.03141} = 7.96 \text{ m/s}$$

Velocity, $$V_2 = \frac{Q}{A_2} = \frac{0.25}{.12564} = 1.99 \text{ m/s}$$

Loss of head due to enlargement is given by equation (11.5) as

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2g} = \textbf{1.816 m of water. Ans.}$$

Problem 11.9 *At a sudden enlargement of a water main from 240 mm to 480 mm diameter, the hydraulic gradient rises by 10 mm. Estimate the rate of flow.* (J.N.T.U., S 2002)

Solution. Given :

Dia. of smaller pipe, $$D_1 = 240 \text{ mm} = 0.24 \text{ m}$$

$\therefore$ Area, $$A_1 = \frac{\pi}{4} D_1^{\,2} = \frac{\pi}{4} (.24)^2$$

Dia. of large pipe, $$D_2 = 480 \text{ mm} = 0.48 \text{ m}$$

$\therefore$ Area, $$A_2 = \frac{\pi}{4} (0.48)^2$$

Rise of hydraulic gradient*, *i.e.*, $$\left(z_2 + \frac{p_2}{\rho g}\right) - \left(\frac{p_1}{\rho g} + z_1\right) = 10 \text{ mm} = \frac{10}{1000} = \frac{1}{100} \text{ m}$$

Let the rate of flow $= Q$

Applying Bernoulli's equation to both sections, *i.e.*, smaller pipe section, and large pipe section.

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{Head loss due to enlargement} \qquad \text{...}(i)$$

* Please refer Art. 11.5.1.

But head loss due to enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g} \quad \text{...(ii)}$$

From continuity equation, we have $A_1V_1 = A_2V_2$

$$\therefore \quad V_1 = \frac{A_2V_2}{A_1} = \frac{\frac{\pi}{4}D_2^2 \times V_2}{\frac{\pi}{4}D_1^2} = \left(\frac{D_2}{D_1}\right)^2 \times V_2 = \left(\frac{.48}{.24}\right)^2 \times V_2 = 2^2 \times V_2 = 4V_2$$

Substituting this value in (*ii*), we get

$$h_e = \frac{(4V_2 - V_2)^2}{2g} = \frac{(3V_2)^2}{2g} = \frac{9V_2^2}{2g}$$

Now substituting the value of h_e and V_1 in equation (*i*),

$$\frac{p_1}{\rho g} + \frac{(4V_2)^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \frac{9V_2^2}{2g}$$

or

$$\frac{16V_2^2}{2g} - \frac{V_2^2}{2g} - \frac{9V_2^2}{2g} = \left(\frac{p_2}{\rho g} + z_2\right) - \left(\frac{p_1}{\rho g} + z_1\right)$$

But hydraulic gradient rise $= \left(\frac{p_2}{\rho g} + z_2\right) - \left(\frac{p_1}{\rho g} + z_1\right) = \frac{1}{100}$

$$\therefore \quad \frac{16V_2^2}{2g} - \frac{V_2^2}{2g} - \frac{9V_2^2}{2g} = \frac{1}{100} \text{ or } \frac{6V_2^2}{2g} = \frac{1}{100}$$

$$\therefore \quad V_2 = \sqrt{\frac{2 \times 9.81}{6 \times 100}} = 0.1808 \simeq 0.181 \text{ m/s}$$

$\therefore$ Discharge, $Q = A_2 \times V_2$

$$= \frac{\pi}{4} D_2^2 \times V_2 = \frac{\pi}{4} (.48)^2 \times .181 = 0.03275 \text{ m}^3\text{/s}$$

= **32.75 litres/s. Ans.**

Problem 11.10 *The rate of flow of water through a horizontal pipe is 0.25 m³/s. The diameter of the pipe which is 200 mm is suddenly enlarged to 400 mm. The pressure intensity in the smaller pipe is 11.772 N/cm². Determine :*

(*i*) *loss of head due to sudden enlargement,* (*ii*) *pressure intensity in the large pipe,*
(*iii*) *power lost due to enlargement.*

Solution. Given :

Discharge, $Q = 0.25 \text{ m}^3\text{/s}$

Dia. of smaller pipe, $D_1 = 200 \text{ mm} = 0.20 \text{ m}$

$\therefore$ Area, $A_1 = \dfrac{\pi}{4}(.2)^2 = 0.03141 \text{ m}^2$

Dia. of large pipe, $D_2 = 400$ mm $= 0.40$ m

$\therefore$ Area, $A_2 = \dfrac{\pi}{4}(0.4)^2 = 0.12566 \text{ m}^2$

Pressure in smaller pipe, $p_1 = 11.772 \text{ N/cm}^2 = 11.772 \times 10^4 \text{ N/m}^2$

Now velocity, $V_1 = \dfrac{Q}{A_1} = \dfrac{0.25}{.03141} = 7.96$ m/s

Velocity, $V_2 = \dfrac{Q}{A_2} = \dfrac{0.25}{.12566} = 1.99$ m/s

(*i*) Loss of head due to sudden enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2 \times 9.81} = \mathbf{1.816 \ m.} \ \textbf{Ans.}$$

(*ii*) Let the pressure intensity in large pipe $= p_2$.

Then applying Bernoulli's equation before and after the sudden enlargement,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_e$$

But $z_1 = z_2$ (Given horizontal pipe)

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_e \text{ or } \frac{p_2}{\rho g} = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - h_e$$

$$= \frac{11.772 \times 10^4}{1000 \times 9.81} + \frac{7.96^2}{2 \times 9.81} - \frac{1.99^2}{2 \times 9.81} - 1.816$$

$$= 12.0 + 3.229 - 0.2018 - 1.8160$$

$$= 15.229 - 2.0178 = 13.21 \text{ m of water}$$

$$\therefore \quad p_2 = 13.21 \times \rho g = 13.21 \times 1000 \times 9.81 \text{ N/m}^2$$

$$= 13.21 \times 1000 \times 9.81 \times 10^{-4} \text{ N/cm}^2 = \mathbf{12.96 \ N/cm^2.} \ \textbf{Ans.}$$

(*iii*) Power lost due to sudden enlargement,

$$P = \frac{\rho g \,.\, Q \,.\, h_e}{1000} = \frac{1000 \times 9.81 \times 0.25 \times 1.816}{1000} = \mathbf{4.453 \ kW.} \ \textbf{Ans.}$$

Problem 11.11 *A horizontal pipe of diameter 500 mm is suddenly contracted to a diameter of 250 mm. The pressure intensities in the large and smaller pipe is given as 13.734 N/cm² and 11.772 N/cm² respectively. Find the loss of head due to contraction if $C_c = 0.62$. Also determine the rate of flow of water.*

Solution. Given :

Dia. of large pipe, $D_1 = 500$ mm $= 0.5$ m

Area, $A_1 = \dfrac{\pi}{4}(0.5)^2 = 0.1963 \text{ m}^2$

Dia. of smaller pipe, $D_2 = 250$ mm $= 0.25$ m

$\therefore$ Area, $A_2 = \frac{\pi}{4}(.25)^2 = 0.04908 \text{ m}^2$

Pressure in large pipe, $p_1 = 13.734 \text{ N/cm}^2 = 13.734 \times 10^4 \text{ N/m}^2$

Pressure in smaller pipe, $p_2 = 11.772 \text{ N/cm}^2 = 11.772 \times 10^4 \text{ N/m}^2$

$C_c = 0.62$

Head lost due to contraction $= \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1.0\right]^2 = \frac{V_2^2}{2g}\left[\frac{1}{0.62} - 1.0\right]^2 = 0.375\,\frac{V_2^2}{2g}$

From continuity equation, we have $A_1V_1 = A_2V_2$

or
$$V_1 = \frac{A_2V_2}{A_1} = \frac{\frac{\pi}{4}D_2^2 \times V_2}{\frac{\pi}{4}D_1^2} = \left(\frac{D_2}{D_1}\right)^2 \times V_2 = \left(\frac{0.25}{0.50}\right)^2 V_2 = \frac{V_2}{4}$$

Applying Bernoulli's equation before and after contraction,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_c$$

But $z_1 = z_2$ (pipe is horizontal)

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c$$

But $h_c = 0.375\,\frac{V_2^2}{2g}$ and $V_1 = \frac{V_2}{4}$

Substituting these values in the above equation, we get

$$\frac{13.734 \times 10^4}{9.81 \times 1000} + \frac{(V_2/4)^2}{2g} = \frac{11.772 \times 10^4}{1000 \times 9.81} + \frac{V_2^2}{2g} + 0.375\,\frac{V_2^2}{2g}$$

or
$$14.0 + \frac{V_2^2}{16 \times 2g} = 12.0 + 1.375\,\frac{V_2^2}{2g}$$

or
$$14 - 12 = 1.375\,\frac{V_2^2}{2g} - \frac{1}{16}\,\frac{V_2^2}{2g} = 1.3125\,\frac{V_2^2}{2g}$$

or
$$2.0 = 1.3125 \times \frac{V_2^2}{2g} \text{ or } V_2 = \sqrt{\frac{2.0 \times 2 \times 9.81}{1.3125}} = 5.467 \text{ m/s.}$$

(*i*) Loss of head due to contraction, $h_c = 0.375\,\frac{V_2^2}{2g} = \frac{0.375 \times (5.467)^2}{2 \times 9.81}$ = **0.571 m. Ans.**

(*ii*) Rate of flow of water, $Q = A_2V_2 = 0.04908 \times 5.467 = 0.2683 \text{ m}^3\text{/s}$ = **268.3 lit/s. Ans.**

Problem 11.12 *If in the problem 11.11, the rate of flow of water is 300 litres/s, other data remaining the same, find the value of co-efficient of contraction, C_c.*

Solution. Given :

$$D_1 = 0.5 \text{ m}, D_2 = 0.25 \text{ m}, p_1 = 13.734 \times 10^4 \text{ N/m}^2,$$
$$p_2 = 11.772 \times 10^4 \text{ N/m}^2, Q = 300 \text{ lit/s} = 0.3 \text{ m}^3\text{/s}$$

Also from Problem 11.11, $V_1 = \frac{V_2}{4}$, where $V_1 = \frac{Q}{A_1} = \frac{0.30}{\frac{\pi}{4}(0.5)^2} = 1.528$ m/s

$$\therefore \quad V_2 = 4 \times V_1 = 4 \times 1.528 = 6.112 \text{ m/s}$$

From Bernoulli's equation, we have

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c. \qquad [\because z_1 = z_2]$$

or $$\frac{13.734 \times 10^4}{9.81 \times 1000} + \frac{(1.528)^2}{2 \times 9.81} = \frac{11.772 \times (10)^4}{9.81 \times 1000} + \frac{(6.112)^2}{2 \times 9.81} + h_c$$

or $$14.0 + 0.119 = 12.0 + 1.904 + h_c$$

$$14.119 = 13.904 + h_c$$

$$\therefore \quad h_c = 14.119 - 13.904 = 0.215$$

But from equation (11.6), $h_c = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2$

Hence equating the two values of h_c, we get

$$\frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2 = 0.215$$

$$V_2 = 6.112, \therefore \frac{6.112^2}{2 \times 9.81}\left[\frac{1}{C_c} - 1\right]^2 = 0.215$$

or $$\left[\frac{1}{C_c} - 1\right]^2 = \frac{0.215 \times 2.0 \times 9.81}{6.112 \times 6.112} = 0.1129$$

or $$\frac{1.0}{C_c} - 1.0 = \sqrt{0.1129} = 0.336 \text{ or } \frac{1.0}{C_c} = 1.0 + 0.336 = 1.336$$

$$\therefore \quad C_c = \frac{1.0}{1.336} = \mathbf{0.748}. \text{ Ans.}$$

Problem 11.13 *A 150 mm diameter pipe reduces in diameter abruptly to 100 mm diameter. If the pipe carries water at 30 litres per second, calculate the pressure loss across the contraction. Take the co-efficient of contraction as 0.6.*

Solution. Given :

Dia. of large pipe, $D_1 = 150 \text{ mm} = 0.15 \text{ m}$

Area of large pipe, $A_1 = \dfrac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$

Dia. of smaller pipe, $D_2 = 100 \text{ mm} = 1.10 \text{ m}$

Area of smaller pipe, $A_2 = \dfrac{\pi}{4}(.10)^2 = 0.007854 \text{ m}^2$

Discharge, $Q = 30 \text{ litres/s} = .03 \text{ m}^3\text{/s}$

Co-efficient of contraction, $C_c = 0.6$

From continuity equation, we have $A_1V_1 = A_2V_2 = Q$

$$\therefore \quad V_1 = \frac{Q}{A_1} = \frac{0.03}{.01767} = 1.697 \text{ m/s}$$

and
$$V_2 = \frac{Q}{A_2} = \frac{.03}{.007854} = 3.82 \text{ m/s}$$

Applying Bernoulli's equation before and after contraction,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 + h_c \qquad \ldots(i)$$

But $Z_1 = Z_2$

and h_c, the head loss due to contraction is given by equation (11.6) as

$$h_c = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2 = \frac{3.82^2}{2 \times 9.81}\left[\frac{1}{0.6} - 1\right]^2 = 0.33$$

Substituting these values in equation (i), we get

$$\frac{p_1}{\rho g} + \frac{1.697^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{3.82^2}{2 \times 9.81} + 0.33$$

or
$$\frac{p_1}{\rho g} + 0.1467 = \frac{p_2}{\rho g} + .7438 + .33$$

$$\therefore \quad \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = .7438 + .33 - .1467 = 0.9271 \text{ m of water}$$

$$\therefore \quad (p_1 - p_2) = \rho g \times 0.9271 = 1000 \times 9.81 \times 0.9271 \text{ N/m}^2$$
$$= 0.909 \times 10^4 \text{ N/m}^2 = 0.909 \text{ N/cm}^2$$

$\therefore$ Pressure loss across contraction
$= p_1 - p_2 =$ **0.909 N/cm². Ans.**

Problem 11.14 *In Fig. 11.3 below, when a sudden contraction is introduced in a horizontal pipe line from 50 cm to 25 cm, the pressure changes from 10,500 kg/m² (103005 N/m²) to 6900 kg/m² (67689 N/m²). Calculate the rate of flow. Assume co-efficient of contraction of jet to be 0.65.*

Following this if there is a sudden enlargement from 25 cm to 50 cm and if the pressure at the 25 cm section is 6900 kg/m² (67689 N/m²) what is the pressure at the 50 cm enlarged section ?

Solution. Given :

Dia. of large pipe, $D_1 = 50 \text{ cm} = 0.5 \text{ m}$

Area, $A_1 = \frac{\pi}{4}(.5)^2 = 0.1963 \text{ m}^2$

Dia. of smaller pipe, $D_2 = 25 \text{ cm} = 0.25 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4}(.25)^2 = 0.04908 \text{ m}^2$

Pressure in large pipe, $p_1 = 10500 \text{ kg/m}^2$ or 103005 N/m^2

Pressure in smaller pipe, $p_2 = 6900 \text{ kg/m}^2$ or 67689 N/m^2

Co-efficient of contraction, $C_c = 0.65$

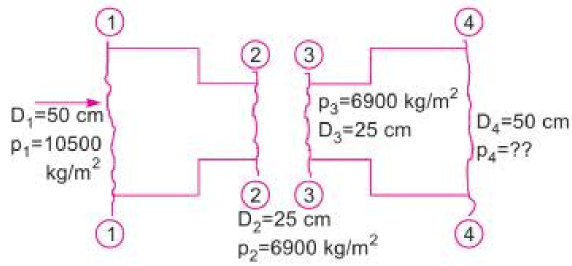


Fig. 11.3

Head lost due to contraction is given by equation (11.6),

$$h_c = \frac{V_2^2}{2g}\left(\frac{1}{C_c} - 1.0\right)^2 = \frac{V_2^2}{2g}\left(\frac{1}{0.65} - 1\right)^2 = 0.2899 \frac{V_2^2}{2g} \quad \ldots(i)$$

From continuity equation, we have

$$A_1V_1 = A_2V_2 \text{ or } V_1 = \frac{A_2V_2}{A_1} = \frac{\frac{\pi}{4}D_2^2 \times V_2}{\frac{\pi}{4}D_1^2}$$

$$= \left(\frac{D_2}{D_1}\right)^2 \times V_2 = \left(\frac{0.50}{0.25}\right)^2 \times V_2 = \frac{V_2}{4} \quad \ldots(ii)$$

Applying Bernoulli's equation at sections 1-1 and 2-2,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 + h_c$$

But $Z_1 = Z_2$ (as pipe is horizontal)

$$\therefore \qquad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c$$

Substituting the values of p_1, p_2, h_c and V_1, we get

$$\frac{103005}{1000 \times 9.81} + \frac{(V_2/4)^2}{2g} = \frac{67689}{1000 \times 9.81} + \frac{V_2^2}{2g} + .2899\ \frac{V_2^2}{2g}$$

or
$$10.5 + \frac{V_2^2}{16 \times 2g} = 6.9 + 1.2899\ \frac{V_2^2}{2g}$$

or
$$10.5 - 6.9 = 1.2899\ \frac{V_2^2}{2g} - \frac{1}{16} \times \frac{V_2^2}{2g} = 1.2274\ \frac{V_2^2}{2g}$$

or
$$3.6 = 1.2274 \times \frac{V_2^2}{2g}$$

$$\therefore \qquad V_2 = \sqrt{\frac{3.6 \times 2 \times 9.81}{1.2274}} = 7.586 \text{ m/s}$$

(*i*) Rate of flow of water, $Q = A_2 V_2 = 0.04908 \times 7.586$

$= \mathbf{0.3723\ m^3/s}$ **or 372.3 lit/s. Ans.**

(*ii*) Applying Bernoulli's equation to sections 3-3 and 4-4,

$$\frac{p_3}{\rho g} + \frac{V_3^2}{2g} + Z_3 = \frac{p_4}{\rho g} + \frac{V_4^2}{2g} + Z_4 + \text{head loss due to sudden enlargement } (h_e)$$

But
$$p_3 = 6900 \text{ kg/m}^2, \text{ or } 67689 \text{ N/m}^2$$
$$V_3 = V_2 = 7.586 \text{ m/s}$$
$$V_4 = V_1 = \frac{V_2}{4} = \frac{7.586}{4} = 1.8965$$
$$Z_3 = Z_4$$

And head loss due to sudden enlargement is given by equation (11.5) as

$$h_e = \frac{(V_3 - V_4)^2}{2g} = \frac{(7.586 - 1.8965)^2}{2 \times 9.81} = 1.65 \text{ m}$$

Substituting these values in Bernoulli's equation, we get

$$\frac{67689}{1000 \times 9.81} + \frac{7.586^2}{2 \times 9.81} = \frac{p_4}{1000 \times 9.81} + \frac{1.8965^2}{2 \times 9.81} + 1.65$$

or
$$6.9 + 2.933 = \frac{p_4}{1000 \times 9.81} + 0.183 + 1.65$$

$$\therefore \quad \frac{p_4}{1000 \times 9.81} = 6.9 + 2.933 - 0.183 - 1.65 = 9.833 - 1.833 = 8.00$$

$$\therefore \quad p_4 = 8 \times 1000 \times 9.81 = \mathbf{78480\ N/m^2}. \text{ Ans.}$$

11.4.3 Loss of Head at the Entrance of a Pipe. This is the loss of energy which occurs when a liquid enters a pipe which is connected to a large tank or reservoir. This loss is similar to the loss of head due to sudden contraction. This loss depends on the form of entrance. For a sharp edge entrance, this loss is slightly more than a rounded or bell mouthed entrance. In practice the value of loss of head at the entrance (or inlet) of a pipe with sharp cornered entrance is taken = $0.5 \frac{V^2}{2g}$, where V = velocity of liquid in pipe.

This loss is denoted by h_i

$$\therefore \quad h_i = 0.5 \frac{V^2}{2g} \quad \text{...(11.8)}$$

11.4.4 Loss of Head at the Exit of Pipe. This is the loss of head (or energy) due to the velocity of liquid at outlet of the pipe which is dissipated either in the form of a free jet (if outlet of the pipe is free) or it is lost in the tank or reservoir (if the outlet of the pipe is connected to the tank or reservoir). This loss is equal to $\frac{V^2}{2g}$, where V is the velocity of liquid at the outlet of pipe. This loss is denoted h_o.

$$\therefore \quad h_o = \frac{V^2}{2g} \quad \text{...(11.9)}$$

where V = velocity at outlet of pipe.

11.4.5 Loss of Head Due to an Obstruction in a Pipe. Whenever there is an obstruction in a pipe, the loss of energy takes place due to reduction of the area of the cross-section of the pipe at the place where obstruction is present. There is a sudden enlargement of the area of flow beyond the obstruction due to which loss of head takes place as shown in Fig. 11.3 (*a*)

Consider a pipe of area of cross-section A having an obstruction as shown in Fig. 11.3 (*a*).

Let a = Maximum area of obstruction

A = Area of pipe

V = Velocity of liquid in pipe

Then $(A - a)$ = Area of flow of liquid at section 1-1.

As the liquid flows and passes through section 1-1, a vena-contracta is formed beyond section 1-1, after which the stream of liquid widens again and velocity of flow at section 2-2 becomes uniform and equal to velocity, V in the pipe. This situation is similar to the flow of liquid through sudden enlargement.

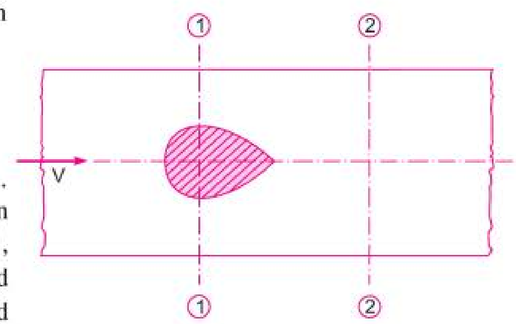


Fig. 11.3 (*a*) *An obstruction in a pipe.*

Let V_c = Velocity of liquid at vena-contracta.

Then loss of head due to obstruction = loss of head due to enlargement from vena-contracta to section 2-2.

$$= \frac{(V_c - V)^2}{2g} \quad \text{...}(i)$$

From continuity, we have $a_c \times V_c = A \times V$ ...(*ii*)

where a_c = area of cross-section at vena-contracta

If C_c = co-efficient of contraction

Then
$$C_c = \frac{\text{area at vena - contracta}}{(A-a)} = \frac{a_c}{(A-a)}$$

$\therefore$
$$a_c = C_c \times (A - a)$$

Substituting this value in (*ii*), we get

$$C_c \times (A - a) \times V_c = A \times V \quad \therefore \quad V_c = \frac{A \times V}{C_c\,(A-a)}$$

Substituting this value of V_c in equation (*i*), we get

$$\text{Head loss due to obstruction} = \frac{(V_c - V)^2}{2g} = \frac{\left(\frac{A \times V}{C_c\,(A-a)} - V\right)^2}{2g} = \frac{V^2}{2g}\left(\frac{A}{C_c\,(A-a)} - 1\right)^2 \quad \text{...(11.10)}$$

11.4.6 Loss of Head due to Bend in Pipe. When there is any bend in a pipe, the velocity of flow changes, due to which the separation of the flow from the boundary and also formation of eddies takes place. Thus the energy is lost. Loss of head in pipe due to bend is expressed as

$$h_b = \frac{kV^2}{2g}$$

where h_b = loss of head due to bend, V = velocity of flow, k = co-efficient of bend

The value of k depends on

(*i*) Angle of bend, (*ii*) Radius of curvature of bend, (*iii*) Diameter of pipe.

11.4.7 Loss of Head in Various Pipe Fittings. The loss of head in the various pipe fittings such as valves, couplings etc., is expressed as

$$= \frac{kV^2}{2g} \quad \text{...(11.11)}$$

where V = velocity of flow, k = co-efficient of pipe fitting.

Problem 11.15 *Water is flowing through a horizontal pipe of diameter 200 mm at a velocity of 3 m/s. A circular solid plate of diameter 150 mm is placed in the pipe to obstruct the flow. Find the loss of head due to obstruction in the pipe if C_c = 0.62.*

Solution. Given :

Dia. of pipe, $D = 200 \text{ mm} = 0.20 \text{ m}$

Velocity, $V = 3.0 \text{ m/s}$

Area of pipe, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.2)^2 = 0.03141 \text{ m}^2$

Dia. of obstruction, $d = 150 \text{ mm} = 0.15 \text{ m}$

$\therefore$ Area of obstruction, $a = \frac{\pi}{4} (.15)^2 = 0.01767 \text{ m}^2$

$$C_c = 0.62$$

The head lost due to obstruction is given by equation (11.10) as

$$= \frac{V^2}{2g}\left(\frac{A}{C_c\,(A-a)} - 1.0\right)^2$$

$$= \frac{3\times 3}{2\times 9.81}\left[\frac{.03141}{0.62\,[.03141 - .01767]} - 1.0\right]^2$$

$$= \frac{9}{2\times 9.81}\,[3.687 - 1.0]^2 = \mathbf{3.311\ m.}\ \textbf{Ans.}$$

Problem 11.16 *Determine the rate of flow of water through a pipe of diameter 20 cm and length 50 m when one end of the pipe is connected to a tank and other end of the pipe is open to the atmosphere. The pipe is horizontal and the height of water in the tank is 4 m above the centre of the pipe. Consider all minor losses and take f = .009 in the formula* $h_f = \dfrac{4\,.\,f\,.\,L\,.\,V^2}{d\times 2g}$.

Solution. Dia. of pipe, d = 20 cm = 0.20 m

Length of pipe, L = 50 m

Height of water, H = 4 m

Co-efficient of friction, f = .009

Let the velocity of water in pipe = V m/s.

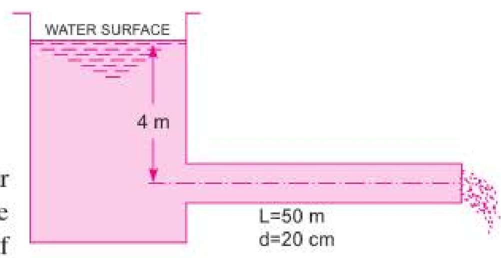


Fig. 11.4

Applying Bernoulli's equation at the top of the water surface in the tank and at the outlet of pipe, we have [Taking point 1 on the top and point 2 at the outlet of pipe].

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{all losses}$$

Considering datum line passing through the centre of pipe

$$0 + 0 + 4.0 = 0 + \frac{V_2^2}{2g} + 0 + (h_i + h_f)$$

or
$$4.0 = \frac{V_2^2}{2g} + h_i + h_f$$

But the velocity in pipe = V, $\therefore V = V_2$

$\therefore$
$$4.0 = \frac{V^2}{2g} + h_i + h_f \qquad ...(i)$$

From equation (11.8), $h_i = 0.5\,\dfrac{V^2}{2g}$ and h_f from equation (11.1) is given as

$$h_f = \frac{4 \, . \, f \, . \, L \, . \, V^2}{d \times 2g}$$

Substituting these values, we have

$$4.0 = \frac{V^2}{2g} + \frac{0.5 \, V^2}{2g} + \frac{4 \, . \, f \, . L \, . \, V^2}{d \times 2g}$$

$$= \frac{V^2}{2g}\left[1.0 + 0.5 + \frac{4 \times .009 \times 50}{0.2}\right] = \frac{V^2}{2g} \; [1.0 + 0.5 + 9.0]$$

$$= 10.5 \times \frac{V^2}{2g}$$

$\therefore$
$$V = \sqrt{\frac{4 \times 2 \times 9.81}{10.5}} = 2.734 \text{ m/sec}$$

$\therefore$ Rate of flow,
$$Q = A \times V = \frac{\pi}{4} \times (0.2)^2 \times 2.734 = 0.08589 \text{ m}^3\text{/s}$$

$$= \mathbf{85.89 \text{ litres/s.}}$$ **Ans.**

Problem 11.17 *A horizontal pipe line 40 m long is connected to a water tank at one end and discharges freely into the atmosphere at the other end. For the first 25 m of its length from the tank, the pipe is 150 mm diameter and its diameter is suddenly enlarged to 300 mm. The height of water level in the tank is 8 m above the centre of the pipe. Considering all losses of head which occur, determine the rate of flow. Take f = .01 for both sections of the pipe.*

Solution. Given :

Total length of pipe, $L = 40$ m

Length of 1st pipe, $L_1 = 25$ m

Dia. of 1st pipe, $d_1 = 150 \text{ mm} = 0.15$ m

Length of 2nd pipe, $L_2 = 40 - 25 = 15$ m

Dia. of 2nd pipe, $d_2 = 300 \text{ mm} = 0.3$ m

Height of water, $H = 8$ m

Co-efficient of friction, $f = 0.01$

Applying Bernoulli's theorem to the free surface of water in the tank and outlet of pipe as shown in Fig. 11.5 and taking reference line passing through the centre of pipe.

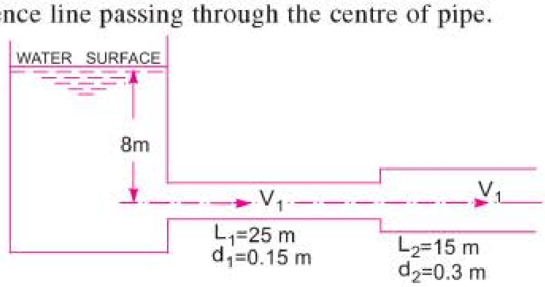


Fig. 11.5

$$0 + 0 + 8 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + 0 + \text{all losses}$$

or
$$8.0 = 0 + \frac{V_2^2}{2g} + h_i + h_{f_1} + h_e + h_{f_2} \quad \ldots(i)$$

where h_i = loss of head at entrance = $0.5 \frac{V_1^2}{2g}$

h_{f_1} = head lost due to friction in pipe 1 = $\frac{4 \times f \times L_1 \times V_1^2}{d_1 \times 2g}$

h_e = loss head due to sudden enlargement = $\frac{(V_1 - V_2)^2}{2g}$

h_{f_2} = Head lost due to friction in pipe 2 = $\frac{4 \times f \times L_2 \times V_2^2}{d_2 \times 2g}$

But from continuity equation, we have

$$A_1 V_1 = A_2 V_2$$

$$\therefore \quad V_1 = \frac{A_2 V_2}{A_1} = \frac{\frac{\pi}{4} d_2^2 \times V_2}{\frac{\pi}{4} d_1^2} = \left(\frac{d_2}{d_1}\right)^2 \times V_2 = \left(\frac{0.3}{.15}\right)^2 \times V_2 = 4V_2$$

Substituting the value of V_1 in different head losses, we have

$$h_i = \frac{0.5\, V_1^2}{2g} = \frac{0.5 \times (4V_2)^2}{2g} = \frac{8V_2^2}{2g}$$

$$h_{f_1} = \frac{4 \times 0.01 \times 25 \times (4V_2)^2}{0.15 \times 2 \times g}$$

$$= \frac{4 \times .01 \times 25 \times 16}{0.15} \times \frac{V_2^2}{2g} = 106.67\, \frac{V_2^2}{2g}$$

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(4V_2 - V_2)^2}{2g} = \frac{9V_2^2}{2g}$$

$$h_{f_2} = \frac{4 \times .01 \times 15 \times V_2^2}{0.3 \times 2g} = \frac{4 \times .01 \times 15}{0.3} \times \frac{V_2^2}{2g} = 2.0 \times \frac{V_2^2}{2g}$$

Substituting the values of these losses in equation (i), we get

$$8.0 = \frac{V_2^2}{2g} + \frac{8V_2^2}{2g} + 106.67\, \frac{V_2^2}{2g} + \frac{9V_2^2}{2g} + 2 \times \frac{V_2^2}{2g}$$

$$= \frac{V_2^2}{2g}\,[1 + 8 + 106.67 + 9 + 2] = 126.67\, \frac{V_2^2}{2g}$$

$$\therefore \quad V_2 = \sqrt{\frac{8.0 \times 2 \times g}{126.67}} = \sqrt{\frac{8.0 \times 2 \times 9.81}{126.67}} = \sqrt{1.2391} = 1.113 \text{ m/s}$$

$\therefore$ Rate of flow, $Q = A_2 \times V_2 = \frac{\pi}{4} (0.3)^2 \times 1.113 = 0.07867 \text{ m}^3\text{/s}$ = **78.67 litres/s. Ans.**

Problem 11.18 *Determine the difference in the elevations between the water surfaces in the two tanks which are connected by a horizontal pipe of diameter 300 mm and length 400 m. The rate of flow of water through the pipe is 300 litres/s. Consider all losses and take the value of f = .008.*

Solution. Given :

Dia. of pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

Length, $L = 400 \text{ m}$

Discharge, $Q = 300 \text{ lit/s} = 0.3 \text{ m}^3/\text{s}$

Co-efficient of friction, $f = 0.008$

Velocity, $$V = \frac{Q}{\text{Area}} = \frac{0.3}{\frac{\pi}{4} \times (.3)^2} = 4.244 \text{ m/s}$$

Let the two tanks are connected by a pipe as shown in Fig. 11.6.

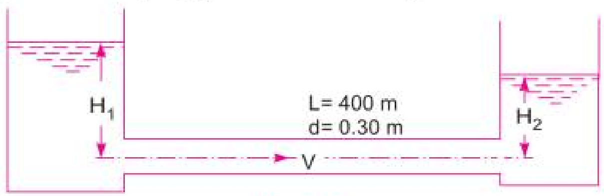


Fig. 11.6

Let H_1 = height of water in 1st tank above the centre of pipe

H_2 = height of water in 2nd tank above the centre of pipe

Then difference in elevations between water surfaces = $H_1 - H_2$

Applying Bernoulli's equation to the free surface of water in the two tanks, we have

$$H_1 = H_2 + \text{losses}$$

$$= H_2 + h_i + H_{f_1} + h_o \qquad \ldots(i)$$

where h_i = Loss of head at entrance $= 0.5 \dfrac{V^2}{2g} = \dfrac{0.5 \times 4.244^2}{2 \times 9.81} = 0.459 \text{ m}$

h_{f_1} = Loss of head due to friction $= \dfrac{4 \times f \times L \times V^2}{d \times 2g} = \dfrac{4 \times .008 \times 400 \times 4.244^2}{0.3 \times 2 \times 9.81} = 39.16 \text{ m}$

h_o = Loss of head at outlet $= \dfrac{V^2}{2g} = \dfrac{4.244^2}{2 \times 9.81} = 0.918 \text{ m}$

Substituting these values in (*i*), we get

$$H_1 = H_2 + 0.459 + 39.16 + 0.918 = H_2 + 40.537$$

$\therefore$ $H_1 - H_2$ = Difference in elevations

= **40.537 m. Ans.**

Problem 11.19 *The friction factor for turbulent flow through rough pipes can be determined by Karman-Prandtl equation,* $\dfrac{1}{\sqrt{f}} = 2 \log_{10} (R_0/k) + 1.74$

where f = friction factor, R_0 *= pipe radius, k = average roughness.*

Two reservoirs with a surface level difference of 20 metres are to be connected by 1 metre diameter pipe 6 km long. What will be the discharge when a cast iron pipe of roughness k = 0.3 mm is used ? What will be the percentage increase in the discharge if the cast iron pipe is replaced by a steel pipe of roughness k = 0.1 mm ? Neglect all local losses.

Solution. Given :

Difference in levels, $h = 20$ m

Dia. of pipe, $d = 1.0$ m $\therefore$ Radius, $R_0 = 0.5$ m $= 500$ mm

Length of pipe, $L = 6$ km $= 6 \times 1000 = 6000$ m

Roughness of cast iron pipe, $k = 0.3$ mm

Roughness of steel pipe, $k = 0.1$ mm

1st Case. ***Cast Iron Pipe.*** First find the value of friction factor using

$$\frac{1}{\sqrt{f}} = 2 \log_{10} (R_0/k) + 1.74 \qquad ...(i)$$

$$= 2\log_{10} (500/0.3) + 1.74 = 8.1837$$

$$\therefore \qquad f = \left(\frac{1}{8.1837}\right)^2 = 0.0149$$

Local losses are to be neglected. This means only head loss due to friction is to be considered. Head loss due to friction is

$$20 = \frac{f \times L \times V^2}{d \times 2g}$$

[Here f is the friction factor and not co-efficient of friction
$\because$ Friction factor = 4 × co-efficient of friction]

$$20 = \frac{.0149 \times 6000 \times V^2}{1.0 \times 2 \times 9.81} = 4.556\ V^2$$

$$\therefore \qquad V = \sqrt{\frac{20}{4.556}} = 2.095 \text{ m/s}$$

$$\therefore \quad \text{Discharge, } Q_1 = V \times \text{Area} = 2.095 \times \frac{\pi}{4} \times d^2 = 2.095 \times \frac{\pi}{4} \times 1^2 = 1.645 \text{ m}^3\text{/s}$$

2nd Case. ***Steel Pipe.*** $k = 0.1$ mm, $R_0 = 500$ mm

Substituting these values in equation (i), we get

$$\frac{1}{\sqrt{f}} = 2 \log_{10} (500/0.1) + 1.74 = 9.1379$$

$$\therefore \qquad f = \left(\frac{1}{9.1379}\right)^2 = 0.0119$$

Head loss due to friction, $20 = \dfrac{f \times L \times V^2}{d \times 2g}$ or $20 = \dfrac{.0119 \times 6000 \times V^2}{1.0 \times 2 \times 9.81} = 3.639\ V^2$

$$\therefore \quad V = \sqrt{\frac{20}{3.639}} = 2.344 \text{ m/s}$$

$\therefore$ Discharge,
$$Q_2 = V \times \text{Area} = 2.344 \times \frac{\pi}{4} \times 1^2 = 1.841 \text{ m}^3\text{/s}$$

percentage increase in the discharge $= \dfrac{Q_2 - Q_1}{Q_1} \times 100 = \dfrac{(1.841 - 1.645)}{1.645} \times 100 =$ **11.91%. Ans.**

Problem 11.20 *Design the diameter of a steel pipe to carry water having kinematic viscosity $\nu = 10^{-6}$ m²/s with a mean velocity of 1 m/s. The head loss is to be limited to 5 m per 100 m length of pipe. Consider the equivalent sand roughness height of pipe $k_s = 45 \times 10^{-4}$ cm. Assume that the Darcy Weisbach friction factor over the whole range of turbulent flow can be expressed as*

$$f = 0.0055 \left[1 + \left(20 \times 10^3 \frac{k_s}{D} + \frac{10^6}{R_e} \right)^{1/3} \right]$$

where D = Diameter of pipe and R_e = Reynolds number.

Solution. Given :

Kinematic viscosity, $\nu = 10^{-6}$ m²/s

Mean velocity, $V = 1$ m/s

Head loss, $h_f = 5$ m in a length $L = 100$ m

Value of $k_s = 45 \times 10^{-4}$ cm $= 45 \times 10^{-6}$ m

Value of
$$f = 0.0055 \left[1 + \left(20 \times 10^3 \frac{k_s}{D} + \frac{10^6}{R_e} \right)^{1/3} \right] \quad \text{...(i)}$$

Using Darcy Weisbach equation, $h_f = \dfrac{4 \times f \times L \times V^2}{D \times 2g}$

or
$$f = \frac{h_f \times D \times 2g}{4 \times L \times V^2} = \frac{5 \times D \times 2 \times 9.81}{4 \times 100 \times 1^2} = 0.2452\, D$$

Now the Reynolds number is given by,

$$R_e = \frac{\rho VD}{\mu} = \frac{V \times D}{\nu} \qquad \left(\because \nu = \frac{\mu}{\rho} \right)$$

$$= \frac{1 \times D}{10^{-6}} = 10^6 D$$

Substituting the values of f, R_e and k_s in equation (i), we get

$$0.2452\, D = 0.0055 \left[1 + \left(20 \times 10^3 \times \frac{45 \times 10^{-6}}{D} + \frac{10^6}{10^6 D} \right)^{1/3} \right]$$

or
$$\frac{0.2452}{0.0055} D = \left[1 + \left(\frac{0.9}{D} + \frac{1}{D} \right)^{1/3} \right]$$

or $$44.58\ D = \left[1 + \left(\frac{1.9}{D}\right)^{1/3}\right] \text{ or } 44.58\ D - 1 = \left(\frac{1.9}{D}\right)^{1/3}$$

or $$(44.58\ D - 1)^3 = \frac{1.9}{D} \text{ or } D\ (44.58\ D - 1)^3 = 1.9 \qquad \ldots(ii)$$

Equation (*ii*) is solved by hit and trial method.

(*i*) Let $D = 0.1$ m, then L.H.S. of equation (*ii*) becomes as

L.H.S. $= 0.1\ (44.58 \times 0.1 - 1)^3 = 0.1 \times 3.458^3 = 4.135$

This is more than the R.H.S.

(*ii*) Let $D = 0.08$ m, then L.H.S. of equation (*ii*) becomes as

L.H.S. $= 0.08\ (44.58 \times 0.08 - 1)^3 = 0.08\ (2.5664)^3 = 1.352$

This is less than the R.H.S.

Hence value of D lies between 0.1 and 0.08

(*iii*) Let $D = 0.085$, then L.H.S. of equation (*ii*) becomes as

L.H.S. $= 0.085\ (44.58 \times 0.085 - 1)^3 = 1.844$

This value is slightly less than R.H.S. Hence increase the value of D slightly.

(*iv*) Let $D = 0.0854$ m, then L.H.S. of equation (*ii*) becomes as

L.H.S. $= 0.0854\ (44.58 \times 0.0854 - 1)^3 = 1.889$

This value is nearly equal to R.H.S.

$\therefore$ Correct value of $D =$ **0.0854 m. Ans.**

Problem 11.21 *A pipe line AB of diameter 300 mm and of length 400 m carries water at the rate of 50 litres/s. The flow takes place from A to B where point B is 30 metres above A. Find the pressure at A if the pressure at B is 19.62 N/cm². Take f = .008.*

Solution. Given :

Dia. of pipe, $d = 300$ mm $= 0.30$ m

Length of pipe, $L = 400$ m

Discharge, $Q = 50$ litres/s $= 0.05\ m^3/s$

$\therefore$ Velocity, $$V = \frac{Q}{\text{Area}} = \frac{0.05}{\frac{\pi}{4}d^2} = \frac{0.05}{\frac{\pi}{4} \times (.3)^2} = 0.7074 \text{ m/s}$$

Pressure at B, $p_B = 19.62\ N/cm^2 = 19.62 \times 10^4\ N/m^2$

$f = .008$

Applying Bernoulli's equations at points A and B and taking datum line passing through A, we have

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + h_f$$

But $V_A = V_B$ [$\because$ Dia. is same]

$z_A = 0,\ z_B = 30$

and $$h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$$

$$\therefore \qquad \frac{p_A}{\rho g} + 0 = \frac{19.62 \times 10^4}{1000 \times 9.81} + 30 + \frac{4 \times .008 \times 400 \times .7074^2}{0.3 \times 2 \times 9.81}$$

$$= 20 + 30 + 1.088 = 51.088$$

$$\therefore \qquad p_A = 51.088 \times 1000 \times 9.81 \text{ N/m}^2$$

$$= \frac{51.088 \times 1000 \times 9.81}{10^4}$$

$$= \mathbf{50.12\ N/cm^2}. \text{ Ans.}$$

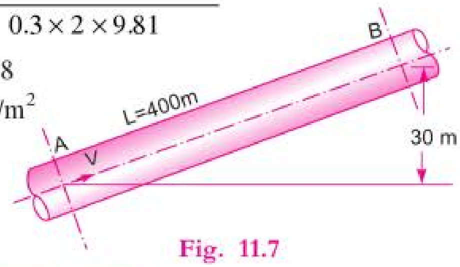


Fig. 11.7

▶ 11.5 HYDRAULIC GRADIENT AND TOTAL ENERGY LINE

The concept of hydraulic gradient line and total energy line is very useful in the study of flow of fluids through pipes. They are defined as :

11.5.1 Hydraulic Gradient Line. It is defined as the line which gives the sum of pressure head $\left(\frac{p}{w}\right)$ and datum head (z) of a flowing fluid in a pipe with respect to some reference line or it is the line which is obtained by joining the top of all vertical ordinates, showing the pressure head (p/w) of a flowing fluid in a pipe from the centre of the pipe. It is briefly written as H.G.L. (Hydraulic Gradient Line).

11.5.2 Total Energy Line. It is defined as the line which gives the sum of pressure head, datum head and kinetic head of a flowing fluid in a pipe with respect to some reference line. It is also defined as the line which is obtained by joining the tops of all vertical ordinates showing the sum of pressure head and kinetic head from the centre of the pipe. It is briefly written as T.E.L. (Total Energy Line).

Problem 11.22 *For the problem 11.16, draw the Hydraulic Gradient Line (H.G.L.) and Total Energy Line (T.E.L.).*

Solution. Given :

$$L = 50 \text{ m}, \ d = 200 \text{ mm} = 0.2 \text{ m}$$

$$H = 4 \text{ m}, \ f = .009$$

Velocity, V through pipe is calculated in problem 11.16 and its value is $V = 2.734$ m/s

Now, $\quad h_i$ = Head lost at entrance of pipe

$$= 0.5\ \frac{V^2}{2g} + \frac{0.5 \times 2.734^2}{2 \times 9.81} = 0.19 \text{ m}$$

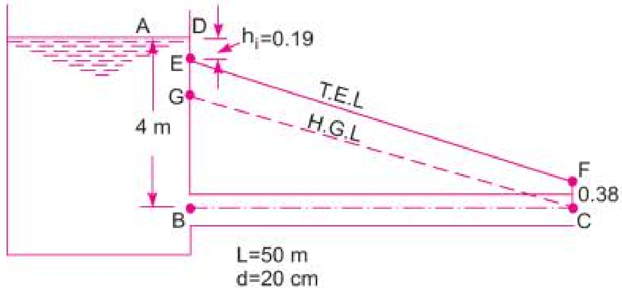


Fig. 11.8

and h_f = Head loss due to friction

$$= \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times 0.009 \times 50 \times (2.734)^2}{0.2 \times 2 \times 9.81} = 3.428 \text{ m}.$$

(*a*) **Total Energy Line (T.E.L.).** Consider three points, *A*, *B* and *C* on the free surface of water in the tank, at the inlet of the pipe and at the outlet of the pipe respectively as shown in Fig. 11.8. Let us find total energy at these points, taking the centre of pipe as reference line.

1. Total energy at $A = \frac{p}{\rho g} + \frac{V^2}{2g} + z = 0 + 0 + 4.0 = 4$ m
2. Total energy at B = Total energy at $A - h_i = 4.0 - 0.19 = 3.81$ m
3. Total energy at $C = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c = 0 + \frac{V^2}{2g} + 0 = \frac{2.734^2}{2 \times 9.81} = 0.38$ m.

Hence total energy line will coincide with free surface of water in the tank. At the inlet of the pipe, it will decrease by h_i (= 0.19 m) from free surface and at outlet of pipe total energy is 0.38 m. Hence in Fig. 11.8,

(*i*) Point *D* represents total energy at *A*

(*ii*) Point *E*, where $DE = h_i$, represents total energy at inlet of the pipe

(*iii*) Point *F*, where $CF = 0.38$ represents total energy at outlet of pipe.

Join *D* to *E* and *E* to *F*. Then *DEF* represents the total energy line.

(*b*) **Hydraulic Gradient Line (H.G.L.).** H.G.L. gives the sum of $(p/w + z)$ with reference to the datum-line. Hence hydraulic gradient line is obtained by subtracting $\frac{V^2}{2g}$ from total energy line. At outlet of the pipe, total energy $= \frac{V^2}{2g}$. By subtracting $\frac{V^2}{2g}$ from total energy at this point, we shall get point *C*, which lies on the centre line of pipe. From *C*, draw a line *CG* parallel to *EF*. Then *CG* represents the hydraulic gradient line.

Problem 11.23 *For the problem 11.17, draw the hydraulic gradient and total energy line.*

Solution. Refer to problem 11.17.

Given :

$$L_1 = 25 \text{ m}, \; d_1 = 0.15 \text{ m}$$

$$L_2 = 15 \text{ m}, \; d_2 = 0.3 \text{ m}, \; f = .01, \; H = 8 \text{ m}$$

The velocity V_2 as calculated in problem 11.17 is

$$V_2 = 1.113 \text{ m/s}$$

$$V_1 = 4V_2 = 4 \times 1.113 = 4.452 \text{ m/s}$$

The various head losses are $h_i = 0.5 \times \frac{V_1^2}{2g} = \frac{0.5 \times 4.452^2}{2 \times 9.81} = 0.50$ m

$$h_{f_1} = \frac{4f \times L_1 \times V_1^2}{d_1 \times 2g} = \frac{4 \times .01 \times 25 \times (4.452)^2}{0.15 \times 2 \times 9.81} = 6.73 \text{ m}$$

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(4.452 - 1.11)^2}{2 \times 9.81} = 0.568 \text{ m}$$

$$h_{f_2} = \frac{4 \times f \times L_2 \times V_2^2}{d_2 \times 2g} = \frac{4 \times .01 \times 15 \times (1.113)^2}{0.3 \times 2 \times 9.81} = 0.126 \text{ m}$$

$$h_o = \frac{V_2^2}{2g} = \frac{1.113^2}{2 \times 9.81} = 0.063 \text{ m}$$

Also $$V_1^2/2g = \frac{4.452^2}{2 \times 9.81} = 1.0 \text{ m}.$$

Total Energy Line

(*i*) Point A lies on free surface of water.

(*ii*) Take $AB = h_i = 0.5$ m.

(*iii*) From B, draw a horizontal line. Take BL equal to the length of pipe, *i.e.*, L_1. From L draw a vertical line downward.

(*iv*) Cut the line $LC = h_{f_1} = 6.73$ m.

(*v*) Join the point B to C. From C, take a line CD vertically downward equal to $h_e = 0.568$ m.

(*vi*) From D, draw DM horizontal and from point F which is lying on the centre of the pipe, draw a vertical line in the upward direction, meeting at M. From M, take a distance $ME = h_{f_2} = 0.126$. Join DE.

Then line $ABCDE$ represents the total energy line.

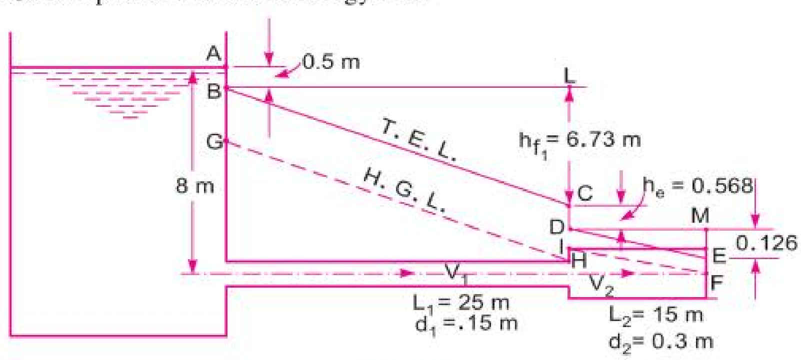


Fig. 11.9

Hydraulic Gradient Line (H.G.L.)

(*i*) From B, take $BG = \dfrac{V_1^2}{2g} = 1.0$ m.

(*ii*) Draw the line GH parallel to the line BC.

(*iii*) From F, draw a line FI parallel to the line ED.

(*iv*) Join the point H and I.

Then the line $GHIF$ represents the hydraulic gradient line (H.G.L.).

Problem 11.24 *For Problem 11.18, draw the hydraulic gradient and total energy line.*

Solution. Refer to Problem 11.18,

Given :

$$d = 300 \text{ mm} = 0.3 \text{ m}$$
$$L = 400 \text{ m}, \; Q = 300 \text{ litres/s} = 0.3 \text{ m}^3/\text{s}$$
$$f = .008$$

Let $H_1 = 50$ m. But $H_1 - H_2 = 40.537$ m (Calculated in Problem 11.18)

$\therefore$ $H_2 = 50 - 40.537 = 9.463$ m.

The calculated losses are :

(*i*) $h_i = 0.459$ m (*ii*) $h_{f_1} = 39.16$ m

(*iii*) $h_o = 0.918$ m

(*a*) **T.E.L.**

(*i*) Point *A* is on the free surface of water in 1st tank. From *A*, take $AB = h_i = 0.459$ m.

(*ii*) Draw a horizontal line *BF*. Take *BF* equal to the length of pipe. From *F*, draw a vertical line in the downward direction. Cut $FC = h_{f_1} = 39.16$ m.

(*iii*) Join *BC*. From *C* take $CD = h_0 = 0.918$ m. The point *D* should coincide with free surface of water in 2nd tank. Then line *ABCD* is the total energy line.

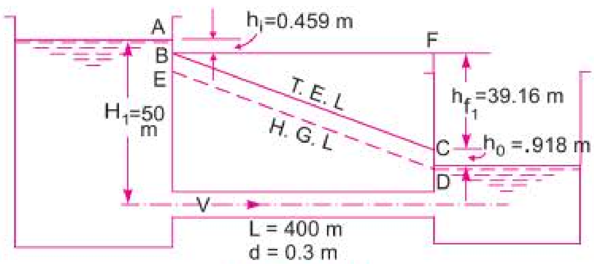


Fig. 11.10

(*b*) **H.G.L.** From *D*, draw a line *DE* parallel to line *BC*. Then *DE* is the H.G.L.

Or

From *B*, take $BE = \dfrac{V^2}{2g} = 0.918$ m and from *E* draw a line *ED* parallel to *BC*. The point *D* should coincide with free surface of water in the 2nd tank. Then line *ED* represents the H.G.L.

Problem 11.25 *The rate of flow of water pumped into a pipe ABC, which is 200 m long, is 20 litres/s. The pipe is laid on an upward slope of 1 in 40. The length of the portion AB is 100 m and its diameter is 100 mm, while the length of the portion BC is also 100 m but its diameter is 200 mm. The change of diameter at B is sudden. The flow is taking place from A to C, where the pressure at A is 19.62 N/cm² and end C is connected to a tank. Find the pressure at C and draw the hydraulic gradient and total energy line. Take f = .008.*

Solution. Given :

Length of pipe, $ABC = 200$ m

Discharge, $Q = 20$ litres/s $= 0.02$ m^3/s

Slope of pipe, $i = 1 \text{ in } 40 = \dfrac{1}{40}$

Length of pipe, $AB = 100$ m, Dia. of pipe $AB = 100$ mm $= 0.1$ m

Length of pipe, $BC = 100$ m, Dia. of pipe $BC = 200$ mm $= 0.2$ m

Pressure at *A*, $p_A = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$

Co-efficient of friction, $f = .008$

Velocity of water in pipe *AB*, $V_1 = \dfrac{\text{Discharge}}{\text{Area of } AB} = \dfrac{0.02}{\frac{\pi}{4}(.1)^2} = 2.54$ m/s

Velocity of water in pipe BC, $V_2 = \dfrac{Q}{\text{Area of } BC} = \dfrac{0.02}{\frac{\pi}{4}(.2)^2} = 0.63$ m/s

Applying Bernoulli's equation to points A and C,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{total loss from } A \text{ to } C \quad \text{...}(i)$$

Total loss from A to C = Loss due to friction in pipe AB + loss of head due to enlargement at B + loss of head due to friction in pipe BC. ...(*ii*)

Now loss of head due to friction in pipe AB,

$$h_{f_1} = \frac{4fLV^2}{d \times 2g} = \frac{4 \times .008 \times 100 \times (2.54)^2}{0.1 \times 2 \times 9.81} = 10.52 \text{ m}$$

Loss of head due to friction in pipe BC,

$$h_{f_2} = \frac{4 \times .008 \times 100 \times (0.63)^2}{0.2 \times 2 \times 9.81} = 0.323 \text{ m}$$

Loss of head due to enlargement at B,

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(2.54 - .63)^2}{2 \times 9.81} = 0.186 \text{ m}$$

$\therefore$ Total loss from A to C $= h_{f_1} + h_e + h_{f_2} = 10.52 + .186 + .323 = 11.029 \simeq 11.03$ m

Substituting this value in (*i*), we get

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + 11.03 \quad \text{...}(iii)$$

Taking datum line passing through A, we have

$$z_A = 0$$

$$z_c = \frac{1}{40} \times \text{total length of pipe} = \frac{1}{40} \times 200 = 5 \text{ m}$$

Also $p_A = 19.62 \times 10^4$ N/m^2

$$V_A = V_1 = 2.54 \text{ m/s}, \; V_c = V_2 = 0.63 \text{ m/s}$$

Substituting these values in (*iii*), we get

$$\frac{19.62 \times (10)^4}{1000 \times 9.81} + \frac{(2.54)^2}{2 \times 9.81} + 0 = \frac{p_c}{\rho g} + \frac{(0.63)^2}{2 \times 9.81} + 5.0 + 11.03$$

or
$$20 + 0.328 = \frac{p_c}{\rho g} + 0.02 + 5.0 + 11.03$$

$\therefore$
$$20.328 = \frac{p_c}{\rho g} + 16.05$$

$\therefore$ $$\frac{p_c}{\rho g} = 20.328 - 16.05 = 4.278 \text{ m}$$

or $$p_c = 4.278 \times 1000 \times 9.81 \text{ N/m}^2$$

$$= \frac{4.278 \times 1000 \times 9.81}{10^4} \text{ N/cm}^2 = \mathbf{4.196 \ N/cm^2.} \textbf{ Ans.}$$

Hydraulic Gradient and Total Energy Line

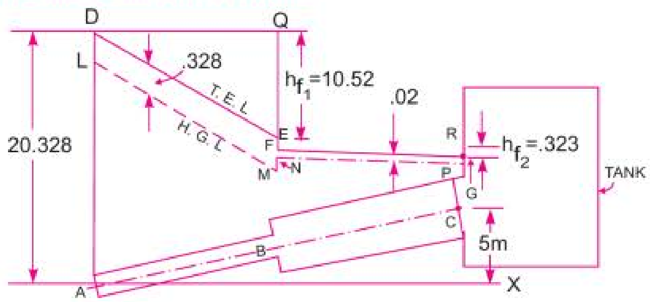


Fig. 11.11

Pipe AB. Assuming the datum line passing through A, then total energy at A

$$= \frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{(2.54)^2}{2 \times 9.81} + 0 = 20 + 0.328$$

$$= 20.328 \text{ m}$$

Total energy at B $= \text{Total energy at } A - h_{f_1} = 20.328 - 10.52 = 9.808 \text{ m}$

Also $$V_c^2/2g = \frac{(0.63)^2}{2 \times 9.81} = 0.02.$$

Total Energy Line. Draw a horizontal line AX as shown in Fig. 11.11. The centre-line of the pipe is drawn in such a way that slope of pipe is 1 in 40. Thus the point C will be at a height of $\frac{1}{40} \times 200 = 5$ m from the line AX. Now draw a vertical line AD equal to total energy at A, *i.e.*, AD = 20.328 m. From point D, draw a horizontal line and from point B, a vertical line, meeting at Q. From Q, take vertical distance $QE = h_{f_1} = 10.52$ m. Join DE. From E, take $EF = h_e = 0.186$ m. From F, draw a horizontal line and from C, a vertical line meeting at R. From R take $RG = h_{f_2} = 0.323$ m. Join F to G. Then $DEFG$ represents the total energy line.

Hydraulic Gradient Line. Draw the line LM parallel to the line DE at a distance in the downward direction equal to 0.328 m. Also draw the line PN parallel to the line GF at a distance of $\frac{V_c^2}{2g} = 0.02$. Join point M to N. Then line $LMNP$ represents the hydraulic gradient line.

Problem 11.26 *A pipe line, 300 mm in diameter and 3200 m long is used to pump up 50 kg per second of an oil whose density is 950 kg/m³ and whose kinematic viscosity is 2.1 stokes. The centre of the pipe line at the upper end is 40 m above than that at the lower end. The discharge at the upper end is atmospheric. Find the pressure at the lower end and draw the hydraulic gradient and the total energy line.*

Solution. Given :

Dia. of pipe, $d = 300$ mm $= 0.3$ m

Length of pipe, $L = 3200$ m

Mass, $M = 50$ kg/s $= \rho \cdot Q$

$\therefore$ Discharge, $Q = \dfrac{50}{\rho} = \dfrac{50}{950} = 0.0526$ m^3/s

$\because$ Density, $\rho = 950$ kg/m^3

Kinematic viscosity, $\nu = 2.1$ stokes $= 2.1$ cm^2/s

$= 2.1 \times 10^{-4}$ m^2/s

Height of upper end $= 40$ m

Pressure at upper end $=$ atmospheric $= 0$

Reynolds number, $R_e = \dfrac{V \times d}{\nu}$, where $V = \dfrac{\text{Discharge}}{\text{Area}} = \dfrac{0.0526}{\frac{\pi}{4}(0.3)^2} = 0.744$ m/s

$$\therefore \quad R_e = \frac{0.744 \times 0.30}{2.1 \times 10^{-4}} = 1062.8$$

$\therefore$ Co-efficient of friction, $f = \dfrac{16}{R_e} = \dfrac{16}{1062.8} = 0.015$

Head lost due to friction, $h_f = \dfrac{4 \times f \times L \times V^2}{d \times 2g}$

$$= \frac{4 \times 0.015 \times 3200 \times (0.744)^2}{0.3 \times 2 \times 9.81} = 18.05 \text{ m of oil}$$

Applying the Bernoulli's equation at the lower and upper end of the pipe and taking datum line passing through the lower end, we have

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f$$

But $z_1 = 0$, $z_2 = 40$ m, $V_1 = V_2$ as diameter is same

$p_2 = 0$, $h_f = 18.05$ m

$\therefore$ Substituting these values, we have

$$\frac{p_1}{\rho g} = 40 + 18.05 = 58.05 \text{ m of oil}$$

$\therefore \quad p_1 = 58.05 \times \rho g = 58.05 \times 950 \times 9.81$ [$\because$ ρ for oil $= 950$]

$= 540997$ N/m^2 $= \dfrac{540997}{10^{-4}}$ N/cm^2 $=$ **54.099 N/cm^2. Ans.**

H.G.L. and T.E.L.

$$\frac{V^2}{2g} = \frac{(.744)^2}{2 \times 9.81} = 0.0282 \text{ m}$$

$$\frac{p_1}{\rho g} = 58.05 \text{ m of oil}$$

$$\frac{p_2}{\rho g} = 0$$

Draw a horizontal line *AX* as shown in Fig. 11.12. From *A*, draw the centre line of the pipe in such a way that point *C* is a distance of 40 m above the horizontal line. Draw a vertical line *AB* through *A* such that *AB* = 58.05 m. Join *B* with *C*. Then *BC* is the hydraulic gradient line.

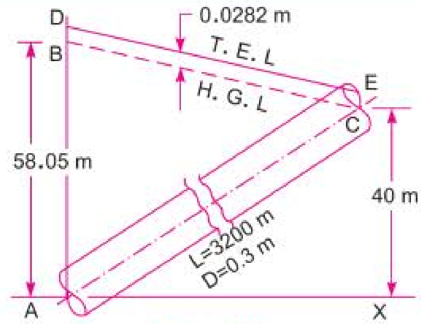


Fig. 11.12

Draw a line *DE* parallel to *BC* at a height of 0.0282 m above the hydraulic gradient line. Then *DE* is the total energy line.

▶ 11.6 FLOW THROUGH SYPHON

Syphon is a long bent pipe which is used to transfer liquid from a reservoir at a higher elevation to another reservoir at a lower level when the two reservoirs are separated by a hill or high level ground as shown in Fig. 11.13.

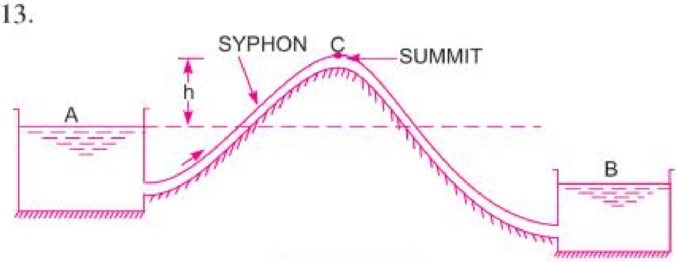


Fig. 11.13

The point *C* which is at the highest of the syphon is called the summit. As the point *C* is above the free surface of the water in the tank *A*, the pressure at *C* will be less than atmospheric pressure. Theoretically, the pressure at *C* may be reduced to – 10.3 m of water but in actual practice this pressure is only – 7.6 m of water or 10.3 – 7.6 = 2.7 m of water absolute. If the pressure at *C* becomes less than 2.7 m of water absolute, the dissolved air and other gases would come out from water and collect at the summit. The flow of water will be obstructed. Syphon is used in the following cases :

1. To carry water from one reservoir to another reservoir separated by a hill or ridge.
2. To take out the liquid from a tank which is not having any outlet.
3. To empty a channel not provided with any outlet sluice.

Problem 11.27 *A syphon of diameter 200 mm connects two reservoirs having a difference in elevation of 20 m. The length of the syphon is 500 m and the summit is 3.0 m above the water level in the upper reservoir. The length of the pipe from upper reservoir to the summit is 100 m. Determine the discharge through the syphon and also pressure at the summit. Neglect minor losses. The co-efficient of friction, f = .005.*

Solution. Given :

Dia. of syphon, $d = 200 \text{ mm} = 0.20 \text{ m}$

Difference in level of two reservoirs, $H = 20 \text{ m}$

Length of syphon, $L = 500 \text{ m}$

Height of summit from upper reservoir, $h = 3.0$ m
Length of syphon upto summit, $L_1 = 100$ m
Co-efficient of friction, $f = .005$

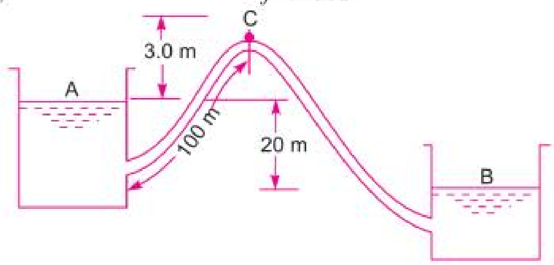


Fig. 11.14

If minor losses are neglected then the loss of head takes place only due to friction.

Applying Bernoulli's equation to points A and B,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + \text{Loss of head due to friction from } A \text{ to } B$$

or $$0 + 0 + z_A = 0 + 0 + z_B + h_f \quad [\because p_A = p_B = \text{atmospheric pressure, } V_A = V_B = 0]$$

$\therefore$ $$z_A - z_B = h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$$

But $$z_A - z_B = 20 \text{ m}$$

$\therefore$ $$20 = \frac{4 \times .005 \times 500 \times V^2}{0.20 \times 2 \times 9.81} = 2.548\ V^2$$

$\therefore$ $$V = \sqrt{\frac{20}{2.548}} = 2.80 \text{ m/s}$$

$\therefore$ Discharge, $$Q = \text{Velocity} \times \text{Area}$$

$$= 2.80 \times \frac{\pi}{4}(.2)^2 = 0.0879 \text{ m}^3\text{/s} = \mathbf{87.9 \text{ litres/s.}} \text{ Ans.}$$

Pressure at Summit. Applying Bernoulli's equation to points A and C,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{Loss of head due to friction between } A \text{ and } C$$

or $$0 + 0 + 0 = \frac{p_c}{\rho g} + \frac{V^2}{2g} + 3.0 + h_{f_1} \qquad \text{[Taking datum line passing through } A]$$

$\therefore$ $$0 = \frac{p_c}{\rho g} + \frac{2.8^2}{2 \times 9.81} + 3.0 + \frac{4 \times .005 \times 100 \times (2.8)^2}{0.2 \times 2 \times 9.81} \quad [V_c = V = 2.80]$$

$$= \frac{p_c}{\rho g} + 0.399 + 3.0 + 4.00 = \frac{p_c}{\rho g} + 7.399$$

$\therefore$ $$\frac{p_c}{\rho g} = \mathbf{-\ 7.399 \text{ m of water.}} \text{ Ans.}$$

Problem 11.28 *A syphon of diameter 200 mm connects two reservoirs having a difference in elevation of 15 m. The total length of the syphon is 600 m and the summit is 4 m above the water level in the upper reservoir. If the separation takes place at 2.8 m of water absolute, find the maximum length of syphon from upper reservoir to the summit. Take f = .004 and atmospheric pressure = 10.3 m of water.*

Solution. Given :

Dia. of syphon, $d = 200$ mm $= 0.2$ m

Difference of level in two reservoirs $= 15$ m

Total length of pipe $= 600$ m

Height of summit from upper reservoir $= 4$ m

Pressure head at summit, $\dfrac{p_c}{\rho g} = 2.8$ m of water absolute

Atmospheric pressure head, $\dfrac{p_c}{\rho g} = 10.3$ m of water absolute

Co-efficient of friction, $f = .004$

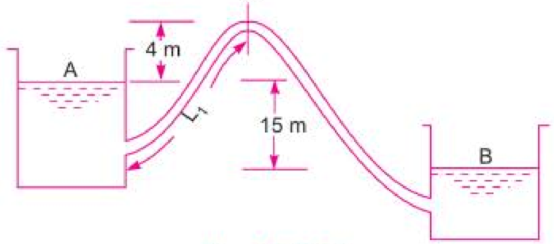


Fig. 11.15 (*a*)

Applying Bernoulli's equation to points A and C and taking the datum line passing through, A,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{Loss of head due to friction between } A \text{ and } C$$

Substituting the values of pressures in terms of absolute, we have

$$10.3 + 0 + 0 = 2.8 + \frac{V^2}{2g} + 4.0 + h_{f_1} \qquad [\because V_c = \text{velocity in pipe} = V]$$

$$\therefore \qquad h_{f_1} = 10.3 - 2.8 - 4.0 - \frac{V^2}{2g} = 3.5 - \frac{V^2}{2g} \qquad ...(i)$$

Applying Bernoulli's equation to points A and B and taking datum line passing through B,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + \text{Loss of head due to friction from } A \text{ to } B$$

But $\qquad \dfrac{p_A}{\rho g} = \dfrac{p_B}{\rho g} =$ atmospheric pressure

$$V_A = 0,\ V_B = 0,\ z_A = 15,\ z_B = 0$$

$$\therefore \qquad 0 + 0 + 15 = 0 + 0 + 0 + h_f$$

$$\therefore \qquad h_f = 15 \text{ or } \frac{4 \times f \times L \times V^2}{d \times 2g} = 15$$

or $$\frac{4 \times .004 \times 600 \times V^2}{0.2 \times 2 \times 9.81} = 15 \text{ or } V = \sqrt{\frac{15 \times 0.2 \times 2 \times 9.81}{4 \times .004 \times 600}} = 2.47 \text{ m/s}$$

Substituting this value of V in equation (*i*), we get

$$h_{f_1} = 3.5 - \frac{2.47^2}{2 \times 9.81} = 3.5 - 0.311 = 3.189 \text{ m} \qquad \text{...}(ii)$$

But $$h_{f_2} = \frac{4 \times f \times L_1 \times V^2}{d \times 2g} \qquad \text{...}(iii)$$

where L_1 = inlet leg of syphon or length of syphon from upper reservoir to the summit.

$$h_{f_1} = \frac{4 \times .004 \times L_1 \times (2.47)^2}{0.2 \times 2 \times 9.81} = 0.0248 \times L_1$$

Substituting this value in equation (*ii*),

$$0.0248\, L_1 = 3.189$$

$$\therefore \qquad L_1 = \frac{3.189}{.0248} = \mathbf{128.58 \text{ m.}} \textbf{ Ans.}$$

Problem 11.29 *A syphon of diameter 200 mm connects two reservoirs whose water surface level differ by 40 m. The total length of the pipe is 8000 m. The pipe crosses a ridge. The summit of ridge is 8 m above the level of water in the upper reservoir. Determine the minimum depth of the pipe below the summit of the ridge, if the absolute pressure head at the summit of syphon is not to fall below 3.0 m of water. Take f = 0.006 and atmospheric pressure head = 10.3 m of water. The length of syphon from the upper reservoir to the summit is 500 m. Find the discharge also.*

Solution. Given :

Dia. of syphon, $d = 200$ mm $= 0.20$ m

Difference in levels of two reservoirs, $H = 40$ m

Total length of pipe, $L = 8000$ m

Height of ridge summit from water level in upper reservoir = 8 m

Let the depth of the pipe below the summit of ridge = x m

$\therefore$ Height of syphon from water surface in the upper reservoir = $(8 - x)$ m

Pressure head at C, $\dfrac{p_c}{\rho g} = 3.0$ m of water absolute

Atmospheric pressure head, $\dfrac{p_a}{\rho g} = 10.3$ m of water

Co-efficient of friction $f = .006$

Length of syphon from upper reservoir to the summit, $L_1 = 500$ m

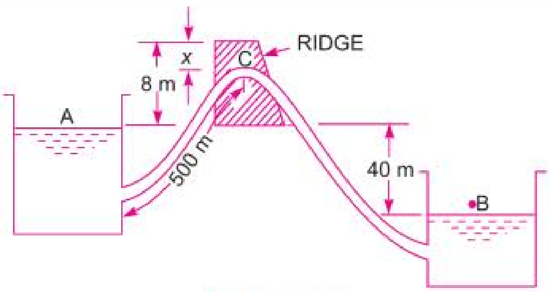


Fig. 11.15 (*b*)

Applying Bernoulli's equation to points *A* and B and taking datum line passing through *B*, we have

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + \text{head loss due to friction } A \text{ to } B$$

or
$$0 + 0 + 40 = 0 + 0 + 0 + \frac{4 \times f \times L \times V^2}{d \times 2g}$$

$\therefore$
$$40 = \frac{4 \times 0.006 \times 8000 \times V^2}{0.2 \times 2 \times 9.81}$$

$\therefore$
$$V = \sqrt{\frac{40 \times 0.2 \times 2 \times 9.81}{4 \times .006 \times 8000}} = 0.904 \text{ m/s}$$

Now applying Bernoulli's equation to points *A* and *C* and assuming datum line passing through *A*, we have

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{head loss due to friction from } A \text{ to } C$$

Substituting $\frac{p_A}{\rho g}$ and $\frac{p_c}{\rho g}$ in terms of absolute pressure

$$10.3 + 0 + 0 = 3.0 + \frac{V^2}{2g} + (8 - x) + \frac{4 \times f \times L_1 \times V^2}{d \times 2g}$$

or
$$10.3 = 3.0 + \frac{(0.904)^2}{2 \times 9.81} + (8 - x) + \frac{4 \times .006 \times 500 \times (0.904)^2}{0.2 \times 2 \times 9.81}$$

$$= 3.0 + 0.041 + (8 - x) + 2.499 = 13.54 - x$$

$\therefore$
$$x = 13.54 - 10.3 = \mathbf{3.24\ m.}\ \textbf{Ans.}$$

Discharge,
$$Q = \text{Area} \times \text{Velocity} = \frac{\pi}{4} \times (.2)^2 \times 0.904 = \mathbf{0.0283\ m^3/s.}\ \textbf{Ans.}$$

▶ 11.7 FLOW THROUGH PIPES IN SERIES OR FLOW THROUGH COMPOUND PIPES

Pipes in series or compound pipes are defined as the pipes of different lengths and different diameters connected end to end (in series) to form a pipe line as shown in Fig. 11.16.

Let, L_1, L_2, L_3 = length of pipes 1, 2 and 3 respectively
d_1, d_2, d_3 = diameter of pipes 1, 2, 3 respectively
V_1, V_2, V_3 = velocity of flow through pipes 1, 2, 3
f_1, f_2, f_3 = co-efficient of frictions for pipes 1, 2, 3
H = difference of water level in the two tanks.

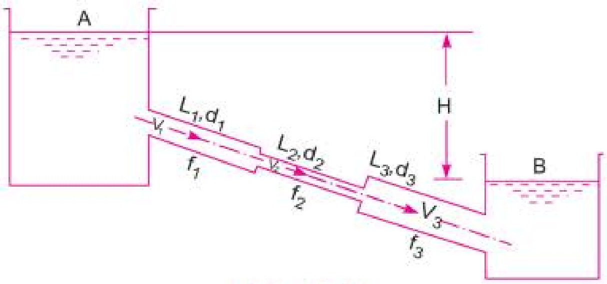


Fig. 11.16

The discharge passing through each pipe is same.

$\therefore$ $$Q = A_1V_1 = A_2V_2 = A_3V_3$$

The difference in liquid surface levels is equal to the sum of the total head loss in the pipes.

$\therefore$ $$H = \frac{0.5\,V_1^2}{2g} + \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{0.5\,V_2^2}{2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g} + \frac{V_3^2}{2g} \quad ...(11.12)$$

If minor losses are neglected, then above equation becomes as

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g} \quad ...(11.13)$$

If the co-efficient of friction is same for all pipes

i.e., $f_1 = f_2 = f_3 = f$, then equation (11.13) becomes as

$$H = \frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} + \frac{4fL_3V_3^2}{d_3 \times 2g}$$

$$= \frac{4f}{2g}\left[\frac{L_1V_1^2}{d_1} + \frac{L_2V_2^2}{d_2} + \frac{L_3V_3^2}{d_3}\right] \quad ...(11.14)$$

Problem 11.30 *The difference in water surface levels in two tanks, which are connected by three pipes in series of lengths 300 m, 170 m and 210 m and of diameters 300 mm, 200 mm and 400 mm respectively, is 12 m. Determine the rate of flow of water if co-efficient of friction are .005, .0052 and .0048 respectively, considering : (i) minor losses also (ii) neglecting minor losses.*

Solution. Given :

Difference of water level, $H = 12$ m

Length of pipe 1, $L_1 = 300$ m and dia., $d_1 = 300$ mm= 0.3 m

Length of pipe 2, $L_2 = 170$ m and dia., $d_2 = 200$ mm = 0.2 m

Length of pipe 3, $L_3 = 210$ m and dia., $d_3 = 400$ mm $= 0.4$ m

Also, $f_1 = .005$, $f_2 = .0052$ and $f_3 = .0048$

(*i*) **Considering Minor Losses.** Let V_1, V_2 and V_3 are the velocities in the 1st, 2nd and 3rd pipe respectively.

From continuity, we have $A_1V_1 = A_2V_2 = A_3V_3$

$$\therefore \qquad V_2 = \frac{A_1V_1}{A_2} = \frac{\frac{\pi}{4}d_1^2}{\frac{\pi}{4}d_2^2}V_1 = \frac{d_1^2}{d_2^2}V_1 = \left(\frac{0.3}{.2}\right)^2 \times V_1 = 2.25\ V_1$$

and
$$V_3 = \frac{A_1V_1}{A_3} = \frac{d_1^2}{d_3^2}V_1 = \left(\frac{0.3}{0.4}\right)^2 V_1 = 0.5625\ V_1$$

Now using equation (11.12), we have

$$H = \frac{0.5\,V_1^2}{2g} + \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{0.5\,V_2^2}{2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g} + \frac{V_3^2}{2g}$$

Substituting V_2 and V_3, $\quad 12.0 = \dfrac{0.5\,V_1^2}{2g} + \dfrac{4 \times .005 \times 300 \times V_1^2}{0.3 \times 2g} + \dfrac{0.5 \times (2.25 V_1^2)^2}{2g}$

$$+\, 4 \times 0.0052 \times 170 \times \frac{(2.25\,V_1)^2}{0.2 \times 2g} + \frac{(2.25\,V_1 - .562\,V_1)^2}{2g} + \frac{4 \times .0048 \times 210 \times (.5625\,V_1)^2}{0.4 \times 2g} + \frac{(.5625\,V_1)^2}{2g}$$

or
$$12.0 = \frac{V_1^2}{2g}\ [0.5 + 20.0 + 2.53 + 89.505 + 2.847 + 3.189 + 0.316]$$

$$= \frac{V_1^2}{2g}\ [118.887]$$

$$\therefore \qquad V_1 = \sqrt{\frac{12 \times 2 \times 9.81}{118.887}} = 1.407 \text{ m/s}$$

$\therefore$ Rate of flow, Q = Area $\times$ Velocity $= A_1 \times V_1$

$$= \frac{\pi}{4}(d_1)^2 \times V_1 = \frac{\pi}{4}(.3)^2 \times 1.407 = 0.09945 \text{ m}^3\text{/s}$$

= 99.45 litres/s. Ans.

(*ii*) **Neglecting Minor Losses.** Using equation (11.13), we have

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g}$$

or
$$12.0 = \frac{V_1^2}{2g}\left[\frac{4 \times .005 \times 300}{0.3} + \frac{4 \times .0052 \times 170 \times (2.25)^2}{0.2} + \frac{4 \times .0048 \times 210 \times (.5625)^2}{0.4}\right]$$

$$= \frac{V_1^2}{2g}\,[20.0 + 89.505 + 3.189] = \frac{V_1^2}{2g} \times 112.694$$

$\therefore$ $$V_1 = \sqrt{\frac{2 \times 9.81 \times 12.0}{112.694}} = 1.445 \text{ m/s}$$

$\therefore$ Discharge, $Q = V_1 \times A_1 = 1.445 \times \frac{\pi}{4}\,(.3)^2 = 0.1021 \text{ m}^3\text{/s} =$ **102.1 litres/s. Ans.**

Problem 11.30 (A). *Three pipes of 400 mm, 200 mm and 300 mm diameters have lengths of 400 m, 200 m, and 300 m respectively. They are connected in series to make a compound pipe. The ends of this compound pipe are connected with two tanks whose difference of water levels is 16 m. If co-efficient of friction for these pipes is same and equal to 0.005, determine the discharge through the compound pipe neglecting first the minor losses and then including them.*

Solution. Given :

Difference of water levels, $H = 16$ m

Length and dia. of pipe 1, $L_1 = 400$ m and $d_1 = 400$ mm $= 0.4$ m

Length and dia. of pipe 2, $L_2 = 200$ m and $d_2 = 200$ mm $= 0.2$ m

Length and dia. of pipe 3, $L_3 = 300$ m and $d_3 = 300$ mm $= 0.3$ m

Also $f_1 = f_2 = f_3 = 0.005$

(*i*) **Discharge through the compound pipe first neglecting minor losses.**

Let V_1, V_2 and V_3 are the velocities in the 1st, 2nd and 3rd pipe respectively.

From continuity, we have $A_1V_1 = A_2V_2 = A_3V_3$

$\therefore$ $$V_2 = \frac{A_1V_1}{A_2} = \frac{\frac{\pi}{4}d_1^2}{\frac{\pi}{4}d_2^2} \times V_1 = \frac{d_1^2}{d_2^2}\,V_1 = \left(\frac{0.4}{0.2}\right)^2 V_1 = 4V_1$$

and $$V_3 = \frac{A_1V_1}{A_3} = \frac{\frac{\pi}{4}d_1^2}{\frac{\pi}{4}d_3^2} \times V_1 = \frac{d_1^2}{d_3^2}\,V_1 = \left(\frac{0.4}{0.2}\right)^2 V_1 = 1.77V_1$$

Now using equation (11.13), we have

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g}$$

or $$16 = \frac{4 \times 0.005 \times 400 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 200 \times (4V_1)^2}{0.2 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 300}{0.3 \times 2 \times 9.81} \times (1.77\,V_1)^2$$

$$= \frac{V_1^2}{2 \times 9.81}\left(\frac{4 \times 0.005 \times 400}{0.4} + \frac{4 \times 0.005 \times 200 \times 16}{0.2} + \frac{4 \times 0.005 \times 300 \times 3.157}{0.3}\right)$$

$$16 = \frac{V_1^2}{2 \times 9.81}\,(20 + 320 + 63.14) = \frac{V_1^2}{2 \times 9.81} \times 403.14$$

$\therefore$ $$V_1 = \sqrt{\frac{16 \times 2 \times 9.81}{403.14}} = 0.882 \text{ m/s}$$

$\therefore$ Discharge, $\quad Q = A_1 \times V_1 = \frac{\pi}{4}(0.4)^2 \times 0.882 = \mathbf{0.1108\ m^3/s.}$ **Ans.**

(*ii*) **Discharge through the compound pipe considering minor losses also.**

Minor losses are :

(*a*) At inlet,
$$h_i = \frac{0.5\,V_1^2}{2g}$$

(*b*) Between 1st pipe and 2nd pipe, due to contraction,
$$h_c = \frac{0.5\,V_2^2}{2g} = \frac{0.5\left(4V_1^2\right)}{2g} \qquad (\because V_2 = 4V_1)$$
$$= \frac{0.5 \times 16 \times V_1^2}{2g} = 8 \times \frac{V_1^2}{2g}$$

(*c*) Between 2nd pipe and 3rd pipe, due to sudden enlargement,
$$h_e = \frac{\left(V_2 - V_3\right)^2}{2g} = \frac{\left(4V_1 - 1.77V_1\right)^2}{2g} \qquad (\because V_3 = 1.77\,V_1)$$
$$= (2.23)^2 \times \frac{V_1^2}{2g} = 4.973\,\frac{V_1^2}{2g}$$

(*d*) At the outlet of 3rd pipe, $h_o = \frac{V_3^2}{2g} = \frac{\left(1.77V_1\right)^2}{2g} = 1.77^2 \times \frac{V_1^2}{2g} = 3.1329\,\frac{V_1^2}{2g}$

The major losses are
$$= \frac{4f_1 \times L_1 \times V_1^2}{d_1 \times 2g} + \frac{4f_2 \times L_2 \times V_2^2}{d_2 \times 2g} + \frac{4f_3 \times L_3 \times V_3^2}{d_3 \times 2g}$$
$$= \frac{4 \times 0.005 \times 400 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 200 \times \left(4V_1\right)^2}{0.2 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 300 \times \left(1.77V_1\right)^2}{0.3 \times 2 \times 9.81}$$
$$= 403.14 \times \frac{V_1^2}{2 \times 9.81}$$

$\therefore$ Sum of minor losses and major losses
$$= \left[\frac{0.5\,V_1^2}{2g} + 8 \times \frac{V_1^2}{2g} + 4.973\,\frac{V_1^2}{2g} + 3.1329\,\frac{V_1^2}{2g}\right] + 403.14\,\frac{V_1^2}{2g}$$
$$= 419.746\,\frac{V_1^2}{2g}$$

But total loss must be equal to H (or 16 m)

$$\therefore \qquad 419.746 \times \frac{V_1^2}{2g} = 16 \quad \therefore \quad V_1 = \sqrt{\frac{16 \times 2 \times 9.81}{419.746}} = 0.864 \text{ m/s}$$

$\therefore$ Discharge, $\quad Q = A_1V_1 = \frac{\pi}{4}(0.4)^2 \times 0.864 = \mathbf{0.1085\ m^3/s.}$ **Ans.**

11.8 EQUIVALENT PIPE

This is defined as the pipe of uniform diameter having loss of head and discharge equal to the loss of head and discharge of a compound pipe consisting of several pipes of different lengths and diameters. The uniform diameter of the equivalent pipe is called equivalent size of the pipe. The length of equivalent pipe is equal to sum of lengths of the compound pipe consisting of different pipes.

Let L_1 = length of pipe 1 and d_1 = diameter of pipe 1
L_2 = length of pipe 2 and d_2 = diameter of pipe 2
L_3 = length of pipe 3 and d_3 = diameter of pipe 3
H = total head loss
L = length of equivalent pipe
d = diameter of the equivalent pipe

Then $L = L_1 + L_2 + L_3$

Total head loss in the compound pipe, neglecting minor losses

$$H = \frac{4f_1L_1V_1^2}{d_1 \times 2g} + \frac{4f_2L_2V_2^2}{d_2 \times 2g} + \frac{4f_3L_3V_3^2}{d_3 \times 2g} \quad \text{...(11.14A)}$$

Assuming $$f_1 = f_2 = f_3 = f$$

Discharge, $$Q = A_1V_1 = A_2V_2 = A_3V_3 = \frac{\pi}{4} d_1^2V_1 = \frac{\pi}{4} d_2^2V_2 = \frac{\pi}{4} d_3^2V_3$$

$\therefore$ $$V_1 = \frac{4Q}{\pi d_1^2},\ V_2 = \frac{4Q}{\pi d_2^2} \text{ and } V_3 = \frac{4Q}{\pi d_3^2}$$

Substituting these values in equation (11.14A), we have

$$H = \frac{4fL_1 \times \left(\frac{4Q}{\pi d_1^2}\right)^2}{d_1 \times 2g} + \frac{4fL_2\left(\frac{4Q}{\pi d_2^2}\right)^2}{d_2 \times 2g} + \frac{4fL_3\left(\frac{4Q}{\pi d_3^2}\right)^2}{d_3 \times 2g}$$

$$= \frac{4 \times 16fQ^2}{\pi^2 \times 2g}\left[\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right] \quad \text{...(11.15)}$$

Head loss in the equivalent pipe, $H = \dfrac{4f\,.\,L\,.\,V^2}{d \times 2g}$ [Taking same value of f as in compound pipe]

where $V = \dfrac{Q}{A} = \dfrac{Q}{\frac{\pi}{4}d^2} = \dfrac{4Q}{\pi d^2}$

$\therefore$ $$H = \frac{4f\,.\,L\,.\left(\frac{4Q}{\pi d^2}\right)^2}{d \times 2g} = \frac{4 \times 16Q^2 f}{\pi^2 \times 2g}\left[\frac{L}{d^5}\right] \quad \text{...(11.16)}$$

Head loss in compound pipe and in equivalent pipe is same hence equating equations (11.15) and (11.16), we have

$$\frac{4\times 16 fQ^2}{\pi^2\times 2g}\left[\frac{L_1}{d_1^5}+\frac{L_2}{d_2^5}+\frac{L_3}{d_3^5}\right]=\frac{4\times 16Q^2 f}{\pi^2\times 2g}\left[\frac{L}{d^5}\right]$$

or $$\frac{L_1}{d_1^5}+\frac{L_2}{d_2^5}+\frac{L_3}{d_3^5}=\frac{L}{d^5} \quad \text{or} \quad \frac{L}{d^5}=\frac{L_1}{d_1^5}+\frac{L_2}{d_2^5}+\frac{L_3}{d_3^5} \qquad \text{...(11.17)}$$

Equation (11.17) is known as Dupuit's equation. In this equation $L = L_1 + L_2 + L_3$ and d_1, d_2 and d_3 are known. Hence the equivalent size of the pipe, *i.e.*, value of d can be obtained.

Problem 11.31 *Three pipes of lengths 800 m, 500 m and 400 m and of diameters 500 mm, 400 mm and 300 mm respectively are connected in series. These pipes are to be replaced by a single pipe of length 1700 m. Find the diameter of the single pipe.*

Solution. Given :

Length of pipe 1, $L_1 = 800$ m and dia., $d_1 = 500$ mm $= 0.5$ m

Length of pipe 2, $L_2 = 500$ m and dia., $d_2 = 400$ mm $= 0.4$ m

Length of pipe 3, $L_3 = 400$ m and dia., $d_3 = 300$ mm $= 0.3$ m

Length of single pipe, $L = 1700$ m

Let the diameter of equivalent single pipe $= d$

Applying equation (11.17), $$\frac{L}{d^5}=\frac{L_1}{d_1^5}+\frac{L_2}{d_2^5}+\frac{L_3}{d_3^5}$$

or $$\frac{1700}{d^5}=\frac{800}{.5^5}+\frac{500}{.4^5}+\frac{400}{0.3^5}=25600+48828.125+164609=239037$$

$\therefore$ $$d^5=\frac{1700}{239037}=.007118$$

$\therefore$ $$d=(.007188)^{0.2}=0.3718=\textbf{371.8 mm. Ans.}$$

11.9 FLOW THROUGH PARALLEL PIPES

Consider a main pipe which divides into two or more branches as shown in Fig. 11.17 and again join together downstream to form a single pipe, then the branch pipes are said to be connected in parallel. The discharge through the main is increased by connecting pipes in parallel.

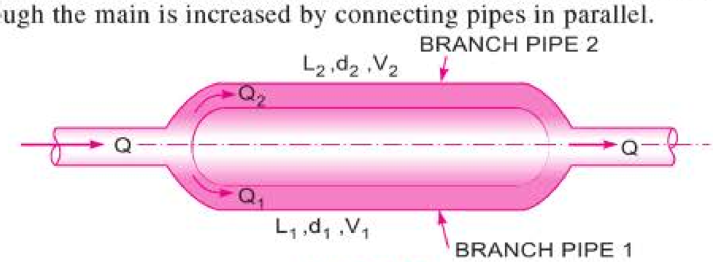


Fig. 11.17

The rate of flow in the main pipe is equal to the sum of rate of flow through branch pipes. Hence from Fig. 11.17, we have

$$Q = Q_1 + Q_2 \qquad \text{...(11.18)}$$

In this, arrangement, the loss of head for each branch pipe is same.

$\therefore$ Loss of head for branch pipe 1 = Loss of head for branch pipe 2 or $$\frac{4f_1L_1V_1^2}{d_1 \times 2g} = \frac{4f_2L_2V_2^2}{d_2 \times 2g} \quad \text{...(11.19)}$$

If $$f_1 = f_2, \text{ then } \frac{L_1V_1^2}{d_1 \times 2g} = \frac{L_2V_2^2}{d_2 \times 2g} \quad \text{...(11.20)}$$

Problem 11.32 *A main pipe divides into two parallel pipes which again forms one pipe as shown in Fig. 11.17. The length and diameter for the first parallel pipe are 2000 m and 1.0 m respectively, while the length and diameter of 2nd parallel pipe are 2000 m and 0.8 m. Find the rate of flow in each parallel pipe, if total flow in the main is 3.0 m³/s. The co-efficient of friction for each parallel pipe is same and equal to .005.*

Solution. Given :

Length of pipe 1, $L_1 = 2000$ m

Dia. of pipe 1, $d_1 = 1.0$ m

Length of pipe 2, $L_2 = 2000$ m

Dia. of pipe 2, $d_2 = 0.8$ m

Total flow, $Q = 3.0$ m^3/s

$f_1 = f_2 = f = .005$

Let Q_1 = discharge in pipe 1

Q_2 = discharge in pipe 2

From equation (11.18), $Q = Q_1 + Q_2 = 3.0$ ...(*i*)

Using equation (11.19), we have

$$\frac{4f_1L_1V_1^2}{d_1 \times 2g} = \frac{4f_2L_2V_2^2}{d_2 \times 2g}$$

$$\frac{4 \times .005 \times 2000 \times V_1}{1.0 \times 2 \times 9.81} = \frac{4 \times .005 \times 2000 \times V_2^2}{0.8 \times 2 \times 9.81}$$

or $$\frac{V_1^2}{1.0} = \frac{V_2^2}{0.8} \text{ or } V_1{}^2 = \frac{V_2^2}{0.8}$$

$\therefore$ $$V_1 = \frac{V_2}{\sqrt{0.8}} = \frac{V_2}{.894} \quad \text{...(ii)}$$

Now $$Q_1 = \frac{\pi}{4} d_1{}^2 \times V_1 = \frac{\pi}{4} (1)^2 \times \frac{V_2}{.894} \quad \left[\because V_1 = \frac{V_2}{.894}\right]$$

and $$Q_2 = \frac{\pi}{4} d_2{}^2 \times V_2 = \frac{\pi}{4} (.8)^2 \times V_2 = \frac{\pi}{4} \times .64 \times V_2$$

Substituting the value of Q_1 and Q_2 in equation (*i*), we get

$$\frac{\pi}{4} \times \frac{V_2}{0.894} + \frac{\pi}{4} \times .64\ V_2 = 3.0 \text{ or } 0.8785\ V_2 + 0.5026\ V_2 = 3.0$$

or $$V_2[.8785 + .5026] = 3.0 \text{ or } V = \frac{3.0}{1.3811} = 2.17 \text{ m/s.}$$

Substituting this value in equation (*ii*),

$$V_1 = \frac{V_2}{.894} = \frac{2.17}{0.894} = 2.427 \text{ m/s}$$

Hence $$Q_1 = \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} \times 1^2 \times 2.427 = \mathbf{1.906 \text{ m}^3\text{/s.}} \text{ Ans.}$$

$\therefore$ $$Q_2 = Q - Q_1 = 3.0 - 1.906 = \mathbf{1.094 \text{ m}^3\text{/s.}} \text{ Ans.}$$

Problem 11.33 *A pipe line of 0.6 m diameter is 1.5 km long. To increase the discharge, another line of the same diameter is introduced parallel to the first in the second half of the length. Neglecting minor losses, find the increase in discharge if 4f = 0.04. The head at inlet is 300 mm.*

Solution. Given :

Dia. of pipe line, $D = 0.6$ m

Length of pipe line, $L = 1.5 \text{ km} = 1.5 \times 1000 = 1500$ m

$4f = 0.04$ or $f = .01$

Head at inlet, $h = 300 \text{ mm} = 0.3$ m

Head at outlet, = atmospheric head = 0

$\therefore$ Head loss, $h_f = 0.3$ m

Length of another parallel pipe, $L_1 = \frac{1500}{2} = 750$ m

Dia. of another parallel pipe, $d_1 = 0.6$ m

Fig. 11.18 shows the arrangement of pipe system.

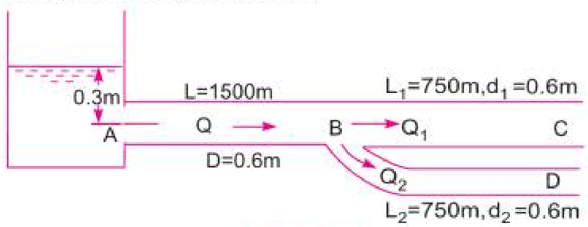


Fig. 11.18

1st Case. Discharge for a single pipe of length 1500 m and dia. = 0.6 m.

This head lost due to friction in single pipe is $h_f = \frac{4fLV^{*2}}{d \times 2g}$

where V^* = velocity of flow for single pipe

or $$0.3 = \frac{4 \times .01 \times 1500 \times V^{*2}}{0.6 \times 2g}$$

$\therefore$ $$V^* = \sqrt{\frac{0.3 \times 0.6 \times 2 \times 9.81}{4 \times .01 \times 1500}} = 0.2426 \text{ m/s}$$

$\therefore$ $$\text{Discharge, } Q^* = V^* \times \text{Area} = 0.2426 \times \frac{\pi}{4} (.6)^2 = 0.0685 \text{ m}^3\text{/s} \quad ...(i)$$

2nd Case. When an additional pipe of length 750 m and diameter 0.6 m is connected in parallel with the last half length of the pipe.

Let Q_1 = discharge in 1st parallel pipe

Q_2 = discharge in 2nd parallel pipe

$\therefore \quad Q = Q_1 + Q_2$

where Q = discharge in main pipe when pipes are parallel.

But as the length and diameters of each parallel pipe is same

$$\therefore \quad Q_1 = Q_2 = Q/2$$

Consider the flow through pipe ABC or ABD

Head loss through ABC = Head lost through AB + head lost through BC ...(*ii*)

But head lost due to friction through ABC = 0.3 m given

Head loss due to friction through $AB = \dfrac{4 \times f \times 750 \times V^2}{0.6 \times 2 \times 9.81}$, where V = velocity of flow through AB

$$= \frac{Q}{\text{Area}} = \frac{Q}{\frac{\pi}{4}(0.6)^2} = \frac{40}{\pi \times .36}$$

$\therefore$ Head loss due to friction through AB

$$= \frac{4 \times .01 \times 750}{0.6 \times 2 \times 9.81} \times \left(\frac{4Q}{\pi \times .36}\right)^2 = 31.87\ Q^2$$

Head loss due to friction through BC

$$= \frac{4 \times f \times L_1 \times V_1^2}{d_1 \times 2g}$$

$$= \frac{4 \times .01 \times 750}{0.6 \times 2 \times 9.81} \times \left[\frac{Q}{2 \times \frac{\pi}{4}(.6)^2}\right] \quad \left[\because V_1 = \frac{\text{Distance}}{\frac{\pi}{4}(.6)^2} = \frac{Q}{2 \times \frac{\pi}{4} \times (.6)^2}\right]$$

$$= \frac{4 \times .01 \times 750}{0.6 \times 2 \times 9.81} \times \frac{16}{4 \times \pi^2 \times .36^2}\ Q^2 = 7.969\ Q^2$$

Substituting these values in equation (*ii*), we get

$$0.3 = 31.87\ Q^2 + 7.969\ Q^2 = 39.839\ Q^2$$

$$\therefore \quad Q = \sqrt{\frac{0.3}{39.839}} = 0.0867\ \text{m}^3/\text{s}$$

$\therefore$ Increase in discharge = $Q - Q^*$ = 0.0867 – 0.0685 = **0.0182 m³/s. Ans.**

Problem 11.34 *A pumping plant forces water through a 600 mm diameter main, the friction head being 27 m. In order to reduce the power consumption, it is proposed to lay another main of appropriate diameter along the side of the existing one, so that two pipes may work in parallel for the entire length and reduce the friction head to 9.6 m only. Find the diameter of the new main if, with the exception of diameter, it is similar to the existing one in every respect.*

Solution. Given :

Dia. of single main pipe, d = 600 mm = 0.6 m

Friction head, h_f = 27 m

Friction head for two parallel pipes = 9.6 m

1st Case.

For a single main [Fig. 11.19 (*a*)]

$$h_f = \frac{4 \,.\, f \,.\, L \,.\, V^2}{d \times 2g} \quad \text{or} \quad 27.0 = \frac{4 \times f \times L \times V^2}{0.6 \times 2 \times 9.81}$$

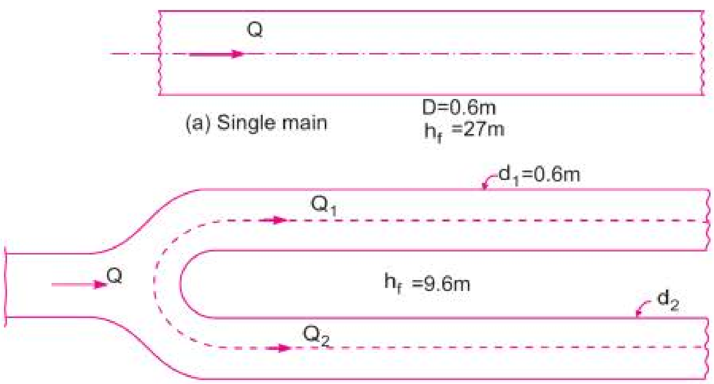
$\therefore$ $$fLV^2 = \frac{27.0 \times 0.6 \times 2 \times 9.81}{4} = \frac{317.844}{5} = 79.461, \text{ where } V = \frac{Q}{A}$$

$\therefore$ $$f.L. \frac{Q^2}{A^2} = 79.461 \qquad \text{...(i)}$$

2nd Case. Two pipes are in parallel [Fig. 11.19 (*b*)]

Loss of head in any one pipe = 9.6 m

$\therefore$ For 1st pipe, $$h_{f_1} = \frac{4 \,.\, f \,.\, L \,.\, V_1^2}{d_1 \times 2g} = 9.6$$



(a) Single main

(b) Two parallel pipes

Fig. 11.19

But $$L_1 = L,\ V_1 = \frac{Q_1}{A_1} = \frac{Q_1}{A} \qquad \left[\because A_1 = A = \frac{\pi}{4}(.6)^2\right]$$

$$d_1 = d = 0.6$$

$\therefore$ $$\frac{4 \,.\, f \,.\, L}{0.6 \times 2 \times 9.81} \frac{Q_1^2}{A^2} = 0.6$$

or $$f \times L \times \frac{Q_1^2}{A^2} = \frac{9.6 \times 0.6 \times 2 \times 9.81}{4} = 28.2528 \qquad \text{...(ii)}$$

For the 2nd pipe, $$h_{f_2} = \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} = 9.6, \qquad \text{where } L_2 = L,\ V_2 = \frac{Q_2}{A_2}$$

$\therefore$ $$\frac{4f \times L \times Q_2^2}{d_2 \times 2g \times A_2^2} = 9.6$$

or $$\frac{f \times L \times Q_2^2}{d_2 \times A_2^2} = \frac{9.6 \times 2 \times 9.81}{4} = 47.088 \qquad \text{...(iii)}$$

Dividing equation (*i*) by equation (*ii*), we get

$$\frac{Q^2}{Q_1^2} = \frac{79.461}{28.2528} = 2.8125$$

$\therefore$ $$\frac{Q}{Q_1} = \sqrt{2.8125} = 1.667$$

$\therefore$ $$Q_1 = \frac{Q}{1.667} = .596\ Q$$

But $$Q_1 + Q_2 = Q$$

$\therefore$ $$Q_2 = Q - Q_1 = Q - .596\ Q = 0.404\ Q$$

Dividing equation (*ii*) by equation (*iii*),

$$\frac{Q_1^2 \times d_2 \times A_2^2}{A^2 \times Q_2^2} = \frac{28.2528}{47.088} = 0.6$$

But $$A_2 = \frac{\pi}{4} d_2^2 \text{ and } A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.6)^2 = \frac{\pi}{4} \times .36$$

$\therefore$ $$\frac{Q_1^2}{Q_2^2} \times \frac{d_2 \times \left(\frac{\pi}{4}\right)^2 \times d_2^4}{\left(\frac{\pi}{4}\right)^2 \times (.36)^2} = 0.6 \quad \text{or} \quad \left(\frac{.596\ Q}{.404\ Q}\right)^2 \times \frac{d_2^5}{.36^2} = 0.6$$

or $$d_2^5 = 0.6 \times .36^2 \times \left(\frac{.404}{.596}\right)^2 = 0.03537$$

$\therefore$ $$d_2 = (.03537)^{1/5} = 0.5125 \text{ m} = \mathbf{512.5\ mm.}\ \textbf{Ans.}$$

Problem 11.35 *A pipe of diameter 20 cm and length 2000 m connects two reservoirs, having difference of water levels as 20 m. Determine the discharge through the pipe.*

If an additional pipe of diameter 20 cm and length 1200 m is attached to the last 1200 m length of the existing pipe, find the increase in the discharge. Take f = .015 and neglect minor losses.

Solution. Given :

Dia. of pipe, $d = 20 \text{ cm} = 0.20 \text{ m}$

Length of pipe, $L = 2000 \text{ m}$

Difference of water levels, $H = 20 \text{ m}$

Co-efficient of friction, $f = 0.015$

1st Case. When a single pipe connects the two reservoirs

$$H = \frac{4 . f . L . V^2}{d \times 2g} = \frac{4f . L}{d \times 2g}\left(\frac{Q}{\frac{\pi}{4}d^2}\right)^2 \qquad \left[\because V = \frac{Q}{\frac{\pi}{4}d^2}\right]$$

$$= \frac{32 f . L . Q^2}{\pi^2 \times g \times d^5}$$

or $$20 = \frac{32 \times .015 \times 2000 \times Q^2}{\pi^2 \times 9.81 \times (0.2)^5} = 30985.07\ Q^2$$

$\therefore$ $$Q = \sqrt{\frac{20}{30985.07}} = \mathbf{0.0254\ m^3/s.}\ \textbf{Ans.}$$

2nd Case.

Let Q_1 = discharge through pipe CD,

Q_2 = discharge through pipe DE,

Q_3 = discharge through pipe DF.

Length of pipe CD, $L_1 = 800$ m and its dia., $d_1 = 0.20$ m

Length of pipe DE, $L_2 = 1200$ m and its dia., $d_2 = 0.20$ m

Length of pipe DF, $L_3 = 1200$ m and its dia., $d_3 = 0.20$ m.

Since the diameters and lengths of the pipes DE and DF are equal. Hence Q_2 will be equal to Q_3. Also for parallel pipes, we have

$$Q_1 = Q_2 + Q_3 = Q_2 + Q_2 = 2Q_2 \qquad [\because Q_2 = Q_3]$$

$\therefore$ $$Q_2 = \frac{Q_1}{2}$$

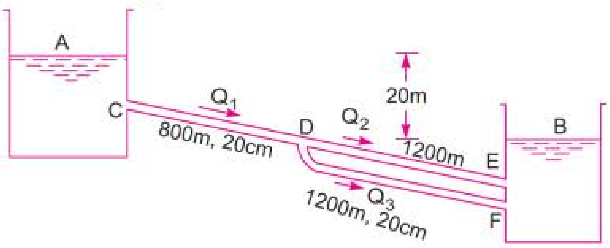


Fig. 11.20

Applying Bernoulli's equation to points A and B and taking the flow through CDE, we have

$$20 = \frac{4f \cdot L_1 \cdot V_1^2}{d_1 \times 2g} + \frac{4f \cdot L_2 \cdot V_2^2}{d_2 \times 2g}$$

where $$V_1 = \frac{Q_1}{\frac{\pi}{4}(.2)^2} = \frac{4Q_1}{\pi \times .04}, V_2 = \frac{Q_2}{\frac{\pi}{4}(.2)^2} = \frac{4Q_2}{\pi \times .04} = \frac{4 \times \frac{Q_1}{2}}{\pi \times .04} = \frac{2Q_1}{\pi \times .04}$$

$$= \frac{4 \times .015 \times 800}{0.2 \times 2 \times 9.81} \times \left(\frac{4Q_1}{\pi \times .04}\right)^2 + \frac{4 \times .015 \times 1200}{0.2 \times 2 \times 9.81} \times \left(\frac{2Q_1}{\pi \times .04}\right)^2$$

$$= 12394\ Q_1^2 + 4647\ Q_1^2 = 17041\ Q_1^2$$

$\therefore$ $$Q_1 = \sqrt{\frac{20}{17041}} = 0.0342\ \text{m}^3/\text{s}$$

Increase in discharge = $Q_1 - Q = 0.0342 - 0.0254 =$ $\mathbf{.0088\ m^3/s.}$ **Ans.**

Problem 11.36 *Two pipes have a length L each. One of them has a diameter D, and the other a diameter d. If the pipes are arranged in parallel, the loss of head, when a total quantity of water Q flows through them is h, but, if the pipes are arranged in series and the same quantity Q flows through them, the loss of head is H. If* $d = \dfrac{D}{2}$, *find the ratio of H to h, neglecting secondary losses and assuming the pipe co-efficient f has a constant value.*

Solution. Given :

Length of pipe 1, $L_1 = L$ and its dia. $d_1 = D$

Length of pipe 2, $L_2 = L$ and its dia., $d_2 = d$

Total discharge $= Q$

Head loss when pipes are arranged in parallel $= h$

Head loss when pipes are arranged in series $= H$

$$d = \frac{D}{2} \text{ and } f \text{ is constant}$$

1st Case. When pipes are connected to parallel

$$Q = Q_1 + Q_2 \qquad ...(i)$$

Loss of head in each pipe $= h$

For pipe AB,
$$\frac{4fL_1V_1^2}{d_1 \times 2g} = h, \text{ where } V_1 = \frac{Q_1}{A_1} = \frac{Q_1}{\frac{\pi}{4}D^2} = \frac{4Q_1}{\pi D^2}$$

$$d_1 = D$$

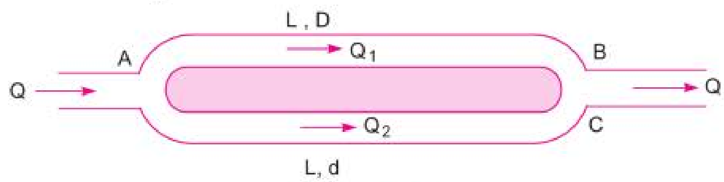


Fig. 11.21

$$\therefore \qquad \frac{4fL \times \left(\frac{4Q_1}{\pi D^2}\right)^2}{D \times 2g} = h \text{ or } \frac{32fLQ_1^2}{\pi^2 D^5 \times g} = h \qquad ...(ii)$$

For pipe AC,
$$\frac{32fLQ_2^2}{\pi^2 d^5 \times g} = h \qquad ...(iii)$$

$$\therefore \qquad \frac{32fLQ_1^2}{\pi^2 D^5 g} = \frac{32fLQ_2^2}{\pi^2 d^5 g} \text{ or } \frac{Q_1^2}{D^5} = \frac{Q_2^2}{d^5}$$

or
$$\left(\frac{Q_1}{Q_2}\right)^2 = \frac{D^5}{d^5} = \frac{(2d)^5}{d^5} \qquad [\because D = 2d]$$
$$= 2^5 = 32$$

$$\therefore \qquad \frac{Q_1}{Q_2} = \sqrt{32} = 5.657 \text{ or } Q_1 = 5.657\ Q_2$$

Substituting the values of Q_1 in equation (*i*), we get

$$Q = 5.657\ Q_2 + Q_2 = 6.657\ Q_2$$

$$\therefore \quad Q_2 = \frac{Q}{6.657} = 0.15\ Q \qquad \text{...}(iv)$$

From (*i*) $\therefore$

$$Q_1 = Q - Q_2 = Q - 0.15\ Q = 0.85\ Q \qquad \text{...}(v)$$

2nd Case. When the pipes are connected in series.

Total loss = Sum of head losses in the two pipes

$$\therefore \quad H = \frac{4f\,.\,L\,.\,V_1^2}{d_1 \times 2g} + \frac{4f\,.\,L\,.\,V_2^2}{d_2 \times 2g}$$

where $V_1 = \dfrac{Q}{\dfrac{\pi}{4}D^2} = \dfrac{4Q}{\pi D^2}$, $V_2 = \dfrac{Q}{\dfrac{\pi}{4}d^2} = \dfrac{4Q}{\pi d^2}$

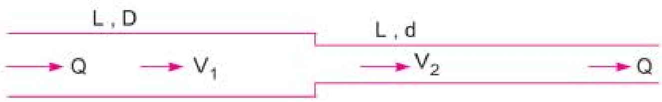


Fig. 11.22

$$\therefore \quad H = \frac{4f\,.\,L \times \left(\dfrac{4Q}{\pi D^2}\right)^2}{D \times 2g} + \frac{4fL\left(\dfrac{4Q}{\pi d^2}\right)^2}{d \times 2g}$$

or

$$H = \frac{32\,fLQ^2}{D^5\pi^2 \times g} + \frac{32\,fLQ^2}{d^5\pi^2 \times g} \qquad \text{...}(vi)$$

From equation (*ii*), $\dfrac{32\,fL}{\pi^2 D^5 \times g} = \dfrac{h}{Q_1^2}$

and from equation (*iii*), $\dfrac{32\,fL}{\pi^2 d^5 \times g} = \dfrac{h}{Q_2^2}$

Substituting these values in equation (*vi*), we have

$$H = Q^2 \times \frac{h}{Q_1^2} + Q^2 \times \frac{h}{Q_2^2} = \frac{Q^2}{Q_1^2}h + \frac{Q^2}{Q_2^2}h = h\left[\frac{Q^2}{Q_1^2} + \frac{Q^2}{Q_2^2}\right]$$

$$\therefore \quad \frac{H}{h} = \frac{Q^2}{Q_1^2} + \frac{Q^2}{Q_2^2}$$

But from equations (*iv*) and (*v*), $Q_1 = .85\ Q$ and $Q_2 = 0.15\ Q$

$$\therefore \quad \frac{H}{h} = \frac{Q^2}{.85^2 Q^2} + \frac{Q^2}{.15^2 Q^2} = \frac{1}{.85^2} + \frac{1}{.15^2} = 1.384 + 44.444 = \mathbf{45.828.}\ \textbf{Ans.}$$

Problem 11.36 (A). *Three pipes of the same length L, diameter D, and friction factor f are connected in parallel. Determine the diameter of the pipe of length L and friction factor f which will carry the same discharge for the same head loss. Use the formula* $h_f = f \times L \times V^2/2g\ D$.

Solution. Given :

Length of each pipe $= L$

Diameter of each pipe $= D$

Friction factor of each pipe $= f$

Head loss, $h_f = f \times L \times V^2/2gD$

When the three pipes are connected in parallel, then head loss in each pipe will be same. And total head loss will be equal to the head loss in each pipe.

Let h_f = Total head loss,

h_{f_1} = Head loss in 1st pipe,

h_{f_2} = Head loss in 2nd pipe, and h_{f_3} = Head loss in 3rd pipe.

Then
$$h_f = h_{f_1} = h_{f_2} = h_{f_3} \text{ or } h_f = \frac{f \times L \times V^2}{2gD} \qquad ...(i)$$

Let Q_1 = Discharge through 1st pipe, Q_2 = Discharge through 2nd pipe,

Q_3 = Discharge through 3rd pipe, and Q = Total discharge.

When the three pipes are connected in parallel, then

$$Q = Q_1 + Q_2 + Q_3 = 3 \times Q_1 \qquad (\because Q_1 = Q_2 = Q_3)$$
$$= 3 \times A_1 \times V_1$$
$$= 3 \times \frac{\pi}{4} D^2 \times V \left(\text{where } A_1 = \frac{\pi}{4} D^2 \text{ and } V_1 = V\right) \qquad ...(ii)$$

For a single pipe (or length L ; friction factor f) which will carry same discharge as the three pipes in parallel

Let d = dia. of the single pipe

v = velocity through single pipe

Then discharge, Q = Area $\times$ Velocity $= \left(\frac{\pi}{4} d^2\right) \times v \qquad ...(iii)$

Equating the two values of discharge, given by equations (*ii*) and (*iii*), we get

$$3 \times \frac{\pi}{4} D^2 \times V = \frac{\pi}{4} d^2 \times v \text{ or } 3 \times \frac{D^2}{d^2} = \frac{v}{V} \qquad ...(iv)$$

The head loss for the single pipe is also equal to the total head loss for three pipes when they are in parallel.

But head loss for the single pipe of length L, dia. d, friction factor f and velocity v is given by

$$h_f = \frac{f \times L \times v^2}{d \times 2g} \qquad ...(v)$$

Equating the two values of h_f given by equations (*i*) and (*v*), we get

$$\frac{f \times L \times V^2}{D \times 2g} = \frac{f \times L \times v^2}{d \times 2g} \text{ or } \frac{V^2}{D} = \frac{v^2}{d}$$

or
$$\frac{d}{D} = \frac{v^2}{V^2} \quad \text{or} \quad \left(\frac{d}{D}\right)^{1/2} = \frac{v}{V}$$

Substituting the value of v/V in equation (*iv*), we get

$$3 \times \frac{D^2}{d^2} = \left(\frac{d}{D}\right)^{1/2} \text{ or } 3 = \left(\frac{d}{D}\right)^{1/2} \times \left(\frac{d}{D}\right)^2 = \left(\frac{d}{D}\right)^{5/2}$$

or $$\frac{d}{D} = 3^{2/5} = 3^{0.4} = 1.55$$

$\therefore$ $$d = \mathbf{1.55\ D.}\ \textbf{Ans.}$$

Hence dia. of single pipe should be 1.55 times the dia. of the three pipes connected in parallel.

Problem 11.37 *For a town water supply, a main pipe line of diameter 0.4 m is required. As pipes more than 0.35 m diameter are not readily available, two parallel pipes of the same diameter were used for water supply. If the total discharge in the parallel pipes is same as in the single main pipe, find the diameter of the parallel pipe. Assume the co-efficient of friction same for all pipes.*

Solution. Given :

Dia. of single main pipe line, $d = 0.4$ m

Let the length of single pipe line $= L$

Co-efficient of friction $= f$

Loss of head due to friction in single pipe $= \dfrac{4fLV^2}{d \times 2g} = \dfrac{4fLV^2}{0.4 \times 2 \times g}$ ...(*i*)

where V = Velocity of flow in the single pipe.

In case of parallel pipe, as the diameters and lengths of the two pipes are same. Hence discharge in each pipe will be half the discharge of single main pipe. As discharge in each parallel pipe is same, hence velocity will also be same.

Let V_* = Velocity in each parallel pipe

d_* = Dia. of each parallel pipe

Then loss of head due to friction in parallel pipes $= \dfrac{4f \times L \times V_*^2}{d_* \times 2g}$ ...(*ii*)

Equating the two losses given by equations (*i*) and (*ii*), we have

$$\frac{4f\,.\,L\,.\,V^2}{0.4 \times 2g} = \frac{4f \times L \times V_*^2}{d_* \times 2g}$$

Cancelling $\dfrac{4fL}{2g}$, $$\frac{V^2}{0.4} = \frac{V_*^2}{d_*^2} \text{ or } \frac{V^2}{V_*^2} = \frac{0.4}{d_*} \quad \text{...(iii)}$$

From continuity

Total flow in single main = sum of flow in two parallel pipes

or Velocity of main × Area = 2 × Velocity in each parallel pipe × Area

$$V \times \frac{\pi}{4}(0.4)^2 = 2 \times V_* \times \frac{\pi}{4} d^2{}_* \text{ or } \frac{V}{V_*} = \frac{2 \times \frac{\pi}{4} d_*^2}{\frac{\pi}{4}(0.4)^2} = \frac{2d_*^2}{0.16}$$

Squaring both sides, $$\frac{V^2}{V_*^2} = \frac{4d_*^4}{0.0256} \quad \text{...(iv)}$$

Comparing equations (*iii*) and (*iv*), we get

$$\frac{0.4}{d_*} = \frac{4d_*^4}{.0256} \quad \text{or} \quad d_*^5 = \frac{0.4 \times .0256}{4} = .00256$$

$\therefore$ $d_* = (.00256)^{1/5} = 0.303 \text{ m} = $ **30.3 cm. Ans.**

$\therefore$ Use two pipes of 30.3 cm diameter.

Problem 11.38 *An old water supply distribution pipe of 250 mm diameter of a city is to be replaced by two parallel pipes of smaller equal diameter having equal lengths and identical friction factor values. Find out the new diameter required.*

Solution. Given :

Dia. of old pipe, $D = 250$ mm $= 0.25$ m

Let d = Dia. of each of parallel pipes

Q = Discharge in old pipe

Q_1 = Discharge in first parallel pipe

Q_2 = Discharge in second parallel pipe

f = Friction factor.

When a single pipe is replaced by two parallel pipes, the head loss will be same in the single pipe and in each of the parallel pipes. Also the discharge in single pipe will be equal to the total discharge in two parallel pipes *i.e.,*

$$h_f = h_{f_1} = h_2 \qquad \text{...(i)}$$

and

$$Q = Q_1 + Q_2 \qquad \text{...(ii)}$$

As the dia. of each parallel pipe is same and also length of each parallel pipe is equal, hence

$$Q_1 = Q_2 \quad \text{or} \quad Q_1 = Q_2 = Q/2$$

Now h_f = Head loss in single pipe

$$= \frac{f \times L \times V^2}{D \times 2g}, \qquad \text{where } f = \text{Friction factor}$$

$$= \frac{f \times L \times \left(\dfrac{Q}{\dfrac{\pi}{4} \times 0.25^2}\right)^2}{0.25 \times 2 \times 9.81} \qquad \left(\because V = \frac{Q}{\text{Area}} = \frac{Q}{\dfrac{\pi}{4} D^2}\right)$$

$$= \frac{f \times L \times (4Q)^2}{0.25 \times 2 \times 9.81 \times \left(\pi \times 0.25^2\right)^2} \qquad \text{...(iii)}$$

h_{f_1} = Head loss in 1st parallel pipe

$$= \frac{f \times L \times (V_1)^2}{d \times 2g} \quad (\because \text{ Dia. of parallel pipe} = d \text{ and } V_1 \text{ is the velocity in 1st parallel pipe})$$

$$= \frac{f \times L \times \left(\dfrac{Q}{2 \times \dfrac{\pi}{4} d^2}\right)^2}{d \times 2g} \qquad \left(\because V_1 = \frac{Q_1}{A_1} = \frac{\dfrac{Q}{2}}{\dfrac{\pi}{4} d^2} \text{ as } Q_1 = \frac{Q}{2}\right)$$

$$= \frac{f \times L \times (4Q)^2}{d \times 2 \times 9.81 \times \left(2 \times \pi \times d^2\right)^2} \quad \text{...(iv)}$$

But $\quad h_f = h_{f_1}$

or $$\frac{f \times L \times (4Q)^2}{0.25 \times 2 \times 9.81 \times (\pi \times 0.25^2)^2} = \frac{f \times L \times (4Q)^2}{d \times 2 \times 9.81 \times \left(2 \times \pi \times d^2\right)^2}$$

or $$d \times (2\pi d^2)^2 = 0.25 \times (\pi \times 0.25^2)^2$$

or $$d \times 4 \times d^4 = 0.25 \times 0.25^4$$

or $$d^5 = \frac{0.25^5}{4} \text{ or } d = \frac{0.25}{(4)^{1/5}} = \frac{0.25}{1.3195} = \mathbf{0.1894\ m \simeq 0.19\ m.\ Ans.}$$

Problem 11.39 *A pipe of diameter 0.4 m and of length 2000 m is connected to a reservoir at one end. The other end of the pipe is connected to a junction from which two pipes of lengths 1000 m and diameter 300 mm run in parallel. These parallel pipes are connected to another reservoir, which is having level of water 10 m below the water level of the above reservoir. Determine the total discharge if f = 0.015. Neglect minor losses.*

Solution. Given :

Dia. of pipe, $\quad d = 0.4$ m

Length of pipe, $\quad L = 2000$ m

Dia. of parallel pipes, $\quad d_1 = d_2 = 300$ mm $= 0.30$ m

Length of parallel pipes, $\quad L_1 = L_2 = 1000$ m

Difference of water level in two reservoir, $H = 10$ m, $f = .015$

Applying Bernoulli's equation to points E and F. Taking flow through ABC.

$$10 = \frac{4fLV^2}{d \times 2g} + \frac{4f \times L_1 \times V_1^2}{d_1 \times 2g}$$

$$= \frac{4 \times .015 \times 2000 \times V^2}{0.4 \times 2 \times 9.81} + \frac{4 \times .015 \times 1000 \times V_1^2}{0.3 \times 2 \times 9.81}$$

$$= 15.29\ V^2 + 10.19\ V_1^2 \quad \text{...(i)}$$

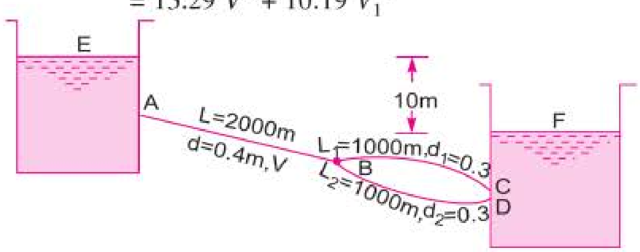


Fig. 11.23

From continuity equation

Discharge through AB = discharge through BC + discharge through BD

or $$\frac{\pi}{4}\, d^2 \times V = \frac{\pi}{4}\, d_1^{\,2} \times V_1 + \frac{\pi}{4}\, d_1^{\,2} V_2$$

But $d_1 = d_2$ and also the lengths of pipes BC and BD are equal and hence discharge through BC and BD will be same. This means $V_1 = V_2$ also

$\therefore$ $$\frac{\pi}{4}\, d^2 V = \frac{\pi}{4}\, d_1^{\,2} \times V_1 + \frac{\pi}{4}\, d_1^{\,2} \times V_1 \qquad [\because d_1 = d_2,\ V_1 = V_2]$$

$$= 2 \times \frac{\pi}{4}\, d_1^{\,2} \times V_1 \ \text{ or } \ d^2 V = 2 d_1^{\,2} V_1$$

or $$(0.4)^2 \times V = 2 \times (0.3)^2 V_1 \text{ or } .16V = 0.18\, V_1$$

$\therefore$ $$V_1 = \frac{0.16}{0.18}\, V = 0.888\, V$$

Substituting this value of V_1 in equation (i), we get

$$10 = 15.29\, V^2 + (10.19)(.888)^2 V^2 = 15.29\, V^2 + 8.035\, V^2 = 23.325\, V^2$$

$\therefore$ $$V = \sqrt{\frac{10}{23.325}} = 0.654 \text{ m/s}$$

$\therefore$ Discharge $= V \times \text{Area}$

$$= 0.654 \times \frac{\pi}{4}\, d^2 = 0.654 \times \frac{\pi}{4}\,(0.4)^2 = \mathbf{.0822\ m^3/s.}\ \textbf{Ans.}$$

Problem 11.40 *Two sharp ended pipes of diameters 50 mm and 100 mm respectively, each of length 100 m are connected in parallel between two reservoirs which have a difference of level of 10 m. If the co-efficient of friction for each pipe is (4f) 0.32, calculate the rate of flow for each pipe and also the diameter of a single pipe 100 m long which would give the same discharge, if it were substituted for the original two pipes.*

Solution. Given :

Dia. of 1st pipe, $d_1 = 50 \text{ mm} = 0.05 \text{ m}$

Length of 1st pipe, $L_1 = 100 \text{ m}$

Dia. of 2nd pipe, $d_2 = 100 \text{ mm} = 0.10 \text{ m}$

Length of 2nd pipe, $L_2 = 100 \text{ m}$

Difference in level in reservoirs, $H = 10$ m

Co-efficient of friction $4f = 0.32$

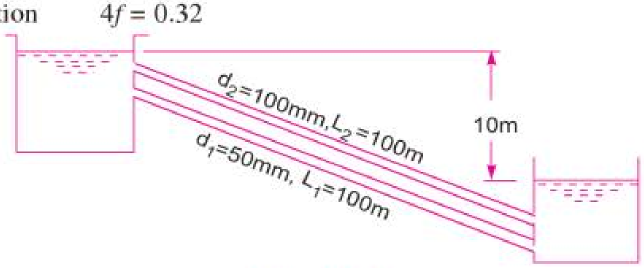


Fig. 11.24

Let V_1 = velocity of flow in pipe 1, and

V_2 = velocity of flow in pipe 2.

When the pipes are connected in parallel, the loss of head will be same in both the pipes.

For the first pipe, loss of head is given as

$$H = \frac{4f \times L_1 \times V_1^2}{d_1 \times 2g} = \frac{0.32 \times 100 \times V_1^2}{0.05 \times 2 \times 9.81} \qquad (\because 4f = .32)$$

or $$10 = 32.619\ V_1^2$$

$\therefore$ $$V_1 = \sqrt{\frac{10}{32.619}} = 0.5535 \text{ m/s}$$

$\therefore$ Rate of flow in 1st pipe, $Q_1 = V_1 \times A_1 = 0.5536 \times \frac{\pi}{4} (d_1)^2$

$$= .5536 \times \frac{\pi}{4} (0.05)^2 = .001087 \text{ m}^3\text{/s} = \mathbf{1.087\ litres/s.}\ \textbf{Ans.}$$

For the 2nd pipe, loss of head is given by,

$$10 = H = \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} = \frac{0.32 \times 100 \times V_2^2}{0.10 \times 2 \times 9.81}$$

$\therefore$ $$V_2 = \sqrt{\frac{10 \times .10 \times 2 \times 9.81}{.32 \times 100}} = 0.783 \text{ m/s}$$

$\therefore$ Rate of flow in 2nd pipe, $Q_2 = A_2 \times V_2 = \frac{\pi}{4} d_2^2 \times V_2$

$$= \frac{\pi}{4} (.1)^2 \times .783 = 0.00615 \text{ m}^3\text{/s} = \mathbf{6.15\ litres/s.}\ \textbf{Ans.}$$

Let D = diameter of a single pipe which is substituted for the two original pipes
L = length of single pipe = 100 m
V = velocity through pipe

The discharge through single pipe,

$$Q = Q_1 + Q_2 = 1.087 + 6.15 = 7.237 \text{ litres/s} = .007237 \text{ m}^3\text{/s}$$

$\therefore$ $$V = \frac{Q}{\text{Area}} = \frac{.007237}{\frac{\pi}{4} D^2} = \frac{4 \times .007237}{\pi D^2} = \frac{.009214}{D^2} \text{ m/s}$$

Loss of head through single pipe is

$$H = \frac{4f \times L \times V^2}{D \times 2g} = \frac{0.32 \times 100 \times \left(\frac{.009214}{D^2}\right)^2}{D \times 2 \times 9.81}$$

or $$10.0 = \frac{.32 \times 100 \times .009214^2}{2 \times 9.81 \times D^5} = \frac{.0001384}{D^5}$$

or $$D^5 = \frac{.0001384}{10} = .00001384$$

$\therefore$ $$D = (.00001384)^{1/5} = 0.1067 \text{ m} = \mathbf{106.7\ mm.}\ \textbf{Ans.}$$

Problem 11.41 *Two reservoir are connected by a pipe line of diameter 600 mm and length 4000 m. The difference of water level in the reservoirs is 20 m. At a distance of 1000 m from the upper reservoir, a small pipe is connected to the pipe line. The water can be taken from the small pipe. Find the discharge to the lower reservoir, if*

(i) No water is taken from the small pipe, and

(ii) 100 litres/s of water is taken from small pipe.

Take f = .005 and neglect minor losses.

Solution. Given :

Dia. of pipe, $d = 600$ mm $= 0.60$ m

Length of pipe, $L = 400$ m

Difference of water level, $H = 20$ m, $f = .005$

(*i*) **No water is taken from small pipe**

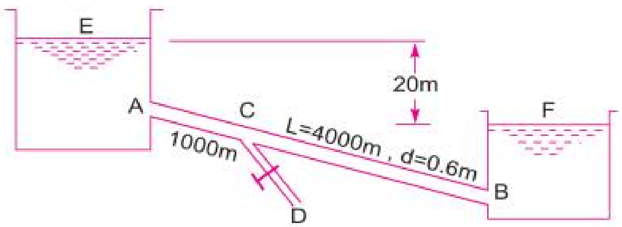


Fig. 11.25

The head loss due to friction in pipe $AB = \dfrac{4f \times L \times V^2}{d \times 2g}$ or $20 = \dfrac{4 \times .005 \times 4000 \times V^2}{0.6 \times 2 \times 9.81}$

$$\therefore \qquad V = \sqrt{\frac{20 \times 0.6 \times 2 \times 9.81}{4 \times .005 \times 4000}} = \sqrt{2.943} = 1.715 \text{ m/s}$$

$\therefore$ Discharge, $Q = \text{Area} \times V = \dfrac{\pi}{4}(0.6)^2 \times 1.715 =$ **0.485 m³/s. Ans.**

(*ii*) **100 litres of water is taken from small pipe**

Let Q_1 = discharge through pipe AC

Q_2 = discharge through pipe CB

Then for parallel pipes $Q_1 = Q_2 + 100$ litres/s $= Q_2 + 0.1$ m^3/s

$\therefore \qquad Q_2 = (Q_1 - 0.1)$ m^3/s ...(*i*)

Length of pipe AC, $L_1 = 1000$ m

Length of pipe CB, $L_2 = 4000 - 1000 = 3000$ m

Applying Bernoulli's equation to points E and F and taking flow through ABC, we have

$$20 = \frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} \qquad ...(ii)$$

where V_1 = velocity through pipe $AC = \dfrac{Q_1}{\dfrac{\pi}{4}(0.6)^2} = \dfrac{4Q_1}{\pi \times .36}$

d_1 = dia. of pipe $AC = 0.6$

V_2 = velocity through pipe $CB = \dfrac{Q_2}{\frac{\pi}{4}(0.6)^2} = \dfrac{4Q_2}{\pi \times .36}$

d_2 = dia. of pipe $CB = 0.6$

Substituting these values in equation (*ii*), we get

$$20 = \frac{4 \times .005 \times 1000}{0.6 \times 2 \times 9.81} \times \left(\frac{4Q_1}{\pi \times .36}\right)^2 + \frac{4 \times .005 \times 3000}{0.6 \times 2 \times 9.81} \times \left(\frac{4Q_2}{\pi \times .36}\right)^2$$

$$20 = 21.25\, Q_1^2 + 63.75\, Q_2^2 \qquad \text{...}(iii)$$

But from (*i*), $Q_2 = Q_1 - 0.1$ or $Q_1 = Q_2 + 0.1$

Substituting the value of Q_1 in equation (*iii*), we get

$$20 = 21.25\, (Q_2 + 0.1)^2 + 63.75\, Q_2^2$$
$$= 21.55\, [Q_2^2 + .01 + 0.2\, Q_2] + 63.75\, Q_2^2$$
$$= 21.25\, Q_2^2 + 0.2125 + 4.250\, Q_2 + 63.75\, Q_2^2$$
$$= 85\, Q_2^2 + 4.25\, Q_2 + .2125$$

or $85\, Q_2^2 + 4.25\, Q_2 - 19.7875 = 0$

This is a quadratic equation in Q_2

$$\therefore \qquad Q_2 = \frac{-4.25 \pm \sqrt{4.25^2 + 4 \times 85 \times 19.7875}}{2 \times 85}$$

$$= \frac{-4.25 \pm \sqrt{18.0625 + 6727.75}}{170} = \frac{-44.25 \pm 82.13}{170} = \frac{82.13 - 4.25}{170}$$

$$= 0.458 \text{ m}^3\text{/s} \qquad \text{(Neglecting negative root)}$$

$\therefore$ Discharge to lower reservoir = Q_2 = **0.458 m³/s. Ans.**

11.10 FLOW THROUGH BRANCHED PIPES

When three or more reservoirs are connected by means of pipes, having one or more junctions, the system is called a branching pipe system. Fig. 11.26 shows three reservoirs at different levels connected to a single junction, by means of pipes which are called branched pipes. The lengths, diameters and co-efficient of friction of each pipes is given. It is required to find the discharge and direction of flow in each pipe. The basic equations used for solving such problems are :

1. **Continuity equation** which means the inflow of fluid at the junction should be equal to the outflow of fluid.
2. **Bernoulli's equation, and**
3. **Darcy-Weisbach equation**

Also it is assumed that reservoirs are very large and the water surface levels in the reservoirs are constant so that steady conditions exist in the pipes. Also minor losses are assumed very small. The flow from reservoir *A* takes place to junction *D*. The flow from junction *D* is towards reservoirs *C*. Now the flow from junction *D* towards reservoir *B* will take place only when piezometric head at *D* $\left(\text{which is equal to } \dfrac{p_D}{\rho g} + Z_D\right)$ is more than the piezometric head at *B* (*i.e.*, Z_B). Let us consider that flow is from *D* to reservoir *B*.

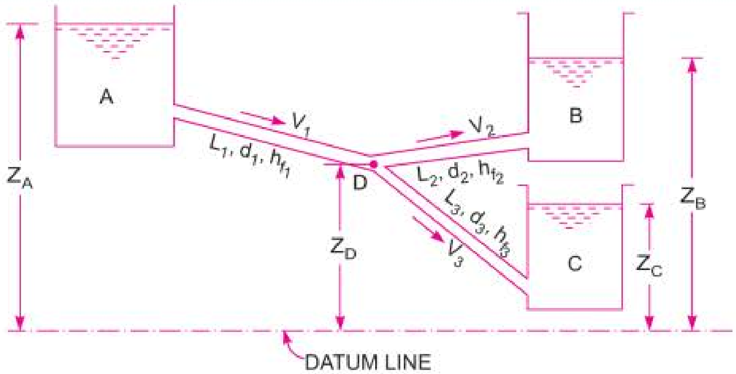


Fig. 11.26

For flow from A to D from Bernoulli's equation

$$Z_A = Z_D + \frac{p_D}{\rho g} + h_{f_1} \quad ...(i)$$

For flow from D to B from Bernoulli's equation

$$Z_D + \frac{p_D}{\rho g} = Z_B + h_{f_2} \quad ...(ii)$$

For flow from D to C from Bernoulli's equation

$$Z_D + \frac{p_D}{\rho g} = Z_C + h_{f_3} \quad ...(iii)$$

From continuity equation,

Discharge through AD = Discharge through DB + Discharge through DC

$$\therefore \quad \frac{\pi}{4} d_1^2 V_1 = \frac{\pi}{4} d_2^2 \times V_2 + \frac{\pi}{4} d_3^2 V_3$$

or
$$d_1^2 V_1 = d_2^2 V_2 + d_3^2 V_3 \quad ...(iv)$$

There are four unknowns *i.e.*, V_1, V_2, V_3 and $\frac{p_D}{\rho g}$ and there are four equations (*i*), (*ii*), (*iii*) and (*iv*). Hence unknown can be calculated.

Problem 11.42 *Three reservoirs A, B and C are connected by a pipe system shown in Fig. 11.27. Find the discharge into or from the reservoirs B and C if the rate of flow from reservoirs A is 60 litres/s. Find the height of water level in the reservoir C. Take f = .006 for all pipes.*

Solution. Given :

Length of pipe AD, $L_1 = 1200$ m

Dia. of pipe AD, $d_1 = 30$ cm $= 0.30$ m

Discharge through AD, $Q_1 = 60$ litres/s $= 0.06$ m^3/s

Height of water level in A from reference line, $Z_A = 40$ m

For pipe DB, length $L_2 = 600$ m, dia., $d_2 = 20$ cm $= 0.20$ m, $Z_B = 38.0$

For pipe DC, length $L_3 = 800$ m, dia., $d_3 = 30$ cm $= 0.30$ m

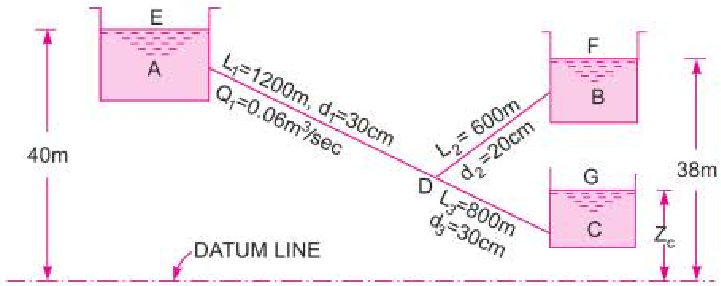


Fig. 11.27

Applying Bernoulli's equations to points E and D, $Z_A = Z_D + \dfrac{p_D}{\rho g} + h_{f_1}$

where $h_{f_1} = \dfrac{4 \,.\, f \,.\, L_1 \,.\, V_1^2}{d_1 \times 2g}$, where $V_1 = \dfrac{Q_1}{\text{Area}} = \dfrac{0.06}{\frac{\pi}{4}(.3)^2} = 0.848$ m/sec

$$h_{f_1} = \frac{4 \times .006 \times 1200 \times .848^2}{0.3 \times 2 \times 9.81} = 3.518 \text{ m}$$

$$\therefore \qquad Z_A = Z_D + \frac{p_D}{\rho g} + 3.518 \text{ or } 40.0 = Z_D + \frac{p_D}{\rho g} + 3.518$$

$$\therefore \qquad \left(Z_D + \frac{p_1}{\rho g}\right) = 40.0 - 3.518 = 36.482 \text{ m}$$

Hence piezometric head at $D = 36.482$. But $Z_B = 38$ m. Hence water flows from B to D.
Applying Bernoulli's equation to points B and D

$$Z_B = \left(Z_D + \frac{p_D}{\rho g}\right) + h_{f_2} \text{ or } 38 = 36.482 + h_{f_2}$$

$$\therefore \qquad h_{f_2} = 38 - 36.482 = 1.518 \text{ m}$$

But
$$h_{f_2} = \frac{4 \,.\, f \,.\, L_2 \,.\, V_2^2}{d_2 \times 2g} = \frac{4 \times .006 \times 600 \times V_2^2}{0.2 \times 2 \times 9.81}$$

$$\therefore \qquad 1.518 = \frac{4 \times .006 \times 600 \times V_2^2}{0.2 \times 2 \times 9.81}$$

$$\therefore \qquad V_2 = \sqrt{\frac{1.518 \times 0.2 \times 2 \times 9.81}{4 \times .006 \times 600}} = 0.643 \text{ m/s.}$$

$\therefore$ Discharge,
$$Q_2 = V_2 \times \frac{\pi}{4}(d_2)^2 = 0.643 \times \frac{\pi}{2} \times (.2)^2$$

$$= 0.0202 \text{ m}^3\text{/s} = \textbf{20.2 litres/s. Ans.}$$

Applying Bernoulli's equation to points D and C

$$Z_D + \frac{p_D}{\rho g} = Z_C + h_{f_3}$$

or
$$36.482 = Z_C + \frac{4f \,.\, L_3 \,.\, V_3^2}{d_3 \times 2g}, \text{ where } V_3 = \frac{Q_3}{\frac{\pi}{4} d_3^2}$$

But from continuity $Q_1 + Q_2 = Q_3$

$\therefore$
$$Q_3 = Q_1 + Q_2 = 0.06 + 0.0202 = 0.0802 \text{ m}^3\text{/s}$$

$\therefore$
$$V_3 = \frac{Q_3}{\frac{\pi}{4}(.3)^2} = \frac{0.0802}{\frac{\pi}{4}(.09)} = 1.134 \text{ m/s}$$

$\therefore$
$$36.482 = Z_C + \frac{4 \times .006 \times 800 \times 1.134^2}{0.3 \times 2 \times 9.81} = Z_C + 4.194$$

$\therefore$
$$Z_C = 36.482 - 4.194 = \mathbf{32.288 \text{ m.}} \textbf{ Ans.}$$

Problem 11.43 *Three reservoirs, A, B and C are connected by a pipe system shown in Fig. 11.28. The lengths and diameters of pipes 1, 2 and 3 are 800 m, 1000 m, 800 m, and 300 mm, 200 mm and 150 mm respectively. Determine the piezometric head at junction D. Take f = .005.*

Solution. Given :

The length of pipe 1, $L_1 = 800$ m and its dia., $d_1 = 300$ mm $= 0.3$ m

The length of pipe 2, $L_2 = 1000$ m and its dia., $d_2 = 200$ mm $= 0.2$ m

The length of pipe 3, $L_3 = 800$ m and its dia., $d_3 = 150$ mm $= 0.15$ m

Height of reservoir, A from datum line, $Z_A = 60$ m

Similarly, $Z_B = 40$ m and $Z_C = 30$ m.

The direction of flow in pipes are shown (given) in Fig. 11.28. Applying Bernoulli's equation to points A and D

$$Z_A = \left(Z_D + \frac{p_D}{\rho g}\right) + h_{f_1}$$

or
$$\left[Z_A - \left(Z_D + \frac{p_D}{\rho g}\right)\right] = h_{f_1} = \frac{4 \times f \times L_1 \times V_1^2}{d_1 \times 2g} = \frac{4 \times .005 \times 800 \times V_1^2}{0.3 \times 2 \times 9.81}$$

or
$$60 - \left(Z_D + \frac{p_D}{\rho g}\right) = 2.718\ V_1^2 \qquad \text{...}(i)$$

Applying Bernoulli's equation to points D and B

$$\left(Z_D + \frac{p_D}{\rho g}\right) = Z_B + h_{f_2} = 40 + \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g}$$

$$= 40 + \frac{4 \times .005 \times 1000 \times V_2^2}{0.2 \times 2 \times 9.81} = 40.0 + 5.09\ V_2^2$$

or
$$\left(Z_D + \frac{p_D}{\rho g}\right) - 40.0 = 5.09\ V_2^2 \qquad \text{...}(ii)$$

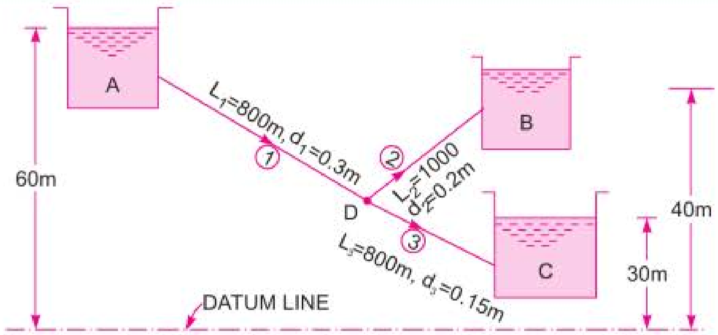


Fig. 11.28

Applying Bernoulli's equation to points D and C

$$\left(Z_D + \frac{p_D}{\rho g}\right) = Z_C + h_{f_3} = 30 + \frac{4f \times L_3 \times V_3^2}{d_3 \times 2g} = 30 + \frac{4 \times .005 \times 800 \times V_3^2}{0.15 \times 2 \times 9.81}$$

or $$\left(Z_D + \frac{p_D}{\rho g}\right) = 30.0 + 5.436\ V_3^2 \quad \text{...(iii)}$$

Adding (*i*) and (*ii*), we have $60 - 40 = 2.718\ V_1^2 + 5.09\ V_2^2$

or $$20 = 2.718\ V_1^2 + 5.09\ V_2^2 \quad \text{...(iv)}$$

Adding (*i*) and (*iii*), we have $60 = 2.718\ V_1^2 + 30.0 + 5.436\ V_3^2$

or $$60 - 30 = 30 = 2.718\ V_1^2 + 5.436\ V_3^2 \quad \text{...(v)}$$

Also from continuity equation, we have

$$Q_1 = Q_2 + Q_3$$

or $$\frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} d_2^2 V_2 + \frac{\pi}{4} d_3^2 V_3 \quad \text{or} \quad d_1^2 V_1 = d_2^2 V_2 + d_3^2 V_3$$

or $$0.3^2 V_1 = 0.2^2 V_2 + 0.15^2 \times V_3 \quad \text{or} \quad .09 V_1 = .04\ V_2 + .0225\ V_3 \quad \text{...(vi)}$$

Now from (*iv*), $$V_2 = \sqrt{\frac{20 - 2.718\ V_1^2}{5.09}} \quad \text{...(vii)}$$

And from (*v*), $$V_3 = \sqrt{\frac{30 - 2.718\ V_1^2}{5.436}} \quad \text{...(viii)}$$

Substituting the value of V_2 and V_3 in (*vi*), we get

$$0.09\ V_1 = .04 \sqrt{\frac{20 - 2.718\ V_1^2}{5.09}} + .0225 \sqrt{\frac{30 - 2.718\ V_1^2}{5.436}}$$

Squaring both sides, we get

$$(0.09\ V_1)^2 = (.04)^2 \times \left(\frac{20 - 2.718\ V_1^2}{5.09}\right) + (0.0225)^2 \times \frac{30 - 2.718\ V_1^2}{5.436} + 2 \times .04$$

$$\times .0225 \times \sqrt{\frac{20 - 2.718\ V_1^2}{5.09}} \times \sqrt{\frac{30 - 2.718\ V_1^2}{5.436}}$$

or $\quad .0081\ V_1^2 = .00628 - .000854\ V_1^2 + .00279 - .000253\ V_1^2 + .0018$

or $\quad .0081\ V_1^2 + .000854\ V_1^2 + .000253\ V_1^2 = .00628 + .00279 + .0018 = .01087$

or $\quad .009207\ V_1^2 = .01087$

$$\therefore \qquad V_1 = \sqrt{\frac{.01087}{.009207}} = 1.086 \text{ m/s}$$

Substituting this value of V_1 in (*vii*) and (*viii*)

$$V_2 = \sqrt{\frac{20 - 2.718 \times V_1^2}{5.09}} = \sqrt{\frac{20 - 2.718 \times 1.086^2}{5.09}} = 1.816 \text{ m/s}$$

$$\therefore \qquad V_3 = \sqrt{\frac{30 - 2.718 \times 1.086^2}{5.436}} = 2.22 \text{ m/s}$$

Piezometric head at $\quad D = Z_D + \dfrac{p_D}{\rho g} = 30.0 + 5.436 \times V_3^2$

$= 30.0 + 5.436 \times (2.22)^2 =$ **56.79 m. Ans.**

Problem 11.44 *A pipe line 60 cm diameter bifurcates at a Y-junction into two branches 40 cm and 30 cm in diameter. If the rate of flow in the main pipe is 1.5 m³/s and mean velocity of flow in 30 cm diameter pipe is 7.5 m/s, determine the rate of flow in the 40 cm diameter pipe.*

Solution. Given :

Dia. of main pipe, $\quad D = 60$ cm $= 0.6$ m

Dia. of branch pipe 1, $\quad D_1 = 40$ cm $= 0.4$ m

Dia. of branch pipe 2, $\quad D_2 = 30$ cm $= 0.3$ m

Velocity in branch pipe 2, $\quad V_2 = 7.5$ m/s

Rate of flow in main pipe, $\quad Q = 1.5\ \text{m}^3/\text{s}$

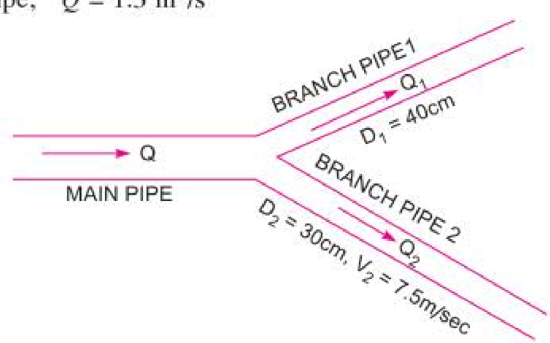


Fig. 11.29

Let $\quad Q_1$ = Rate of flow in branch pipe 1,

Q_2 = Rate of flow in branch pipe 2,

Q = Rate of flow in main pipe,

Now rate of flow in main pipe is equal to the sum of rate of flow in branch pipes.

$$\therefore \qquad Q = Q_1 + Q_2 \qquad \text{...}(i)$$

But Q_2 = Area of branch pipe 2 × Velocity in branch pipe 2

$$= A_2 \times V_2 = \frac{\pi}{4} D_2^2 \times V_2 = \frac{\pi}{4} (0.3)^2 \times 7.5 = 0.53 \text{ m}^3/\text{s}$$

Substituting the values of Q and Q_2 in equation (*i*), we get

$$1.5 = Q_1 + 0.53$$

$$\therefore \quad Q_1 = 1.5 - 0.53 = \mathbf{0.97 \text{ m}^3/\text{s}}. \text{ Ans.}$$

11.11 POWER TRANSMISSION THROUGH PIPES

Power is transmitted through pipes by flowing water or other liquids flowing through them. The power transmitted depends upon (*i*) the weight of liquid flowing through the pipe and (*ii*) the total head available at the end of the pipe. Consider a pipe AB connected to a tank as shown in Fig. 11.30. The power available at the end B of the pipe and the condition for maximum transmission of power will be obtained as mentioned below :

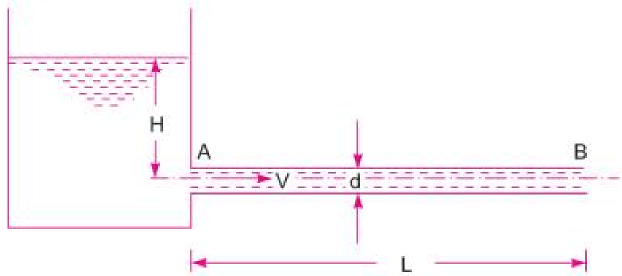


Fig. 11.30 *Power transmission through pipe.*

Let L = length of the pipe,
d = diameter of the pipe,
H = total head available at the inlet of pipe,
V = velocity of flow in pipe,
h_f = loss of head due to friction, and f = co-efficient of friction.

The head available at the outlet of the pipe, if minor losses are neglected

= Total head at inlet – loss of head due to friction

$$= H - h_f = H - \frac{4f \times L \times V^2}{d \times 2g} \qquad \left\{\because\ h_f = \frac{4f \times L \times V^2}{d \times 2g}\right\}$$

Weight of water flowing through pipe per sec,

$$W = \rho g \times \text{volume of water per sec} = \rho g \times \text{Area} \times \text{Velocity}$$

$$= \rho g \times \frac{\pi}{4} d^2 \times V$$

$\therefore$ The power transmitted at the outlet of the pipe

= weight of water per sec × head at outlet

$$= \left(\rho g \times \frac{\pi}{4} d^2 \times V\right) \times \left(H - \frac{4f \times L \times V^2}{d \times 2g}\right) \text{ Watts}$$

$\therefore$ Power transmitted at outlet of the pipe,

$$P = \frac{\rho g}{1000} \times \frac{\pi}{4} d^2 \times V \left(H - \frac{4fLV^2}{d \times 2g}\right) \text{ kW} \qquad ...(11.21)$$

Efficiency of power transmission,

$$\eta = \frac{\text{Power available at outlet of the pipe}}{\text{Power supplied at the inlet of the pipe}}$$

$$= \frac{\text{Weight of water per sec} \times \text{Head available at outlet}}{\text{Weight of water per sec} \times \text{Head at inlet}}$$

$$= \frac{W \times (H - h_f)}{W \times H} = \frac{H - h_f}{H}. \qquad \text{...(11.22)}$$

11.11.1 Condition for Maximum Transmission of Power.

The condition for maximum transmission of power is obtained by differentiating equation (11.21) with respect to V and equating the same to zero.

Thus
$$\frac{d}{dV}(P) = 0$$

or
$$\frac{d}{dV}\left[\frac{\rho g}{1000} \times \frac{\pi}{4} d^2 \left(HV - \frac{4fLV^3}{d \times 2g}\right)\right] = 0$$

or
$$\frac{\rho g}{1000} \times \frac{\pi}{4} d^2 \left(H - \frac{4 \times 3 \times f \times L \times V^2}{d \times 2g}\right) = 0$$

or
$$H - 3 \times \frac{4fLV^2}{d \times 2g} = 0 \text{ or } H - 3 \times h_f = 0 \qquad \left(\because \frac{4fLV^2}{d \times 2g} = h_f\right)$$

$\therefore$
$$H = 3h_f \text{ or } h_f = \frac{H}{3} \qquad \text{...(11.23)}$$

Equating (11.23) is the condition for maximum transmission of power. It states that power transmitted through a pipe is maximum when the loss of head due to friction is one-third of the total head at inlet.

11.11.2 Maximum Efficiency of Transmission of Power.

Efficiency of power transmission through pipe is given by equation (11.22) as

$$\eta = \frac{H - h_f}{H}$$

For maximum power transmission through pipe the condition is given by equation (11.23) as

$$h_f = \frac{H}{3}$$

Substituting the value of h_f in efficiency, we get maximum η,

$$\eta_{max} = \frac{H - H/3}{H} = 1 - \frac{1}{3} = \frac{2}{3} \text{ or } 66.7\%. \qquad \text{...(11.24)}$$

Problem 11.45 *A pipe of diameter 300 mm and length 3500 m is used for the transmission of power by water. The total head at the inlet of the pipe is 500 m. Find the maximum power available at the outlet of the pipe, if the value of f = .006.*

Solution. Given :

Diameter of the pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

Length of the pipe, $L = 3500$ m
Total head at inlet, $H = 500$ m
Co-efficient of friction, $f = .006$
For maximum power transmission, using equation (11.23)

$$h_f = \frac{H}{3} = \frac{500}{3} = 166.7 \text{ m}$$

Now
$$h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .006 \times 3500 \times V^2}{0.3 \times 2 \times 9.81} = 14.27\ V^2$$

Equating the two values of h_f, we get

$$166.7 = 14.27\ V^2 \text{ or } V = \sqrt{\frac{166.7}{14.27}} = 3.417 \text{ m/s}$$

$\therefore$ Discharge, $Q = V \times \text{Area}$

$$= 3.417 \times \frac{\pi}{4}(d)^2 = 3.417 \times \frac{\pi}{4}(.3)^2 = 0.2415 \text{ m}^3\text{/s}$$

Head available at the end of the pipe

$$= H - h_f = H - \frac{H}{3} = \frac{2H}{3} = \frac{2 \times 500}{3} = 333.33 \text{ m}$$

$$\therefore \text{ Maximum power available} = \frac{\rho g \times Q \times \text{head at the end of pipe}}{1000} \text{ kW}$$

$$= \frac{1000 \times 9.81 \times .2415 \times 333.33}{1000} \text{ kW} = \mathbf{689.7\ kW.}\ \textbf{Ans.}$$

Problem 11.46 *A pipe line of length 2000 m is used for power transmission. If 110.3625 kW power is to be transmitted through the pipe in which water having a pressure of 490.5 N/cm² at inlet is flowing. Find the diameter of the pipe and efficiency of transmission if the pressure drop over the length of pipe is 98.1 N/cm². Take f = .0065.*

Solution. Given :
Length of pipe, $L = 2000$ m
Power transmitted $= 110.3625$ kW
Pressure at inlet, $p = 490.5 \text{ N/cm}^2 = 490.5 \times 10^4 \text{ N/m}^2$

$\therefore$ Pressure head at inlet, $H = \dfrac{p}{\rho g} = \dfrac{490.5 \times 10^4}{1000 \times 9.81} = 500$ m [$\because \rho = 1000$]

Pressure drop $= 98.1 \text{ N/cm}^2 = 98.1 \times 10^4 \text{ N/m}^2$

$\therefore$ Loss of head, $h_f = \dfrac{98.1 \times 10^4}{\rho g} = \dfrac{98.1 \times 10^4}{1000 \times 9.81} = 100$ m

Co-efficient of friction, $f = .0065$
Head available at the end of the pipe $= H - h_f = 500 - 100 = 400$ m
Let the diameter of the pipe $= d$

Now power transmitted is given by, $P = \dfrac{\rho g \times Q \times \left(H - h_f\right)}{1000}$ kW

or
$$110.3625 = \frac{1000 \times 9.81 \times Q \times 400}{1000}$$

$\therefore$
$$Q = \frac{110.3625 \times 1000}{1000 \times 9.81 \times 400} = 0.02812$$

But discharge, $Q = \text{Area} \times \text{Velocity} = \dfrac{\pi}{4} d^2 \times V$

$\therefore$
$$\frac{\pi}{4} d^2 \times V = .02812$$

$\therefore$
$$V = \frac{.2812 \times 4}{\pi d^2} = \frac{0.0358}{d^2} \quad \text{...}(i)$$

The head lost due to friction, $h_f = \dfrac{4f \times L \times V^2}{d \times 2g}$

But $h_f = 100$ m

$\therefore$
$$100 = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .0065 \times 2000 \times V^2}{d \times 2 \times 9.81}$$

$$= \frac{2.65 \times V^2}{d} = \frac{2.65}{d} \times \left(\frac{.0358}{d^2}\right)^2 = \frac{.003396}{d^5}$$

$\because$ From equation (i), $V = \dfrac{.0358}{d^2}$

$\therefore$
$$100 = \frac{.003396}{d^5}$$

or
$$d = \left(\frac{.003396}{100}\right)^{1/5} = 0.1277 \text{ m} = \textbf{127.7 mm. Ans.}$$

Efficiency of power transmission is given by equation (11.22),

$$\eta = \frac{H - h_f}{H} = \frac{500 - 100}{500} = \mathbf{0.80 = 80\%}. \textbf{ Ans.}$$

Problem 11.47 *For Problem 11.46, find : (i) the diameter of the pipe corresponding to maximum efficiency of transmission, (ii) diameter of the pipe corresponding to 90% efficiency of transmission.*

Solution. (i) Diameter of pipe corresponding to maximum efficiency.

Let the dia. of pipe for $\eta_{max} = d$

But from equation (11.24), $\eta_{max} = 66.67\% = \dfrac{2}{3}$

or
$$\frac{H - h_f}{H} = \frac{2}{3} \text{ or } \frac{500 - h_f}{500} = \frac{2}{3}$$

or $$h_f = 500 - 500 \times \frac{2}{3} = \frac{1500 - 1000}{3} = \frac{500}{3} = 166.7 \text{ m}$$

The other data given from Problem 11.46,

Power transmitted $= 110.3625$

Length of pipe, $L = 2000$ m

Co-efficient of friction, $f = .0065$

Power transmitted is given by the relation,

$$P = \frac{\rho g \times Q \times (H - h_f)}{1000}$$

or $$110.3625 = \frac{1000 \times 9.81 \times Q \times (500 - 166.7)}{1000}$$

or $$Q = \frac{110.3625 \times 1000}{1000 \times 9.81 \times (500 - 166.7)} = 0.03375 \text{ m}^3/\text{s}$$

But $$Q = \text{area of pipe} \times \text{velocity of flow}$$
$$= \frac{\pi}{4} d^2 \times V \text{ \{where } V = \text{velocity of flow\}}$$

$\therefore$ $$0.03375 = \frac{\pi}{4} d^2 \times V$$

$\therefore$ $$V = \frac{0.03375 \times 4}{\pi \times d^2} = \frac{0.04297}{d^2} \quad \text{...}(i)$$

Now the head lost due to friction, $h_f = \dfrac{4fLV^2}{d \times 2g}$

But $$h_f = 166.7 \text{ m}$$

$\therefore$ $$166.7 = \frac{4 \times .0065 \times 2000 \times V^2}{d \times 2 \times 9.81}$$
$$= \frac{2.65\, V^2}{d} = \frac{2.65}{d} \times \left(\frac{.04297}{d^2}\right)^2 = \frac{.00489}{d^5} \qquad \left(\because V = \frac{.04297}{d^2}\right)$$

$\therefore$ $$d^5 = \frac{.00489}{166.7} = .00002933$$

$\therefore$ $$d = (.00002933)^{1/5} = 0.1240 \text{ m} = \mathbf{124 \text{ mm.}}$$ **Ans.**

(*ii*) Let the diameter of pipe, when efficiency of transmission is 90% = d
$$\eta = 90\% = 0.9$$

But η is given by equation (11.22) as, $\eta = \dfrac{H - h_f}{H} = 0.9$

But $$H = 500 \text{ m}$$

$$\therefore \quad \frac{500 - h_f}{500} = 0.9 \text{ or } 500 - 500 \times 0.9 = h_f \text{ or } 500 - 450 = h_f$$

$$\therefore \quad h_f = 500 - 450 = 50 \text{ m}$$

The other given data is, $P = 110.3625$, $L = 2000$, $f = .0065$

Using relation for power transmission, $P = \dfrac{\rho g \times Q \times (H - h_f)}{1000}$

or
$$110.3625 = \frac{1000 \times 9.81 \times Q \times (500 - 50)}{1000}$$

$$Q = \frac{110.3625 \times 1000}{1000 \times 9.81 \times (500 - 50)} = .025 \text{ m}^3\text{/s}$$

But
$$Q = \frac{\pi}{4} d^2 \times V$$

$$\therefore \quad \frac{\pi}{4} d^2 \times V = .025 \text{ or } V = \frac{.025 \times 4}{\pi d^2} = \frac{0.03183}{d^2} \quad \text{...}(i)$$

Now the head lost due to friction, $h_f = \dfrac{4fLV^2}{d \times 2g}$

or
$$50 = \frac{4 \times .0065 \times 2000 \times}{d \times 2g} \times \left(\frac{.03183}{d^2}\right)^2 = \frac{.002685}{d^5}$$

$$\therefore \quad d^5 = \frac{.002685}{50} = .0000537$$

$$d = (.0000537)^{1/5} = .1399 \text{ m} \simeq \mathbf{140 \text{ mm.}}$$ **Ans.**

▶ 11.12 FLOW THROUGH NOZZLES

Fig. 11.31 shows a nozzle fitted at the end of a long pipe. The total energy at the end of the pipe consists of pressure energy and kinetic energy. By fitting the nozzle at the end of the pipe, the total energy is converted into kinetic energy. Thus nozzles are used, where higher velocities of flow are required. The examples are :

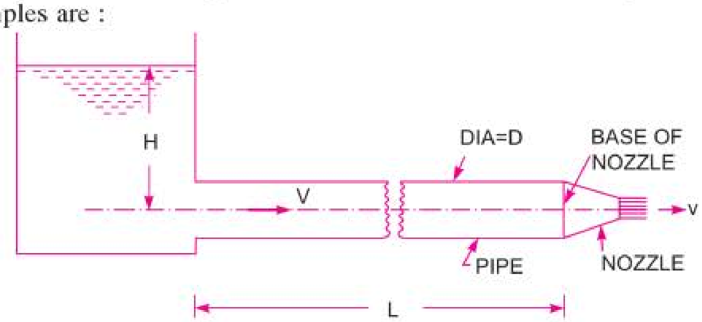


Fig. 11.31 *Nozzle fitted to a pipe.*

1. In case of Pelton turbine, the nozzle is fitted at the end of the pipe (called penstock) to increase velocity.
2. In case of the extinguishing fire, a nozzle is fitted at the end of the hose pipe to increase velocity.

Let D = diameter of the pipe, L = length of the pipe,

A = area of the pipe = $\frac{\pi}{4} D^2$,

V = velocity of flow in pipe,

H = total head at the inlet of the pipe,

d = diameter of nozzle at outlet,

v = velocity of flow at outlet of nozzle,

a = area of the nozzle at outlet = $\frac{\pi}{4} d^2$,

f = co-efficient of friction for pipe.

Loss of head due to friction in pipe, $h_f = \frac{4fLV^2}{2g \times D}$

∴ Head available at the end of the pipe or at the base of nozzle

= Head at inlet of pipe – head lost due to friction

$$= H - h_f = \left(H - \frac{4fLV^2}{2g \times D} \right)$$

Neglecting minor losses and also assuming losses in the nozzle negligible, we have

Total head at inlet of pipe = total head (energy) at the outlet of nozzle + losses

But total head at outlet of nozzle = kinetic head = $\frac{v^2}{2g}$

$$\therefore \quad H = \frac{v^2}{2g} + h_f = \frac{v^2}{2g} + \frac{4fLV^2}{2gD} \quad \left(\because h_f = \frac{4fLV^2}{2gD} \right) \quad ...(i)$$

From continuity equation in the pipe and outlet of nozzle,

$$AV = av$$

$$\therefore \quad V = \frac{av}{A}$$

Substituting this value in equation (i), we get

$$H = \frac{v^2}{2g} + \frac{4fL}{2gD} \times \left(\frac{av}{A} \right)^2 = \frac{v^2}{2g} + \frac{4fLa^2v^2}{2g \times D \times A^2} = \frac{v^2}{2g} \left(1 + \frac{4fLa^2}{DA^2} \right)$$

$$\therefore \quad v = \sqrt{\frac{2gH}{\left(1 + \frac{4fL}{D} \times \frac{a^2}{A^2} \right)}} \quad ...(11.25)$$

∴ Discharge through nozzle = $a \times v$.

11.12.1 Power Transmitted Through Nozzle. The kinetic energy of the jet at the outlet of nozzle $= \frac{1}{2} mv^2$

Now mass of liquid at the outlet of nozzle per second $= \rho av$

$\therefore$ Kinetic energy of the jet at the outlet per sec. $= \frac{1}{2} \rho av \times v^2 = \frac{1}{2} \rho av^3$

$\therefore$ Power in kW at the outlet of nozzle $= (\text{K.E./sec}) \times \frac{1}{1000} = \dfrac{\frac{1}{2}\rho av^3}{1000}$

$\therefore$ Efficiency of power transmission through nozzle,

$$\eta = \frac{\text{Power at outlet of nozzele}}{\text{Power at the inlet of pipe}} = \frac{\dfrac{\frac{1}{2}\rho av^3}{1000}}{\dfrac{\rho g . Q . H}{1000}}$$

$$= \frac{\frac{1}{2}\rho av . v^2}{\rho g . Q . H} = \frac{\frac{1}{2}\rho av . v^2}{\rho g . av . H} \qquad \{\because Q = av\}$$

$$= \frac{v^2}{2gH} = \left[\frac{1}{1 + \frac{4fL}{D} \times \frac{a^2}{A^2}}\right] \qquad ...(11.26)$$

$$\left(\because \text{From equation (11.25), } \frac{v^2}{2gH} = \left[\frac{1}{1 + \frac{4fL}{D}\frac{a^2}{A^2}}\right]\right)$$

11.12.2 Condition for Maximum Power Transmitted Through Nozzle. We know that, the total head at inlet of pipe = total head at the outlet of the nozzle + losses

i.e.,
$$H = \frac{v^2}{2g} + h_f \qquad \left[\begin{array}{l}\because \text{total head at outlet of nozzle} = \dfrac{v^2}{2g} \text{ and} \\ h_f = \dfrac{4 . f . L . V^2}{D \times 2g} = \text{loss of liquid in pipe}\end{array}\right]$$

$$= \frac{v^2}{2g} + \frac{4 . f . L . V^2}{D \times 2g}$$

$$\therefore \qquad \frac{v^2}{2g} = \left(H - \frac{4 . f . L . V^2}{D \times 2g}\right)$$

But power transmitted through nozzle $= \dfrac{\frac{1}{2}\rho av^3}{1000} = \dfrac{\frac{1}{2}\rho av}{1000} \times v^2 = \dfrac{\frac{1}{2}\rho av}{1000}\left[2g\left(H - \dfrac{4 . f . L . V^2}{D \times 2g}\right)\right]$

$$= \frac{\rho gav}{1000}\left[H - \frac{4fLV^2}{D \times 2g}\right] \quad ...(11.27)$$

Now from continuity equation, $AV = av$

$$\therefore \quad V = \frac{av}{A}$$

Substituting the value of V in equation (11.27), we get

$$\text{Power transmitted through nozzle} = \frac{\rho gav}{1000}\left[H - \frac{4fLa^2}{D \times 2g}\frac{v^2}{A^2}\right]$$

The power (P) will be maximum, when $\frac{d(P)}{dv} = 0$

or
$$\frac{d}{dv}\left[\frac{\rho gav}{1000}\left(H - \frac{4fL}{D \times 2g}\frac{a^2v^2}{A^2}\right)\right] = 0$$

or
$$\frac{d}{dv}\left[\frac{\rho ga}{1000}\left(Hv - \frac{4fL}{D \times 2g}\frac{a^2v^3}{A^2}\right)\right] = 0$$

or
$$\left[\frac{\rho ga}{1000}\left(H - 3\frac{4fL}{D \times 2g}\frac{a^2v^2}{A^2}\right)\right] = 0 \text{ or } H - 3 \times \frac{4fL}{D \times 2g} \times V^2 = 0 \quad \left(\because V = \frac{av}{A}\right)$$

or
$$H - 3h_f = 0 \quad \left(\because \frac{4fLV^2}{D \times 2g} = h_f = \text{head loss in pipe}\right)$$

or
$$h_f = \frac{H}{3} \quad ...(11.28)$$

Equation (11.28) gives the condition for maximum power transmitted through nozzle. It states that power transmitted through nozzle is maximum when the head lost due to friction in pipe is one-third the total head supplied at the inlet of pipe.

11.12.3 Diameter of Nozzle for Maximum Transmission of Power Through Nozzle. For maximum transmission of power, the condition is given by equation (11.28) as, $h_f = \frac{H}{3}$

But
$$h_f = \frac{4f \,.\, L \,.\, V^2}{D \times 2g}$$

$$\therefore \quad \frac{4fLV^2}{D \times 2g} = \frac{H}{3} \text{ or } H = 3 \times \frac{4fLV^2}{D \times 2g}$$

But H is also = total head at outlet of nozzle + losses

$$= \frac{v^2}{2g} + h_f = \frac{v^2}{2g} + \frac{4fLV^2}{D \times 2g}$$

Equating the two values of H, we get

$$3 \times \frac{4fLV^2}{D \times 2g} = \frac{v^2}{2g} + \frac{4fLV^2}{D \times 2g} \quad \text{or} \quad \frac{12fLV^2}{D \times 2g} - \frac{4fLV^2}{D \times 2g} = \frac{v^2}{2g}$$

or
$$\frac{8fLV^2}{D \times 2g} = \frac{v^2}{2g} \qquad \text{...}(i)$$

But from continuity, $AV = av$ or $V = \dfrac{av}{A}$.

Substituting this value of V in equation (i), we get

$$\frac{8fL}{D \times 2g} \times \frac{a^2v^2}{A^2} = \frac{v^2}{2g} \quad \text{or} \quad \frac{8fL}{D} \times \frac{a^2}{A^2} = 1 \qquad \left(\text{Divide by } \frac{v^2}{2g}\right) \text{...}(ii)$$

or
$$\frac{8fL}{D} \times \frac{\left(\frac{\pi}{4}d^2\right)^2}{\left(\frac{\pi}{4}D^2\right)^2} = 1 \quad \text{or} \quad \frac{8fL}{D} \times \frac{d^4}{D^4} = 1 \quad \text{or} \quad d^4 = \frac{D^5}{8fL}$$

$\therefore$
$$d = \left(\frac{D^5}{8fL}\right)^{1/4} \qquad \text{...(11.29)}$$

From equation (ii), $\dfrac{8fL}{D} = \dfrac{A^2}{a^2}$

$\therefore$
$$\frac{A}{a} = \sqrt{\frac{8fL}{D}} \qquad \text{...(11.30)}$$

Equation (11.30) gives the ratio of the area of the supply pipe to the area of the nozzle and hence from this equation, the diameter of the nozzle can be obtained.

Problem 11.48 *A nozzle is fitted at the end of a pipe of length 300 m and of diameter 100 mm. For the maximum transmission of power through the nozzle, find the diameter of nozzle. Take f = .009.*

Solution. Given :

Length of pipe, $L = 300$ m

Diameter of pipe, $D = 100$ mm $= 0.1$ m

Co-efficient of friction, $f = .009$

Let the diameter of nozzle $= d$

For maximum transmission of power, the diameter of nozzle is given by relation (11.29) as

$$d = \left(\frac{D^5}{8fL}\right)^{1/4} = \left(\frac{0.1^5}{8 \times .009 \times 300}\right)^{1/4} = 0.02608 \text{ m} = \mathbf{26.08 \text{ mm. Ans.}}$$

Problem 11.49 *The head of water at the inlet of a pipe 2000 m long and 500 mm diameter is 60 m. A nozzle of diameter 100 mm at its outlet is fitted to the pipe. Find the velocity of water at the outlet of the nozzle if f = .01 for the pipe.*

Solution. Given :

Head of water at inlet of pipe, $H = 60$ m

Length of pipe, $L = 2000$ m
Dia. of pipe, $D = 500$ mm $= 0.50$ m
Dia. of nozzle at outlet, $d = 100$ mm $= 0.1$ m
Co-efficient of friction, $f = .01$

The velocity at outlet of nozzle is given by equation (11.25) as

$$v = \sqrt{\frac{2gH}{\left(1 + \frac{4fL}{D} \times \frac{a^2}{A^2}\right)}} = \sqrt{\frac{2 \times 9.81 \times 60}{1 + \frac{4 \times .01 \times 2000}{0.5}\left(\frac{\frac{\pi}{4}d^2}{\frac{\pi}{4}D^2}\right)^2}}$$

$$= \sqrt{\frac{2 \times 9.81 \times 60}{1 + \frac{4 \times .01 \times 2000}{0.5} \times \left(\frac{0.1 \times .1}{0.5 \times .5}\right)^2}} = \textbf{30.61 m/s. Ans.}$$

Problem 11.50 *Find the maximum power transmitted by a jet of water discharging freely out of nozzle fitted to a pipe = 300 m long and 100 mm diameter with co-efficient of friction as 0.01. The available head at the nozzle is 90 m.*

Solution. Given :

Length of pipe, $L = 300$ m
Dia. of pipe, $D = 100$ mm $= 0.1$ m
Co-efficient of friction, $f = .01$
Head available at nozzle, $= 90$ m

For maximum power transmission through the nozzle, the diameter at the outlet of nozzle is given by equation (11.29) as

$$d = \left(\frac{D^5}{8fL}\right)^{1/4} = \left[\frac{(0.1)^5}{8 \times .01 \times 300}\right]^{1/4} = .0254 \text{ m}$$

$\therefore$ Area at the nozzle, $a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.0254)^2 = .0005067 \text{ m}^2$.

The nozzle at the outlet, discharges water into atmosphere and hence the total head available at the nozzle is converted into kinetic head.

$\therefore$ Head available at outlet $= v^2/2g$ or $90 = v^2/2g$

$\therefore$ $v = \sqrt{2 \times 9.81 \times 90} = 42.02$ m/s

Discharge through nozzle, $Q = a \times v = .0005067 \times 42.02 = 0.02129 \text{ m}^3/\text{s}$

$$\therefore \text{ Maximum power transmitted} = \frac{\rho g \times Q \times \text{Head at outlet of nozzle}}{1000}$$

$$= \frac{1000 \times 9.81 \times 0.02129 \times 90}{1000} = \textbf{18.796 kW. Ans.}$$

Problem 11.51 *The rate of flow of water through a pipe of length 2000 m and diameter 1 m is 2 m^3/s. At the end of the pipe a nozzle of outside diameter 300 mm is fitted. Find the power transmitted* *through the nozzle if the head of water at inlet of the pipe is 200 m and co-efficient of friction for pipe is 0.01.*

Solution. Given :

Length of pipe,	$L = 2000$ m
Dia. of pipe,	$D = 1$ m
Discharge,	$Q = 2\ m^3/s$
Dia. of nozzle,	$d = 300$ mm = 0.3 m
Head at inlet of pipe,	$H = 200$ m
Co-efficient of friction,	$f = .01$

Now area of pipe, $A = \dfrac{\pi}{4} D^2 = \dfrac{\pi}{4} \times 1^2 = 0.7854\ m^2$

Velocity of water through pipe, $V = \dfrac{Q}{A} = \dfrac{2.0}{0.7854} = 2.546$ m/s

Power transmitted through nozzle is given by equation (11.27) as

$$P = \frac{\rho g \, . \, a \, . \, v}{1000}\left[H - \frac{4fLV^2}{D \times 2g}\right]$$

$$= \frac{1000 \times 9.81 \times 2.0}{1000}\left[200 - \frac{4 \times .01 \times 2000 \times (2.546)^2}{1 \times 2 \times 9.81}\right] \quad (\because av = Q)$$

$$= \mathbf{3405.43\ kW.}\ \textbf{Ans.}$$

▶ 11.13 WATER HAMMER IN PIPES

Consider a long pipe *AB* as shown in Fig. 11.32 connected at one end to a tank containing water at a height of *H* from the centre of the pipe. At the other end of the pipe, a valve to regulate the flow of water is provided. When the valve is completely open, the water is flowing with a velocity, *V* in the pipe. If now the valve is suddenly closed, the momentum of the flowing water will be destroyed and consequently a wave of high pressure will be set up. This wave of high pressure will be transmitted along the pipe with a velocity equal to the velocity of sound wave and may create noise called knocking. Also this wave of high pressure has the effect of hammering action on the walls of the pipe and hence it is also known as water hammer.

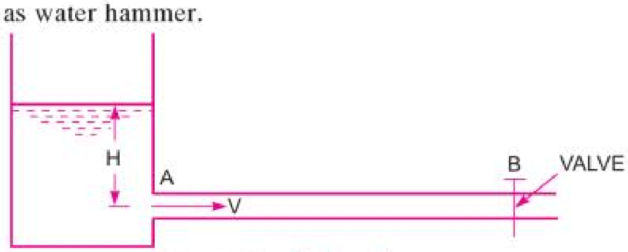


Fig. 11.32 *Water hammer.*

The pressure rise due to water hammer depends upon : (*i*) the velocity of flow of water in pipe, (*ii*) the length of pipe, (*iii*) time taken to close the valve, (*iv*) elastic properties of the material of the pipe. The following cases of water hammer in pipes will be considered :

1. Gradual closure of valve,
2. Sudden closure of valve and considering pipe rigid, and

3. Sudden closure of valve and considering pipe elastic.

11.13.1 Gradual Closure of Valve. Let the water is flowing through the pipe AB shown in Fig. 11.32, and the valve provided at the end of the pipe is closed gradually.

Let A = area of cross-section of the pipe AB,
L = length of pipe,
V = velocity of flow of water through pipe,
T = time in second required to close the valve, and
p = intensity of pressure wave produced.

Mass of water in pipe $AB = \rho \times$ volume of water $= \rho \times A \times L$

The valve is closed gradually in time 'T' seconds and hence the water is brought from initial velocity V to zero velocity in time seconds.

$\therefore$ Retardation of water $= \dfrac{\text{Change of velocity}}{\text{Time}} = \dfrac{V-0}{T} = \dfrac{V}{T}$

$\therefore$ Retarding force $= \text{Mass} \times \text{Retardation} = \rho AL \times \dfrac{V}{T}$ ...(*i*)

If p is the intensity of pressure wave produced due to closure of the valve, the force due to pressure wave,

$= p \times \text{area of pipe} = p \times A$ ...(*ii*)

Equating the two forces, given by equations (*i*) and (*ii*),

$$\rho AL \times \frac{V}{T} = p \times A$$

$$\therefore \quad p = \frac{\rho LV}{T} \quad \text{...(11.31)}$$

Head of pressure, $H = \dfrac{p}{\rho g} = \dfrac{\rho LV}{\rho g \times T} = \dfrac{\rho LV}{\rho \times g \times T}$ or $H = \dfrac{LV}{gT}$ ...(11.32)

(*i*) The valve closure is said to be gradual if $T > \dfrac{2L}{C}$ ...(11.33)

where t = time in sec, C = velocity of pressure wave

(*ii*) The valve closure is said to be sudden if $T < \dfrac{2L}{C}$ ...(11.34)

where C = velocity of pressure wave.

11.13.2 Sudden Closure of Valve and Pipe is Rigid. Equation (11.31) gives the relation between increase of pressure due to water hammer in pipe and the time required to close the valve. If $t = 0$, the increase in pressure will be infinite. But from experiments, it is observed that the increase in pressure due to water hammer is finite, even for a very rapid closure of valve. Thus equation (11.31) is valid only for (*i*) incompressible fluids and (*ii*) when pipe is rigid. But when a wave of high pressure is created, the liquids get compressed to some extent and also pipe material gets stretched. For a sudden closure of valve [the valve of t is small and hence a wave of high pressure is created] the following two cases will be considered :

(*i*) Sudden closure of valve and pipe is rigid, and
(*ii*) Sudden closure of valve and pipe is elastic.

Consider a pipe AB in which water is flowing as shown in Fig. 11.32. Let the pipe is rigid and valve fitted at the end B is closed suddenly.

Let A = Area of cross-section of pipe AB,

L = Length of pipe,

V = Velocity of flow of water through pipe,

p = Intensity of pressure wave produced,

K = Bulk modulus of water.

When the valve is closed suddenly, the kinetic energy of the flowing water is converted into strain energy of water if the effect of friction is neglected and pipe wall is assumed perfectly rigid.

$$\therefore \quad \text{Loss of kinetic energy} = \frac{1}{2} \times \text{mass of water in pipe} \times V^2$$

$$= \frac{1}{2} \times \rho AL \times V^2 \qquad (\because \text{ mass} = \rho \times \text{volume} = \rho \times A \times L)$$

$$\text{Gain of strain energy} = \frac{1}{2}\left(\frac{p^2}{K}\right) \times \text{volume} = \frac{1}{2}\frac{p^2}{K} \times AL$$

Equating loss of kinetic energy to gain of strain energy

$$\therefore \quad \frac{1}{2}\rho AL \times V^2 = \frac{1}{2}\frac{p^2}{K} \times AL$$

or

$$p^2 = \frac{1}{2}\rho AL \times V^2 \times \frac{2K}{AL} = \rho K V^2$$

$$\therefore \quad p = \sqrt{\rho K V^2} = V\sqrt{K\rho} = V\sqrt{\frac{K\rho^2}{\rho}} \qquad \text{...(11.35)}$$

$$= \rho V \times C \qquad \left(\because \sqrt{K/\rho} = C\right) \text{...(11.36)}$$

where C = velocity* of pressure wave.

11.13.3 Sudden Closure of Valve and Pipe is Elastic. Consider the pipe AB in which water is flowing as shown in Fig. 11.32. Let the thickness 't' of the pipe wall is small compared to the diameter D of the pipe and also let the pipe is elastic.

Let E = Modulus of Elasticity of the pipe material,

$\frac{1}{m}$ = Poisson's ratio for pipe material,

p = Increase of pressure due to water hammer,

t = Thickness of the pipe wall,

D = Diameter of the pipe.

When the valve is closed suddenly, a wave of high pressure of intensity p will be produced in the water. Due to this high pressure p, circumferential and longitudinal stresses in the pipe wall will be produced.

Let f_l = Longitudinal stress in pipe

f_c = Circumferential stress in pipe,

The magnitude of these stresses are given as $f_l = \frac{pD}{4t}$ and $f_c = \frac{pD}{2t}$

Now from the knowledge of strength of material we know, strain energy stored in pipe material per unit volume

* For derivation of velocity of pressure wave, please refer to chapter 15.

$$= \frac{1}{2E}\left[f_t^2 + f_c^2 - \frac{2f_t \times f_c}{m}\right]$$

$$= \frac{1}{2E}\left[\left(\frac{pD}{4t}\right)^2 + \left(\frac{pD}{2t}\right)^2 - \frac{2 \times \frac{pD}{4t} \times \frac{pD}{2t}}{m}\right]$$

$$= \frac{1}{2E}\left[\frac{p^2D^2}{16t^2} + \frac{p^2D^2}{4t^2} - \frac{p^2D^2}{4mt^2}\right]$$

Taking $\frac{1}{m} = \frac{1}{4}$ $\left(i.e., \text{ Poisson ratio} = \frac{1}{4}\right)$

$\therefore$ Strain energy stored in pipe material per unit volume

$$= \frac{1}{2E}\left[\frac{p^2D^2}{16t^2} + \frac{p^2D^2}{4t^2} - \frac{p^2D^2}{4t^2 \times 4}\right] = \frac{1}{2E} \times \frac{p^2D^2}{4t^2} = \frac{p^2D^2}{8Et^2}$$

Total volume of pipe material = $\pi D \times t \times L$.

$\therefore$ Total strain energy stored in pipe material

= Strain energy per unit volume $\times$ total volume

$$= \frac{p^2D^2}{8Et^2} \times \pi D \times t \times L = \frac{p^2\pi D^3 L}{8Et}$$

$$= \frac{p^2 \times \pi D^2 \times DL}{8Et} = \frac{p^2 A \times DL}{2Et} \qquad \left(\because \frac{\pi D^2}{4} = \text{Area of pipe} = A\right)$$

Now loss of kinetic energy of water $= \frac{1}{2}\, mV^2 = \frac{1}{2}\, \rho AL \times V^2$ $\qquad (\because m = \rho AL)$

Gain of strain energy in water $= \frac{1}{2}\left(\frac{p^2}{K}\right) \times \text{volume} = \frac{1}{2}\frac{p^2}{K} \times AL$

Then, loss of kinetic energy of water = Gain of strain energy in water + Strain energy stored in pipe material.

$$\therefore \qquad \frac{1}{2}\,\rho AL \times V^2 = \frac{1}{2}\left(\frac{p^2}{K}\right) \times AL + \frac{p^2 A \times DL}{2Et}$$

Divide by AL,

$$\frac{\rho V^2}{2} = \frac{1}{2}\frac{p^2}{K} + \frac{p^2 D}{2Et} = \frac{p^2}{2}\left[\frac{1}{K} + \frac{D}{Et}\right] \text{ or } \rho V^2 = p^2\left[\frac{1}{K} + \frac{D}{Et}\right]$$

$$\therefore \qquad p^2 = \frac{\rho V^2}{\frac{1}{K} + \frac{D}{Et}} \text{ or } p = \sqrt{\frac{\rho V^2}{\frac{1}{K} + \frac{D}{Et}}} = V \times \sqrt{\frac{\rho}{\frac{1}{K} + \frac{D}{Et}}} \qquad \text{...(11.37)}$$

11.13.4 Time Taken by Pressure Wave to Travel from the Valve to the Tank and from Tank to the Valve

Let T = The required time taken by pressure wave

L = Length of the pipe

C = Velocity of pressure wave

Then total distance = $L + L = 2L$

$$\therefore \text{ Time, } \quad T = \frac{\text{Distance}}{\text{Velocity of pressure wave}} = \frac{2L}{C}. \qquad \text{...(11.38)}$$

Problem 11.52 *The water is flowing with a velocity of 1.5 m/s in a pipe of length 2500 m and of diameter 500 mm. At the end of the pipe, a valve is provided. Find the rise in pressure if the valve is closed in 25 seconds. Take the value of C = 1460 m/s.*

Solution. Given :

Velocity of water, $V = 1.5$ m/s

Length of pipe, $L = 2500$ m

Diameter of pipe, $D = 500$ mm $= 0.5$ m

Time to close the valve, $T = 25$ seconds

Value of, $C = 1460$ m/s

Let the rise in pressure $= p$

The ratio, $\dfrac{2L}{C} = \dfrac{2 \times 2500}{1460} = 3.42$

From equation (11.33), we have if $T > \dfrac{2L}{C}$, the closure of valve is said to be gradual.

Here $T = 25$ sec and $\dfrac{2L}{C} = 3.42$

$\therefore$ $T > \dfrac{2L}{C}$ and hence valve is closed gradually.

For gradually closure of valve, the rise in pressure is given by equation (11.31) as

$$p = \frac{\rho VL}{T} = 1000 \times 2500 \times \frac{1.5}{25} = 150000 \text{ N/m}^2$$

$$= \frac{150000}{10^4} \frac{\text{N}}{\text{cm}^2} = \mathbf{15.0} \ \frac{\mathbf{N}}{\mathbf{cm^2}}. \text{ Ans.}$$

Problem 11.53 *If in Problem 11.52, the valve is closed in 2 sec, find the rise in pressure behind the valve. Assume the pipe to be rigid one and take Bulk modulus of water.*
i.e., $K = 19.62 \times 10^4$ N/cm^2.

Solution. Given :

$V = 1.5$ m/s, $L = 2500$ m

$D = 500$ mm $= 0.5$ m

Time to close the valve, $T = 2$ sec

Bulk modulus of water, $K = 19.62 \times 10^4$ N/cm^2

$= 19.62 \times 10^4 \times 10^4$ N/m$^2 = 19.62 \times 10^8$ N/m^2

Velocity of pressure wave is given by,

$$C = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{19.62 \times 10^8}{1000}} = 1400 \text{ m/s} \qquad (\because \rho = 1000)$$

The ratio, $$\frac{2L}{C} = \frac{2 \times 2500}{1400} = 3.57 \qquad \therefore \quad T < \frac{2L}{C}.$$

$\therefore$ From equation (11.34), if $T < \dfrac{2L}{C}$, valve is closed suddenly. For sudden closure of valve, when pipe is rigid, the rise in pressure is given by equation (11.35) or (11.36) as

$$p = V\sqrt{K\rho} = 1.5\sqrt{19.62 \times 10^8 \times 1000} \qquad (\because \rho = 1000)$$

$$= 210.1 \times 10^4 \text{ N/m}^2 = \mathbf{210.1\ N/cm^2.}\ \textbf{Ans.}$$

Problem 11.54 *If in Problem 11.52, the thickness of the pipe is 10 mm and the valve is suddenly closed at the end of the pipe, find the rise in pressure if the pipe is considered to be elastic. Take $E = 19.62 \times 10^{10}$ N/m^2 for pipe material and $K = 19.62 \times 10^4$ N/cm^2 for water. Calculate the circumferential stress and longitudinal stress developed in the pipe wall.*

Solution. Given :

$V = 1.5$ m/s, $L = 2500$ m, $D = 0.5$ m

Thickness of pipe, $t = 10$ mm $= .01$ m

Modulus of elasticity, $E = 19.62 \times 10^{10}$ N/m^2

Bulk modulus, $K = 19.62 \times 10^4$ N/cm^2 $= 19.62 \times 10^8$ N/m^2

For sudden closure of the valve for an elastic pipe, the rise in pressure is given by equation (11.37) as

$$p = V \times \sqrt{\frac{\rho}{\left(\frac{1}{K} + \frac{D}{Et}\right)}} = 1.5 \times \sqrt{\frac{1000}{\left(\frac{1}{19.62 \times 10^8} + \frac{0.5}{19.62 \times 10^{10} \times .01}\right)}}$$

$$= 1.5 \times \sqrt{\frac{1000}{\left(5.09 \times 10^{-10} + 2.54 \times 10^{-10}\right)}}$$

$$= 1715510 \text{ N/m}^2 = \mathbf{171.55\ N/cm^2.}\ \textbf{Ans.}$$

Circumferential stress (f_c) is given by

$$= \frac{p \times D}{2t} = \frac{171.55 \times 0.5}{2 \times .01} = 4286.9 \text{ N/m}^2$$

Longitudinal stress is given by, $f_l = \dfrac{p \times D}{4t} = \dfrac{171.55 \times 0.5}{4 \times .01} = \mathbf{2143.45\ N/m^2.}$ **Ans.**

Problem 11.55 *A valve is provided at the end of a cast iron pipe of diameter 150 mm and of thickness 10 mm. The water is flowing through the pipe, which is suddenly stopped by closing the valve. Find the maximum velocity of water, when the rise of pressure due to sudden closure of valve is 196.2 N/cm^2. Take K for water as 19.62×10^4 N/cm^2 and E for cast iron pipe as 11.772×10^6 N/cm^2.*

Solution. Given :

Diameter of pipe, $D = 150$ mm $= 0.15$ m

Thickness of pipe, $t = 10$ mm $= .01$ m

Rise of pressure, $p = 196.2 \text{ N/cm}^2 = 196.2 \times 10^4 \text{ N/m}^2$

Bulk modulus, $K = 19.62 \times 10^4 \text{ N/cm}^2 = 19.62 \times 10^8 \text{ N/m}^2$

Modulus of elasticity, $E = 11.772 \times 10^6 \text{ N/cm}^2 = 11.772 \times 10^{10} \text{ N/m}^2$

For sudden closure of valve and when pipe is elastic, the pressure rise is given by equation (11.37) as

$$p = V \times \sqrt{\frac{\rho}{\left(\frac{1}{K} + \frac{D}{Et}\right)}} = V \times \sqrt{\frac{1000}{\left(\frac{1}{19.62 \times 10^8} + \frac{0.15}{11.772 \times 10^{10} \times .01}\right)}}$$

or
$$196.2 \times 10^4 = V \times \sqrt{\frac{1000}{5.09 \times 10^{-10} + 1.274 \times 10^{-10}}}$$

$$= V \times \sqrt{\frac{1000}{6.364 \times 10^{-10}}} = V \times 125.27 \times 10^4$$

$$\therefore \quad V = \frac{196.2 \times 10^4}{125.27 \times 10^4} = 1.566 \text{ m/s}$$

$\therefore$ Maximum velocity $=$ **1.566 m/s. Ans.**

▶ 11.14 PIPE NETWORK

A pipe network is an interconnected system of pipes forming several loops or circuits. The pipe network is shown in Fig. 11.33. The examples of such networks of pipes are the municipal water distribution systems in cities and laboratory supply system. In such system, it is required to determine the distribution of flow through the various pipes of the network. The following are the necessary conditions for any network of pipes :

(*i*) The flow into each junction must be equal to the flow out of the junction. This is due to continuity equation.

(*ii*) The algebraic sum of head losses round each loop must be zero. This means that in each loop, the loss of head due to flow in clockwise direction must be equal to the loss of head due to flow in anticlockwise direction.

(*iii*) The head loss in each pipe is expressed as $h_f = rQ^n$. The value of r depends upon the length of pipe, diameter of pipe and co-efficient of friction of pipe. The value of n for turbulent flow is 2. We know that,

$$h_f = \frac{4 \times f \times L \times V^2}{D \times 2g} = \frac{4fL \times \left(\frac{Q}{A}\right)^2}{D \times 2g} \qquad \left(\because V = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}D^2}\right)$$

$$= \frac{4fL \times Q^2}{D \times 2g \times \left(\frac{\pi}{4}D^2\right)^2} = \frac{4fL \times Q^2}{D \times 2g \times \left(\frac{\pi}{4}\right)^2 \times D^4}$$

$$= \frac{4f \times L \times Q^2}{2g \times \left(\frac{\pi}{4}\right)^2 \times D^5}$$

$$= rQ^2 \qquad \ldots(11.39) \quad \left(\text{where } \frac{4f \times L}{2g \times \left(\frac{\pi}{4}\right)^2 \times D^5} = r\right)$$

This head loss will be positive, when the pipe is a part of loop and the flow in the pipe is clockwise.

Generally, the pipe network problems are difficult to solve analytically. Hence the methods of successive approximations are used. '**Hardy Cross Method**' is one such method which is commonly used.

11.14.1 Hardy Cross Method.

The procedure for Hardy Cross Method is as follows :

1. In this method a trial distribution of discharges is made arbitrarily but in such a way that continuity equation is satisfied at each function (or node).
2. With the assumed values of Q, the head loss in each pipe is calculated according to equation (11.39).
3. Now consider any loop (or circuits). The algebraic sum of head losses round each loop must be zero. This means that in each loop, the loss of head due to flow in clockwise direction must be equal to the loss of head due to flow in anticlockwise direction.
4. Now calculate the net head loss around each loop considering the head loss to be positive in clockwise flow and to be negative in anticlockwise flow.

If the net head loss due to assumed values of Q round the loop is zero, then the assumed values of Q in that loop is correct. But if the net head loss due to assumed values of Q is not zero, then the assumed values of Q are corrected by introducing a correction ΔQ for the flows, till the circuit is balanced.

The correction factor ΔQ* is obtained by

$$\Delta Q = \frac{-\sum r Q_0^n}{\sum r n Q_0^{n-1}} \qquad \ldots(11.40)$$

For turbulent flow, the value of $n = 2$ and hence above correction factor becomes as

$$\Delta Q = \frac{-\sum r Q_0^2}{\sum 2r Q_0} \qquad \ldots(11.41)$$

5. If the value of ΔQ comes out to be positive, then it should be added to the flows in the clockwise direction ($\because$ the flows in clockwise direction in a loops are considered positive) and subtracted from the flows in the anticlockwise direction.
6. Some pipes may be common to two circuits (or two loops), then the two corrections are applied to these pipes.

---

* Let for any pipe Q_0 = assumed discharge and Q = correct discharge, then

$$Q = Q_0 + \Delta Q$$

$\therefore$ Head loss for the pipe, $\quad h_f = rQ^2 = r(Q_0 + \Delta Q)^2$.

For complete circuit, the net head loss, $\Sigma h_f = \Sigma\,(rQ^2) = \Sigma r\,(Q_0 + \Delta Q)^2 = \Sigma r(Q_0^2 + 2Q_0\,\Delta Q + \Delta Q^2)$ $= \Sigma r\,({Q_0}^2 + 2Q_0\,\Delta Q)$ As ΔQ is small compared with Q_0 and hence ΔQ^2 can be neglected.

$$\therefore \qquad \Sigma\, rQ^2 = \Sigma\, r{Q_0}^2 + \Sigma r \times 2Q_0\Delta Q$$

For the correct distribution, the net head loss for a circuit should be zero (*i.e.*, $\Sigma h_f = \Sigma\,(rQ^2) = 0$)

$$\therefore \qquad \Sigma r Q_0^2 + \Sigma r \times 2Q_0\Delta Q = 0$$

or $\Sigma rQ_0^2 + \Delta Q\ \Sigma r \times 2Q_0 = 0$ [As ΔQ is same for one circuit, hence it can be taken out of the summation]

$$\therefore \qquad \Delta Q = \frac{-\Sigma\, r\, Q_0^2}{\Sigma\, 2r\, Q_0}.$$

7. After the corrections have been applied to each pipe in a loop and to all loops, a second trial calculation is made for all loops. The procedure is repeated till ΔQ becomes negligible.

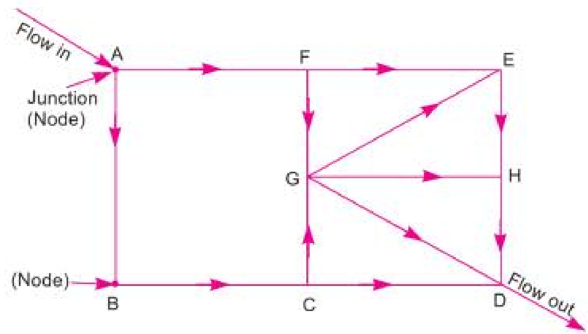


Fig. 11.33 *Pipe network.*

[Loops are : *ABCGFA, FEGF, GEHG, GHDG and GCDG*]

Problem 11.56 *Calculate the discharge in each pipe of the network shown in Fig. 11.34. The pipe network consists of 5 pipes. The head loss h_f in a pipe is given by $h_f = rQ^2$. The values of r for various pipes and also the inflow or outflows at nodes are shown in the figure.*

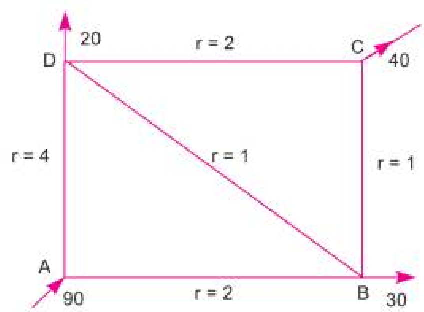


Fig. 11.34

Solution. Given :

Inflow at node A = 90, outflow at B = 30, at C = 40 and at D = 20.

Values of r for AB = 2, for BC = 1, for CD = 2, for AD = 4 and for BD = 1.

For the first trial, the discharges are assumed as shown in Fig. 11.34 (a) so that continuity is satisfied at each node (*i.e.*, flow into a node = flow out of the node). For this distribution of discharge, the corrections ΔQ for the loops ABD and BCD are calculated.

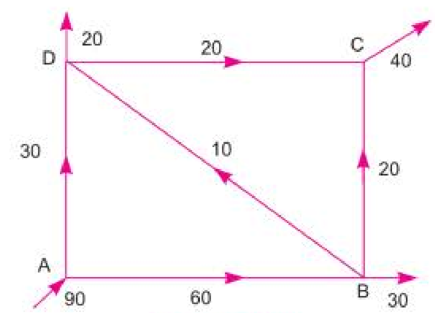


Fig. 11.34(a)

First Trial

Loop ADB

Pipe	r	Q_0	$h_f = rQ_0^2$	$2rQ_0$
AD	4	30	$4 \times 30^2 = 3600$	$2 \times 4 \times 30 = 240$
DB	1	10	$-1 \times 10^2 = -100$	$2 \times 1 \times 10 = 20$
AB	2	60	$-2 \times 60^2 = -7200$	$2 \times 2 \times 60 = 240$
			$\Sigma rQ_0^2 = -3700,$	$\Sigma 2rQ_0 = 500,$

$$\therefore \qquad \Delta Q = \frac{-\Sigma r Q_0^2}{\Sigma 2r Q_0} = \frac{-(-3700)}{500} = 7.4$$

In the loop ADB, the head loss h_f is negative in pipes DB and AB as the direction of discharges in these pipes is anticlockwise.

As ΔQ is positive for loop ADB, hence it should be added to the flow in the clockwise direction and subtractedfrom the flow in the anticlockwise direction. Hence the corrected flow for second trial for loop ADB will be as follows :

Pipe $AD = 30 + 7.4 = 37.4$ (flow is clockwise)
Pipe $AB = 60 - 7.4 = 52.6$ (flow is anticlockwise)
Pipe $BD = 10 - 7.4 = 2.\ 6$ (flow is anticlockwise)

Loop DCB

Pipe	r	Q_0	$h_f = rQ_0^2$	$2rQ_0$
DC	2	20	$2 \times 20^2 = 800$	$2 \times 2 \times 20 = 80$
CB	1	20	$-1 \times 20^2 = -400$	$2 \times 1 \times 20 = 40$
BD	1	10	$1 \times 10^2 = 100$	$2 \times 1 \times 10 = 20.$
			$\Sigma 2\ rQ_0^2 = 500$	$\Sigma 2rQ_0 = 140$

$$\therefore \quad \Delta Q = \frac{-\Sigma r Q_0^2}{\Sigma 2r Q_0} \frac{-500}{140} = -3.57 \simeq -3.6.$$

The head loss in pipe BC for loop DCB is negative as the direction of discharge in pipe BC in anticlockwise.

As ΔQ is negative for loop DCB, hence it should be subtracted from the flow in the clockwise direction and added to the flow in the anticlockwise direction. Hence corrected flow for second trial for loop DCB will be as follows :

Pipe $DC = 20 - 3.6 = 16.4$
Pipe $BC = 20 + 3.6 = 23.6$
Pipe $BD^* = 2.6 - 3.6 = -1$

Note. The pipe BD is common to two loops (*i.e.*, loop ADB and loop DCB). Hence this pipe will get two corrections. After the two corrections, the resultant flow in pipe BD is negative in loop DCB. Hence the direction of flow will be anticlockwise in pipe BD for loop DCB.

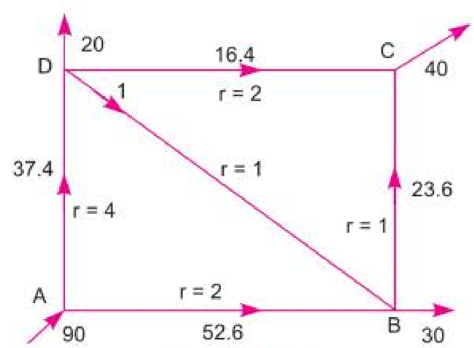


Fig. 11.34 (*b*)

The distribution of discharges in various pipes for second trial is shown in Fig. 11.34 (*b*). For second trial the correction ΔQ for loops ADB and DCB are calculated as follows :

Loop ADB

Pipe	r	Q_0	$h_f = rQ_0^2$	$2rQ_0$
AD	4	37.4	$4 \times 37.4^2 = 5595$	$2 \times 4 \times 37.4 = 299.2$
DB	1	1	$1 \times 1^2 = 1$	$2 \times 1 \times 1 = 2$
AB	2	52.6	$-2 \times 52.6^2 = -5533.5$	$2 \times 2 \times 52.6 = 210.4$
			$\Sigma rQ_0^2 = 62.54,$	$\Sigma 2rQ_0 = 511.6$

$$\therefore \Delta Q = \frac{-\sum r Q_0^2}{\sum 2r Q_0} = \frac{62.54}{-511.6}$$

$$= -0.122 \simeq -0.1$$

As ΔQ is negative, hence it should be subtracted from the flow in the clockwise direction and added to the flow in the anticlockwise direction

As the correction (ΔQ) is small (*i.e.*, $\Delta Q = -0.1$), this correction is applied and further trials are discontinued.

Hence corrected flow for

loop *ADB* will be as follows :

For pipe *AD*, $Q_0 = 37.4 - 0.1 = 37.3$ (as flow is clockwise)

For pipe *DB*, $Q_0 = 1 - 0.1 = 0.9$ (as flow is clockwise)

For pipe *AB*, $Q_0 = 52.6 + 0.1 = 52.7$ (as flow is anti-clockwise)

Loop DCB

Pipe	r	Q_0	$h_f = rQ_0^2$	$2rQ_0$
DC	2	16.4	$2 \times 16.4^2 = 537.9$	$2 \times 2 \times 16.4 = 65.6$
CB	1	23.6	$-1 \times 23.6^2 - 556.9$	$2 \times 1 \times 23.6 = 47.2$
BD	1	1	$-1 \times 1^2 = -1$	$2 \times 1 \times 1 = 2$
			$\Sigma rQ_0^2 = -20,$	$\Sigma 2rQ_0 = 114.8$

$$\therefore \Delta Q = \frac{-\sum r Q_0^2}{\sum 2r Q_0} = \frac{-(-20)}{114.8}$$

$$= \frac{20}{114.8} = 0.174$$

$$\simeq 0.2$$

As ΔQ is positive, hence it should be added to the flow in the clockwise direction and subtracted from the flow in the anticlockwise direction.

As the correction (ΔQ) is small (*i.e.*, $\Delta Q = 0.2$), this correction is applied and further trials are discontinued.

Hence corrected flow

for loop *DCB* will be as follows :

For pipe *DC*, $Q_0 = 16.4 + 0.2 = 16.6$ (clockwise flow)

For pipe *CB*, $Q_0 = 23.6 - 0.2 = 23.4$ (anticlockwise flow)

For pipe *BD*, $Q_0 = 0.9 - 0.2 = 0.7$ (anticlockwise flow)

The final distribution of discharges in each pipe is as follows :

Discharge in pipe $AD = 37.3$ from A to D

$AB = 52.7$ from A to B

$DB = 0.7$ from D to B

$DC = 16.6.$ from D to C

$BC = 23.4$ from B to C

The final discharge in each pipe is shown in Fig. 11.34 (*c*)

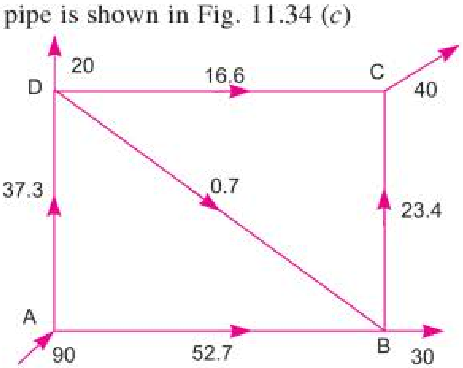


Fig. 11.34 (*c*)

Note. The pipe *DB* is common to two loop (*i.e.*, loops *ADB* and loop *DBC*). Hence this pipe will get two corrections. For loop *ADB*, the correction $\Delta Q = -0.1$ and hence the corrected flow in pipe *DB* is $1 - 0.1 = 0.9$. Now again, the correction is applied to pipe *DB* when we consider loop *DBC*. For loop *DBC*, the correction $\Delta Q = 0.2$ but flow is anticlockwise and hence the final correct flow in pipe *DB* will be $0.9 - 0.2 = 0.7$.

HIGHLIGHTS

1. The energy loss in pipe is classified as major energy loss and minor energy losses. Major energy loss is due to friction while minor energy losses are due to sudden expansion of pipe, sudden contraction of pipe, bend in pipe and an obstruction in pipe.

2. Energy loss due to friction is given by Darcy Formula, $h_f = \dfrac{4fLV^2}{d \times 2g}$.

3. The head loss due to friction in pipe can also be calculated by Chezy's formula.

$$V = C\sqrt{mi} \quad \text{Chezy's formula}$$

where C = Chezy's Constant

m = Hydrualic mean depth = $\dfrac{d}{4}$ (for pipe running full)

V = Velocity of flow

i = Loss of head per unit length = $\dfrac{h_f}{L}$

$\therefore \quad h_f = L \times i$, where i is obtained from Chezy's formula.

4. Loss of head due to sudden expansion of pipe, $h_e = \dfrac{(V_1 - V_2)^2}{2g}$

where V_1 = Velocity in small pipe, V_2 = Velocity in large pipe.

5. Loss of head due to sudden contraction of pipe, $h_c = \left(\dfrac{1}{C_c} - 1\right)\dfrac{V_2^2}{2g}$

where C_c = co-efficient of contraction $= 0.375\,\dfrac{V_2^2}{2g}$ ...(For $C_c = 0.62$)

$= 0.5\,\dfrac{V_2^2}{2g}$ ...(if value of C_c is not given)

6. Loss of head at the entrance of a pipe, $h_i = 0.5\,\dfrac{V^2}{2g}$.

7. Loss of head at the exit of pipe, $h_o = \dfrac{V^2}{2g}$.

8. The line representing the sum of pressure head and datum head with respect to some reference line is called hydraulic gradient line (H.G.L.) while the line representing the sum of pressure head, datum head and velocity head with respect to some reference line is known as total energy line (T.E.L.).

9. Syphon is a long bent pipe used to transfer liquids from a reservoir at a higher level to another reservoir at a lower level, when the two reservoirs are separated by a high level ground.

10. The maximum vacuum created at the summit of syphon is only 7.4 m of water.

11. When pipes of different lengths and different diameters are connected end to end, pipes are called in series or compound pipes. The rate of flow through each pipe connected in series is same.

12. A single pipe of uniform diameter, having same discharge and same loss of head as compound pipe consisting of several pipes of different lengths and diameters, is known as equivalent pipe. The diameter of equivalent pipe is called equivalent size of the pipe.

13. The equivalent size of the pipe is obtained from

$$\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

where L = equivalent length of pipe = $L_1 + L_2 + L_3$

d_1, d_2, d_3 = are diameters of pipes connected in series

d = equivalent size of the pipes.

14. When the pipes are connected in parallel, the loss of head in each pipe is same. The rate of flow in main pipe is equal to sum of the rate of flow in each pipe, connected in parallel.

15. For solving problems for branched pipes, the three basic, equations *i.e.*, continuity, Bernoulli's and Darcy's equations are used.

16. Power transmitted in kW through pipe is given by $P = \dfrac{\rho g \times Q \times (H - h_f)}{1000}$

where Q = discharge through pipe = area × velocity = $\dfrac{\pi}{4} d^2 \times V$

H = total head at inlet of pipe

h_f = head lost due to friction

$= \dfrac{4fLV^2}{d \times 2g}$, where L = Length of pipe

In S.I. units, power transmitted is given by, Power = $\dfrac{\rho g \times Q \times (H - h_f)}{1000}$ kW.

17. Efficiency of power transmission through pipes, $\eta = \dfrac{H - h_f}{H}$.

18. Condition for maximum transmission of power through pipe,, $h_f = \dfrac{H}{3}$ and maximum efficiency $= 66.67\%$.

19. The velocity of water at the outlet of the nozzle is $v = \sqrt{\dfrac{2gH}{\left(1 + \dfrac{4fL}{D} \times \dfrac{a^2}{A^2}\right)}}$

where H = head at the inlet of the pipe, L = length of the pipe,

D = diameter of the pipe, a = area of the nozzle outlet,

A = area of the pipe.

20. The power transmitted through nozzle, $P = \dfrac{\rho g \times Q}{1000}\left[H - \dfrac{4fLV^2}{D \times 2g}\right]$

and the efficiency of power transmission through nozzle, $\eta = \dfrac{1}{\left[1 + \dfrac{4fL}{D} \times \dfrac{a^2}{A^2}\right]}$

21. Condition for maximum power transmission through nozzle, $h_f = \dfrac{H}{3}$.

22. Diameter of nozzle for maximum power transmission through nozzle is, $d = \left(\frac{D^5}{8fL}\right)^{1/4}$

where d = diameter of the nozzle at outlet, D = diameter of the pipe,
L = length of the pipe, f = co-efficient of friction for pipe.

23. When a liquid is flowing through a long pipe fitted with a valve at the end of the pipe and the valve is closed suddenly, a pressure wave of high intensity is produced behind the valve. This pressure wave of high intensity is having the effect of hammering action on the walls of the pipe. This phenomenon is known as water hammer.

24. The intensity of pressure rise due to water hammer is given by

$$p = \frac{\rho LV}{T} \quad \text{... when valve is closed gradually.}$$

$$= V\sqrt{K\rho} \quad \text{... when valve is closed suddenly and pipe is assumed rigid}$$

$$= V \times \sqrt{\frac{\rho}{\frac{1}{K} + \frac{D}{Et}}} \quad \text{... when valve is closed suddenly and pipe is elastic.}$$

where L = Length of pipe, V = Velocity of flow,
T = Time required to close the valve, K = Bulk modulus of water,
D = Diameter of the pipe, E = Modulus of elasticity for pipe material.
t = Thickness of the pipe wall,

25. If the time required to close the valve :

$$T > \frac{2L}{C} \quad \text{... the valve closure is said to be gradual,}$$

$$T < \frac{2L}{C} \quad \text{... the valve closure is said to be sudden}$$

where L = length of pipe,

C = velocity of pressure wave produced due to water hammer = $\sqrt{\frac{K}{\rho}}$.

EXERCISE

(A) THEORETICAL PROBLEMS

1. How will you determine the loss of head due to friction in pipes by using (*i*) Darcy Formula and (*ii*) Chezy's formula ?
2. (*a*) What do you understand by the terms : Major energy loss and minor energy losses in pipes ?
   (*b*) What do you understand by total energy line, hydraulic gradient line, pipes in series, pipes in parallel and equivalent pipe ?
3. (*a*) Derive an expression for the loss of head due to : (*i*) Sudden enlargement and (*ii*) Sudden contraction of a pipe.
   (*b*) Obtain expression for head loss in a sudden expansion in the pipe. List all the assumptions made in the derivation.
4. Define and explain the terms : (*i*) Hydraulic gradient line and (*ii*) Total energy line.

5. Show that the loss of head due to sudden expansion in pipe line is a function of velocity head.
6. What is a syphon ? On what principle it works ?
7. What is a compound pipe ? What will be loss of head when pipes are connected in series ?
8. Explain the terms : (*i*) Pipes in parallel (*ii*) Equivalent pipe and (*iii*) Equivalent size of the pipe.
9. Find an expression for the power transmission through pipes. What is the condition for maximum transmission of power and corresponding efficiency of transmission ?
10. Prove that the head loss due to friction is equal to one-third of the total head at inlet for maximum power transmission through pipes or nozzles.
11. Prove that the velocity through nozzle is given by $v = \sqrt{\dfrac{2gH}{1 + \dfrac{4fL}{D} \times \dfrac{a^2}{A^2}}}$

    where a = Area of nozzle at outlet, A = Area of the pipe.
12. Show that the diameter of the nozzle for maximum transmission of power is given by $d = \left(\dfrac{D^5}{8fL}\right)^{1/4}$

    where D = Diameter of pipe, L = Length of pipe.
13. Find an expression for the ratio of the outlet area of the nozzle to the area of the pipe for maximum transmission of power.
14. Explain the phenomenon of Water Hammer. Obtain an expression for the rise of pressure when the flowing water in a pipe is brought to rest by closing the valve gradually.
15. Show that the pressure rise due to sudden closure of a valve at the end of a pipe, through which water is flowing is given by $p = V\sqrt{\dfrac{d}{\dfrac{1}{K} + \dfrac{D}{Et}}}$

    where V = Velocity of flow, D = Diameter of pipe, E = Young's Modulus, K = Bulk Modulus and t = Thickness of pipe.
16. Three pipes of different diameters and different lengths are connected in series to make a compound pipe. The ends of this compound pipe are connected with two tanks whose difference of water level is H. If co-efficient of friction for these pipes is same, then derive the formula for the total head loss, neglecting first the minor losses and then including them.
17. For the two cases of flow in a sudden contraction in a pipeline and flow in a sudden expansion in a pipe line, draw the flow pattern, piezometric grade line and total energy line.
18. What do you mean by "equivalent pipe" and "flow through parallel pipes" ?
19. (*a*) Define and explain the terms : (*i*) Hydraulic gradient line and (*ii*) total energy line.
    (*b*) What do you mean by equivalent pipe ? Obtain an expression for equivalent pipe.

    (Delhi University, December 2002)

(B) NUMERICAL PROBLEMS

1. Find the head loss due to friction in a pipe of diameter 250 mm and length 60 m, through which water is flowing at a velocity of 3.0 m/s using (*i*) Darcy formula and (*ii*) Chezy's Formula for which $C = 55$. Take ν for water = .01 stoke. [**Ans.** (*i*) 1.182, (*ii*) 2.856]
2. Find the diameter of a pipe of length 2500 m when the rate of flow of water through the pipe is 0.25 m^3/s and head loss due to friction is 5 m. Take $C = 50$ in Chezy's formula. [**Ans.** 605 mm]

3. An oil of Kinematic Viscosity 0.5 stoke is flowing through a pipe of diameter 300 mm at the rate of 320 litres per sec. Find the head lost due to friction for a length of 60 m of the pipe. [**Ans.** 5.14 m]
4. Calculate the rate of flow of water through a pipe of diameter 300 mm, when the difference of pressure head between the two ends of a pipe 400 m apart is 5 m of water. Take the value of f = .009 in the formula

$$h_f = \frac{4fLV^2}{d \times 2g}$$ [**Ans.** 0.101 m^3/s]

5. The discharge through a pipe is 200 litres/s. Find the loss of head when the pipe is suddenly enlarged from 150 mm to 300 mm diameter. [**Ans.** 3.672 m]
6. The rate of flow of water through a horizontal pipe is 0.3 m^3/s. The diameter of the pipe is suddenly enlarged from 250 mm to 500 mm. The pressure intensity in the smaller pipe is 13.734 N/cm^2. Determine : (*i*) loss of head due to sudden enlargement, (*ii*) pressure intensity in the large pipe and (*iii*) power lost due to enlargement. [**Ans.** (*i*) 1.07 m, (*ii*) 14.43 N/cm^2, (*iii*) 3.15 kW]
7. A horizontal pipe of diameter 400 mm is suddenly contracted to a diameter of 200 mm. The pressure intensities in the large and smaller pipe is given as 14.715 N/cm^2 and 12.753 N/cm^2 respectively. If C_c = 0.62, find the loss of head due to contraction. Also determine the rate of flow of water. [**Ans.** (*i*) 0.571 m, (*ii*) 171. 7 litres/s]
8. Water is flowing through a horizontal pipe of diameter 300 mm at a velocity of 4 m/s. A circular solid plate of diameter 200 mm is placed in the pipe to obstruct the flow. If C_c = 0.62, find the loss of head due to obstruction in the pipe. [**Ans.** 2.953 m]
9. Determine the rate of flow of water through a pipe of diameter 10 cm and length 60 cm when one end of the pipe is connected to a tank and other end of the pipe is open to the atmosphere. The height of water in the tank from the centre of the pipe is 5 cm. Pipe is given as horizontal and value of f = .01. Consider minor losses. [**Ans.** 15.4 litres/s]
10. A horizontal pipe-line 50 m long is connected to a water tank at one end and discharges freely into the atmosphere at the other end. For the first 30 m of its length from the tank, the pipe is 200 mm diameter and its diameter is suddenly enlarged to 400 mm. The height of water level in the tank is 10 m above the centre of the pipe. Considering all minor losses, determine the rate of flow. Take f = .01 for both sections of the pipe. [**Ans.** 164.13 litres/s]
11. Determine the difference in the elevations between the water surfaces in the two tanks which are connected by a horizontal pipe of diameter 400 mm and length 500 m. The rate of flow of water through the pipe is 200 litres/s. Consider all losses and take the value of f = .009. [**Ans.** 11.79 m]
12. For the problems 9, 10 and 11 draw the hydraulic gradient lines (H.G.L.) and total energy lines (T.E.L.)
13. A syphon of diameter 150 mm connects two reservoirs having a difference in elevation of 15 m. The length of the syphon is 400 m and summit is 4.0 m above the water level in the upper reservoir. The length of the pipe from upper reservoir to the summit is 80 m. Determine the discharge through the syphon and also pressure at the summit. Neglect minor losses. The co-efficient of friction, f = .005. [**Ans.** 41.52 litres/s, – 7.281 m of water]
14. A syphon of diameter 200 mm connects two reservoirs having a difference in elevation as 20 m. The total length of the syphon is 800 m and the summit is 5 m above the water level in the upper reservoir. If the separation takes place at 2.8 m of water absolute find the maximum length of syphon from upper reservoir to the summit. Take f = .004 and atmospheric pressure = 10.3 m of water. [**Ans.** 87.52 m]
15. Three pipes of lengths 800 m, 600 m and 300 m and of diameters 400 mm, 300 mm and 200 mm respectively are connected in series. The ends of the compound pipe is connected to two tanks, whose water surface levels are maintained at a difference of 15 m. Determine the rate of flow of water through the pipes if f = .005. What will be diameter of a single pipe of length 1700 m and f = .005, which replaces the three pipes ? [**Ans.** 0.0848 m^3/s, 266.5 mm]

16. Two pipes of lengths 2500 m each and diameters 80 cm and 60 cm respectively, are connected in parallel. The co-efficient of friction for each pipe is 0.006. The total flow is equal to 250 litres/s. Find the rate of flow in each pipe. [**Ans.** 0.1683 m^3/s, 0.0817 m^3/s]
17. A pipe of diameter 300 mm and length 1000 m connects two reservoirs, having difference of water levels as 15 m. Determine the discharge through the pipe. If an additional pipe of diameter 300 mm and length 600 m is attached to the last 600 m length of the existing pipe, find the increase in the discharge. Take $f = .02$ and neglect minor losses. [**Ans.** 0.0742 m^3/s, 0.0258 m^3/s]
18. Two sharp ended pipes of diameters 60 mm and 100 mm respectively, each of length 150 m are connected in parallel between two reservoirs which have a difference of level of 15 m. If co-efficient of friction for each pipe is 0.08, calculate the rate of flow for each pipe and also the diameter of a single pipe 150 m long which would give the same discharge if it were substituted for the original two pipes. [**Ans.** 0.0017, .00615, 110 mm]
19. Three reservoirs A, B and C are connected by a pipe system having length 700 m, 1200 m and 500 m and diameters 400 mm, 300 mm and 200 mm respectively. The water levels in reservoir A and B from a datum line are 50 m and 45 m respectively. The level of water in reservoir C is below the level of water in reservoir B. Find the discharge into or from the reservoirs B and C if the rate of flow from reservoir A is 150 litres per sec. Find the height of water level in the reservoir C. Take $f = .005$ for all pipes. [**Ans.** .005 m^3/s, .095 m^3/s, 24.16 m]
20. A pipe of diameter 300 mm and length 3000 m is used for the transmission of power by water. The total head at the inlet of the pipe is 400 m. Find the maximum power available at the outlet of the pipe. Take $f = .005$. [**Ans.** 667.07 kW]
21. A pipe line of length 2100 m is used for transmitting 103 kW. The pressure at the inlet of the pipe is 392.4 N/cm^2. If the efficiency of transmission is 80%, find the diameter of the pipe. Take $f = .005$. [**Ans.** 136 mm]
22. A nozzle is fitted at the end of a pipe of length 400 m and of diameter 150 mm. For the maximum transmission of power through the nozzle, find the diameter of the nozzle. Take $f = .008$. [**Ans.** 41.5 mm]
23. The head of water at the inlet of a pipe of length 1500 m and of diameter 400 mm is 50 m. A nozzle of diameter 80 mm at the outlet, is fitted to the pipe. Find the velocity of water at the outlet of the nozzle if $f = .01$ for the pipe. [**Ans.** 28.12 m/s]
24. The rate of flow of water through a pipe of length 1500 m and diameter 800 mm is 2 m^3/s. At the end of the pipe a nozzle of outside diameter 200 mm is fitted. Find the power transmitted through the nozzle if the head of water at the inlet of the pipe is 180 m and $f = .01$ for pipe. [**Ans.** 2344.7 kW]
25. The water is flowing with a velocity of 2 m/s in a pipe of length 2000 m and of diameter 600 mm. At the end of the pipe, a valve is provided. Find the rise in pressure if the valve is closed in 20 seconds. Take the value of $C = 1420$ m/s. [**Ans.** 20 N/cm^2]
26. If the valve in the problem 25 is closed in 1.5 sec, find the rise in pressure. Take bulk modulus of water $= 19.62 \times 10^4$ N/cm^2 and consider pipe as rigid one. [**Ans.** 186.75 N/cm^2]
27. If in the problem 25, the thickness of the pipe is 10 mm and the valve is closed suddenly. Find the rise in pressure if the pipe is considered to be elastic. Take value of $E = 19.62 \times 10^6$ N/cm^2 for pipe material and $K = 19.62 \times 10^4$ N/cm^2 for water. Calculate the circumferential stress and longitudinal stress developed in the pipe wall. [**Ans.** $p = 221.47$ N/cm^2, $f_c = 6644.1$ N/cm^2, $f_l = 3322$ N/cm^2]
28. The difference in water surface levels in two tanks, which are connected by two pipes in series of lengths 600 m and 400 m and of diameters 30 cm and 20 cm respectively, is 15 m. Determine the rate of flow of water if the co-efficient of friction is 0.005 for both the pipes. Neglect minor losses.
29. Water is flowing vertically downwards through a 10 cm diameter pipe at the rate of 50 l.p.s. At a particular location the pipe suddenly enlarges to 20 cm diameter. A point P is located 50 cm above the section of enlargement and another point Q is located 50 cm below it in the enlarged portion. A pressure gauge connected at P gives a reading of 19.62 N/cm^2. Calculate the pressure at location Q neglecting friction loss between P and Q but considering the loss due to sudden enlargement. What

neglecting friction loss between P and Q but considering the loss due to sudden enlargement. What will be the pressure at Q if the same discharge flows upwards assuming that the pressure P remains the same ? Consider the loss due to contraction with $C_c = 0.60$ but neglect friction loss between P and Q.
[**Ans.** 21.36 N/cm^2, 23.4 N/cm^2] (*A.M.I.E., Summer, 1985*)

30. Two tanks are connected with the help of two pipes in series. The lengths of the pipes are 1000 m and 800 m whereas the diameters are 400 mm and 200 mm respectively. The co-efficient of friction for both the pipes is 0.008. The difference of water level in the two tanks is 15 m. Find the rate of flow of water through the pipes, considering all losses. Also draw the total energy line and hydraulic gradient lines for the system. (*Delhi University, May 1998*) [**Ans.** 0.0464 m^3/s]

[**Hint.** $L_1 = 1000$ m ; $L_2 = 800$ m ; $d_1 = 400$ mm $= 0.4$ m ; $d_2 = 200$ mm $= 0.2$ m, $f = 0.008$; $H = 15$ m.

Now $$H = h_i + hf_1 + h_c + hf_2 + h_0,$$

$$H = \frac{0.5\,V_1^2}{2g} + \frac{4f \times L_1 \times V_1^2}{d \times 2g} + \frac{0.5\,V_2^2}{2g} + \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} + \frac{V_2^2}{2g}$$

or $$15 = \frac{0.5\,V_1^2}{2 \times 9.81} + \frac{4 \times 0.008 \times 1000 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{0.5\,V_2^2}{2 \times 9.81} + \frac{4 \times 0.008 \times 800 \times V_2^2}{0.2 \times 2 \times 9.81} + \frac{V_2^2}{2g}$$

Also $$A_1V_1 = A_2V_2 \text{ or } V_2 = 4V_1$$

$\therefore$ $$15 = 0.02548V_1^2 + 4.0775V_1^2 + 0.02548V_2^2 + 6.524V_2^2 + 0.05097V_2^2$$
$$= 4.103V_1^2 + 6.6V_2^2 = 4.103V_1^2 + 6.6 \times (4V_1)^2$$
$$= 4.103V_1^2 + 105.607V_1^2 = 109.71V_1^2$$

$\therefore$ $$V_1 = \sqrt{\frac{15}{109.71}} = 0.3697 \text{ m/s} \;\therefore\; Q = A_1V_1 = \frac{\pi}{4}(.4)^2 \times 0.3697 = 0.0464 \text{ m}^3\text{/s}$$]

31. A pipe of diameter 25 cm and length 2000 m connects two reservoirs, having difference of water level 25 m. Determine the discharge through the pipe. If an additional pipe of diameter 25 cm and length 1000 m is attached to the last 1000 m length of the existing pipe, find the increase in discharge. Take $f = 0.015$. Neglect minor losses. (*Delhi University, December 2002*) [**Ans.** (*i*) 49.62 l/s, (*ii*) 13.14 l/s]